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Local Structure of Cellular Automata

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Abstract

From the definition of a cellular automaton (S, Q, f, ν) with S a discrete cellular space, Q a finite set of cell states, f an n-ary local function $f(x_1, ..., x_n)$ and ν a neighborhood function ν : $\{1, ..., n\} \to S$, we pick up a pair (f, ν) called the *local structure*. Introducing the local structure has revealed new aspects of changing (permuting) the neighborhood and corresponding variables of a local function. Particularly, it is proved that if (f, ν) and (f', ν') are two reduced local structures which are equivalent, then there is a permutation π such that $\nu^\pi = \nu'$.

Key words cellular automaton, local structure, neighborhood, permutation, equivalence, isomorphism, reversibility

1 Introduction

Most studies on cellular automata (CA for short) first assume some standard neighborhood (von Neumann, Moore) or its modifications and then investigate the global behaviors and mathematical properties or look for a local function that would meet a given problem, say, the self-reproduction [8], the Game of Life [1] and so on. In 2003, however, H.Nishio and M.Margenstern began a general study of the neighborhood in its own right, where the neighborhood N can be an arbitrary finite subset of the space S and particularly discussed the problem if N generates (*fills*) S or not [6]. On the other hand, as for the dynamics of CA, it has been shown that some properties depend on the choice of the neighborhood, while others do not [4].

Following such research on the neighborhood of CA, the notion of *the neighborhood function*, though not so named, was first introduced by T. Worsch and H. Nishio (2007) for achieving universality of CA by changing the neighborhood [11, 10]. The notion of the neighborhood function has been used for the research of mathematical properties such as injectivity/surjectivity of CA with the neighborhood being changed [7, 5] .

In this paper we newly define *the local structure* (f, ν), i.e. from the definition of a cellular automaton (S, Q, f, ν) with S a discrete cellular space, Q a finite set of cell states, f an n-ary local function $f(x_1, ..., x_n)$ and ν a neighborhood (function) $\nu : \{1, ..., n\} \to S$, we pick up a pair (f, ν) called the *local structure* for investigating the significance of neighborhoods relative to local functions. After giving definitions, we give some basic results including a lemma : *If* (f, ν) *and* (f', ν') *are two reduced local structures which are equivalent, then there is a permutation* π *such that* $\nu^\pi = \nu'$. A corollary to this lemma gives another simple proof for the first theorem shown in [7, 5]: *By changing the neighborhood function, infinitely many different global CA functions are induced by any single local function which is not constant.*

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2 Definitions

2.1 Cellular Automaton CA (S, Q, f, ν)

A cellular automaton (CA for short) is defined by a 4-tuple (S, Q, f, ν):

- S: a discrete cellular space such as $\mathbb{Z}^d$, hyperbolic space ...
- Q: a finite set of the states of each cell.
- $f : Q^n \to Q$: a local function in $n \geq 1$ variables.
- ν: an injective map from $\{1, ..., n\}$ to S, called a *neighborhood function*, which connects the i-th variable of f to $\nu(i)$. That is, $(\nu(1), ..., \nu(n))$ becomes a neighborhood of size n.

In order to study effects of changing the neighborhood (function), we define the pair (f, ν) as a *local structure* of CA and investigate its mathematical properties.

2.2 Local Structure (f, ν)

In this paper we assume that S is a d-dimensional Euclidean grid $\mathbb{Z}^d$ $(d \geq 1)$ with the additive operator $+$.

Definition 1 *[neighborhood]*

For $n \in \mathbb{N}$, a neighborhood (function) is a mapping $\nu : \mathbb{N}_n \to \mathbb{Z}^d$, where $\mathbb{N}_n = \{1, 2, \ldots, n\}$.

This can equivalently be seen as a list ν with n components; $(\nu_1, \ldots, \nu_n)$, where $\nu_i = \nu(i), 1 \leq i \leq n$.

The set of all neighborhoods of size n will be denoted as $\mathcal{N}_n$.

Definition 2 *[local structure, reduced]*

A pair (f, ν) of a local function $f : Q^n \to Q$ and a neighborhood $\nu \in \mathcal{N}_n$ is called a local structure. We call n the arity of the local structure.

A local structure is called reduced, if and only if the following conditions are fulfilled:

- *f depends on all arguments.*
- *ν is injective, i.e. $\nu_i \neq \nu_j$ for $i \neq j$ in the list of neighborhood ν.*

Each local structure induces the *global function* $F : Q^{\mathbb{Z}^d} \to Q^{\mathbb{Z}^d}$ or the dynamics of CA. Every element $c \in Q^{\mathbb{Z}^d}$ is called a *(global) configuration*. For any global configuration $c \in Q^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$, let $c(x)$ be the state of cell x in c. Then F is given by $F(c)(x) = f(c(x + \nu_1), c(x + \nu_2), ..., c(x + \nu_n))$.

2.3 Equivalence

Definition 3 *[equivalence]*

Two local structures (f, ν) and (f', ν') are called equivalent, if and only if they induce the same global function. In that case we sometimes write $(f, \nu) \approx (f', \nu')$.

Lemma 1
For each local structure (f,ν) there is an equivalent reduced local structure (f',ν').

Proof.
Let n denote the arity of (f,ν). Assume that (f,ν) is not reduced.

We assume that $n \geq 1$ and show how to construct an equivalent local structure (f',ν') with arity $n-1$.

Case 1: ν is not injective. Then clearly $n \geq 2$. Let i and j be indices such that $i < j$ and $\nu_i = \nu_j$. Define $\nu' \in \mathcal{N}_{n-1}$ as

$$\nu'_k = \begin{cases} \nu_k & \text{iff } k < j \\ \nu_{k+1} & \text{iff } k \geq j, \end{cases} \tag{1}$$

i.e. drop the j-th component of ν, and define $f' : Q^{n-1} \to Q$ by

$$f'(q_1,\ldots,q_{n-1}) = f(q_1,\ldots,q_{j-1},q_i,q_j,\ldots,q_{n-1}) \tag{2}$$

For any configuration $c \in Q^{\mathbb{Z}^d}$, we have

$$\begin{aligned} F'(c)(0) &= f'(c(\nu'_1),\ldots,c(\nu'_{n-1})) \\ &= f(c(\nu'_1),\ldots,c(\nu'_{j-1}),c(\nu'_i),c(\nu'_j),\ldots,c(\nu'_{n-1})) \\ &= f(c(\nu_1),\ldots,c(\nu_{j-1}),c(\nu_i),c(\nu_{j+1}),\ldots,c(\nu_n)) \\ &= f(c(\nu_1),\ldots,c(\nu_{j-1}),c(\nu_j),c(\nu_{j+1}),\ldots,c(\nu_n)) \\ &= F(c)(0) \end{aligned} \tag{3}$$

Since application of local functions commutes with shifts, it follows $F'(c)(x) = F(c)(x)$ for all $x \in \mathbb{Z}^d$.

Case 2: f does not depend on all arguments. Then clearly $n \geq 1$. Assume that it does not depend on argument x_i, $1 \leq i \leq n$. Define $\nu' \in \mathcal{N}_{n-1}$ as

$$\nu'_k = \begin{cases} \nu_k & \text{iff } k < i \\ \nu_{k+1} & \text{iff } k \geq i, \end{cases} \tag{4}$$

and

$$f'(q_1,\ldots,q_{n-1}) = f(q_1,\ldots,q_{i-1},q,q_{i+1},\ldots,q_{n-1}), \tag{5}$$

for any $q \in Q$. Since f does not depend on the i-th argument, f' is well defined.

For any configuration $c \in Q^{\mathbb{Z}}$, we have

$$\begin{aligned} F'(c)(0) &= f'(c(\nu'_1),\ldots,c(\nu'_{i-1}),c(\nu'_i),c(\nu'_{i+1}),\ldots,c(\nu'_{n-1})) \\ &= f(c(\nu'_1),\ldots,c(\nu'_{i-1}),q,c(\nu'_i),c(\nu'_{i+1}),\ldots,c(\nu'_{n-1})) \\ &= f(c(\nu_1),\ldots,c(\nu_{i-1}),q,c(\nu_{i+1}),c(\nu_{i+2}),\ldots,c(\nu_n)) \\ &= f(c(\nu_1),\ldots,c(\nu_{i-1}),c(\nu_i),c(\nu_{i+1}),c(\nu_{i+2}),\ldots,c(\nu_n)) \\ &= F(c)(0) \end{aligned} \tag{6}$$

By *downward induction* on n, we reach an equivalent reduced local structure of less arity than n. ■

The construction above does *not imply* that the equivalent reduced local structure itself is *unique*. In fact in general *it is not*: As a simple example consider the local function $f(x_1,x_2)$ over $GF(2)$: $(x_1,x_2) \mapsto x_1 + x_2 \,(\text{mod.}2)$. Since the order of the arguments x_i does not matter for the value $f(x_1,x_2)$, the local structures $(f,(0,1))$ and $(f,(1,0))$ are equivalent. At the same time both are obviously reduced.

2.4 Permutation of Local Structure

Definition 4 *[permutation of local structure]*
Let π denote a permutation of the numbers in $\mathbb{N}_n$.

- *For a neighborhood ν, denote by ν^π the neighborhood defined by $\nu^\pi_{\pi(i)} = \nu_i$.*
- *For an n-tuple $\ell \in Q^n$, denote by ℓ^π the permutation of ℓ such that $\ell^\pi(i) = \ell(\pi(i))$ for $1 \le i \le n$.*

  For a local function $f : Q^n \to Q$, denote by f^π the local function $f^\pi : Q^n \to Q$ such that $f^\pi(\ell) = f(\ell^\pi)$ for all ℓ.

In the first part of the definition we have preferred the given specification to the equally possible $\nu^\pi_i = \nu_{\pi(i)}$, because the former leads to a slightly simpler formulation of the following lemma.

3 Results

Lemma 2
(f,ν) and (f^π,ν^π) are equivalent for any permutation π.

Proof:
For any configuration c:

$$
\begin{aligned}
F^\pi(c)(0) &= f^\pi(c(\nu^\pi_1),\ldots,c(\nu^\pi_n)) \\
&= f(c(\nu^\pi_{\pi(1)}),\ldots,c(\nu^\pi_{\pi(n)})) \\
&= f(c(\nu_1),\ldots,c(\nu_n)) \\
&= F(c)(0)
\end{aligned} \tag{7}
$$

■

We are now going to show that for *reduced* local structures, the relationship *via a permutation* is the *only* possibility to get equivalence.

Lemma 3
If (f,ν) and (f',ν') are two reduced local structures which are equivalent, then there is a permutation π such that $\nu^\pi = \nu'$.

Proof:
Assume that there is an x which does not appear in ν' but does appear in ν, say at position i. Since (f,ν) is reduced, f does depend on its i-th argument and there are two configurations c and $\overline{c}$, which do *only* differ at cell x, such that $F(c)(0) \neq F(\overline{c})(0)$. Since ν' does not contain x, it is clear that $F'(c)(0) = F'(\overline{c})(0)$. It is therefore impossible that $F(c)(0) = F'(c)(0)$ and simultaneously $F(\overline{c})(0) = F'(\overline{c})(0)$. Hence $F(c) \neq F'(c)$ and $F \neq F'$. ■

Lemma 4
If (f,ν) and (f',ν') are two reduced local structures which are equivalent, then there is a permutation π such that $(f^\pi,\nu^\pi) = (f',\nu')$.

Proof:
By Lemma 3 we already know that ν and ν' are permutations of each other: $\nu' = \nu^{\pi}$ for some π. But it is clear that different local functions induce different global functions, if they use the same neighborhood. Hence if $f' \neq f^{\pi}$, then $(f', \nu^{\pi}) \not\approx (f^{\pi}, \nu^{\pi}) \approx (f, \nu)$. ■

By choosing different neighborhoods which are not permutations of each other, one immediately gets the following corollary, which claims the same thing as Theorem 1 of H.Nishio, MCU2007 [5]: *By changing the neighborhood function ν, infinitely many different global CA functions are induced by any single local function $f_3(x, y, z)$ which is not constant.* Proof was given for 1-dimensional CA by concretely showing biinfinite words which correspond to different neighborhoods.

Corollary 1
For each reduced non-constant local function f, there are infinitely many reduced neighborhoods ν, such that the local structures (f, ν) induce pairwise different global CA functions.

4 Isomorphism

Since the above definition of equivalence is *too strong*, we will consider a weaker notion *isomorphism* which allows permutation of the set of cell states.

In the same space S, consider two CA A and B having different local structures (f_A, ν_A) and (f_B, ν_B), where f_A and f_B are defined on possibly different domains; $f : Q_A^n \to Q_A$ and $f_B : Q_B^{n'} \to Q_B$.

Definition 5
If $|Q_A| = |Q_B|$, then we can consider a bijection $\varphi : Q_A \to Q_B$. Two CA A and B are called isomorphic under φ denoted by $A \underset{\varphi}{\sim} B$, if and only if the following diagram commutes for all global configurations. Note that bijection φ naturally extends to $\varphi : Q_A^{\mathbf{Z}^d} \to Q_B^{\mathbf{Z}^d}$.

$$\begin{array}{ccc} c_A & \xrightarrow{\varphi} & c_B \\ {\scriptstyle F_A}\downarrow & & \downarrow{\scriptstyle F_B} \\ c'_A & \xrightarrow{\varphi} & c'_B \end{array} \tag{8}$$

where c_A (c_B) is a global configuration of A (B) and c'_A (c'_B) is the next configuration of $c_A(c_B)$.

Both equivalence and isomorphism of local structures are evidently equivalence relations.

From the definitions of equivalence and isomorphism among local structures, we have

Lemma 5
If $(f_A, \nu_A) \approx (f_B, \nu_B)$, then $(f_A, \nu_A) \underset{\varphi}{\sim} (f_B, \nu_B)$ for any φ. The converse is not always true.

Example: We consider 6 Elementary CA which are show to be reversible in page 436 of [9]. Rules 15, 51 and 85 are equivalent (and isomorphic) each other. Rules 170, 204 and 240 are equivalent (and isomorphic). However, rules 15 and 240 (resp. 51 and 204, 85 and 170) are *not equivalent but isomorphic* under $\varphi : 0 \mapsto 1, 1 \mapsto 0$. Summing up those 6 Elementary reversible CA are all isomorphic.

For the isomorphism too, the following lemma is proved in the same manner as Lemma 3.

Lemma 6 (Lemma 3')
If (f_A, ν_A) and (f_B, ν_B) are two reduced local structures which are φ-isomorphic under a bijection $\varphi : Q_A \rightarrow Q_B$, then there is a permutation π such that $\nu_A^{\pi} = \nu_B$.

Proof.
Assume that there is an x which does not appear in ν_B but does appear in ν_A, say at position i. Since (f_A, ν_A) is reduced, f_A does depend on its i-th argument and there are two configurations c_A and $\overline{c_A}$, which do *only* differ at cell x, such that $F(c_A)(0) \neq F(\overline{c_A})(0)$.

Since ν_B does not contain x, clearly $F_B(\varphi(c_A))(0) = F_B(\overline{\varphi(c_A)})(0)$. It is therefore impossible that $F_A(c_A)(0) = F_B(\varphi(c_A))(0)$ and simultaneously $F_A(\overline{c_A})(0) = F_B(\overline{\varphi(c_A)})(0)$. Hence $F_A(c_A) \neq F_B(\varphi(c_A))$ and $F_A \neq F_B$. ■

5 Concluding Remarks

In this paper we focused on a pair (f, ν) called *local structure* and examined equivalence or *sameness* of CA computation with respect to *permutations* of the neighborhood ν and the local function f as well as the state set. In this respect we notice some past definitions of equivalence, isomorphism and homomorphism of CA [3] [2].

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