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ON CONCEPT LATTICE APPROXIMATION

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ABSTRACT. In these notes we motivate the need of approximating concept lattices by lattices of small size or by lattices of nicer structure. We also present some approachs towards an approximation theory for concept lattices. The main goal is to support Formal Concept Analysis (FCA) technics in Knowledge Discovery and Knowledge Management.

1. FCA AND KNOWLEDGE DISCOVERY

1.1. Information systems, contexts and concepts. The elementary way to encode information is to describe, by means of a relation that some objects have some properties. This sets up a binary relation I between the set G of objects and the set M of properties. The triple (G, M, I) is called a **formal context.** $(g, m) \in$ I reads "*the object g has the property m*"; we also write $g\,\mathrm{I}\,m$. Figure 1 (taken from [GS]) shows us the members of the Star Alliance and their flying destinations. Some interesting patterns are formed by objects sharing the same properties. In Knowledge discovery, many techniques are based on the formalization of such patterns, namely that of *concept*. A **formal concept** of a context (G, M, I) is a pair (A, B) such that B is exactly the set of all properties shared by the objects in A (denoted by A^I or A'), and A is the set of all objects that have all the properties in B (denoted by B^I or B'). We called A the **extent** of the concept (A, B) and B the **intent** of the concept (A, B). The **concept hierarchy** states that *a concept is more general if its extent is larger* or equivalently *if its intent is smaller*. Formally, $A \subseteq C \iff: (A, B) \leq (C, D) :\iff B \supseteq D$. This defines an order relation on the set $\mathfrak{B}(G, M, I)$ of all concepts of (G, M, I). The structure of this poset is more richer. We call a poset $(L, \leq)$ **lattice** if $\min\{x, y\}$ and $\max\{x, y\}$[1] exist for any pair $\{x, y\} \subseteq L$. A poset $(L, \leq)$ is a **complete lattice** if $\min Z$ and $\max Z$ (also denoted by $\bigwedge Z$ and $\bigvee Z$ resp.) exist for every subset Z of L. Equivalently, a **lattice** is an algebra $(L, \wedge, \vee)$ of type $(2, 2)$ such that $\wedge$ and $\vee$ are idempotent, commutative, associative and satisfy the absorption laws: $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$. It holds: $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and $x \wedge y = x \iff x \leq y \iff x \vee y = y$.

Theorem 1.1. [Wi82] *$(\mathfrak{B}(G, M, I), \leq)$ is a complete lattice, in which infimum and supremum are given by*

$$\bigwedge_{k\in K} (A_k, B_k) = \left(\bigcap_{k\in K} A_k, \left(\bigcup_{k\in K} B_k\right)''\right); \quad \bigvee_{k\in K} (A_k, B_k) = \left(\left(\bigcup_{k\in K} A_k\right)'', \bigcap_{k\in K} B_k\right).$$

$\mathfrak{B}(G, M, I)$ is called the **concept lattice** of the context (G, M, I). For $g \in G$ and $m \in M$ we set $g' := \{g\}'$ and $m' := \{m\}'$. We define some special *building block* concepts: $\gamma g := (g'', g')$ (**object concept**) and $\mu m := (m', m'')$ (**attribute**

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[1] $\min\{x, y\}$ (also denoted by $x \wedge y$) is the greatest element below x and y. $\max\{x, y\}$ (also denoted by $x \vee y$) is the smallest element above x and y.

Star Alliance	Latin America	Europe	Canada	Asia Pacific	Middle East	Africa	Mexico	Caribbean	US
Air Canada	×	×	×	×	×		×	×	×
Air New Zealand		×		×					×
All Nippon Airways		×		×					×
Ansett Australia				×					
The Austrian Airlines Group		×	×	×	×	×			×
British Midland		×							
Lufthansa	×	×	×	×	×	×	×		×
Mexicana	×		×				×	×	×
Scandinavian Airlines	×	×		×		×			×
Singapore Airlines		×	×	×	×	×			×
Thai Airways International	×	×		×				×	×
United Airlines	×	×	×	×			×	×	×
VARIG	×	×		×		×	×		×

FIGURE 1. A formal context about the destinations of Star Alliance members

concept). For any concept $(A, B) \in \mathfrak{B}(G, M, \mathrm{I})$ we have $\bigvee\{\gamma g \mid g \in A\} = (A, B) = \bigwedge\{\mu m \mid m \in B\}$. Thus $\{\gamma g \mid g \in G\}$ is $\bigvee$-dense and $\{\mu m \mid m \in M\}$ is $\bigwedge$-dense in $\mathfrak{B}(G, M, \mathrm{I})$. Conversely each complete lattice is (a copy of) a concept lattice of a certain context. In fact,

Theorem 1.2. [Wi82] *a complete lattice L is isomorphic to a concept lattice of a context (G, M, I) iff there are maps $\alpha : G \to L$ and $\beta : M \to L$ such that $\alpha(G)$ is $\bigvee$-dense in L, $\beta(M)$ is $\bigwedge$-dense in L and $g \,\mathrm{I}\, m \iff \alpha(g) \le \beta(m)$.*

Theorem 1.1 and Theorem 1.2 form the basic theorem of *Formal Concept Analysis*. The Mathematical foundations have been documented in a monograph by Bernhard Ganter and Rudolf Wille [GW99].

1.2. **Concept lattices and their diagrams.** Finite concept lattices can be represented by labeled Hasse diagrams. Each node represent a concept. The label g is written underneath of γg and m above μm. The extent of a concept represented by a node a is given by all labels in G from a downwards, and the intent by all labels in M from a upwards. Figure 2 presents the diagram of the concept lattice of the context of Figure 1. Diagrams are valuable tools for visualization of data[2]. However drawing a good diagram is a big challenge. Quite often, the size of the lattice is large and its structure complex. Thus we need tools to "approximate" by reducing the size or by making the structure nicer.

1.3. **Galois connections and closure operators.**

Definition 1.1. On a poset $(P, \le)$ a **closure operator** is a map $c : P \to P$ that satisfies $x \le c(y) \iff c(x) \le c(y)$ and a **kernel operator** a map $k : P \to P$ that satisfies $k(x) \le y \iff k(x) \le k(y)$.

[2]For example we can read on Figure 2 that *each member of the Star Alliance who flies to US and Europe also flies to Asia Pacific*, or that *each member who flies to Mexico and Asia Pacific also flies to Latin America, US, Canada and Europe.*

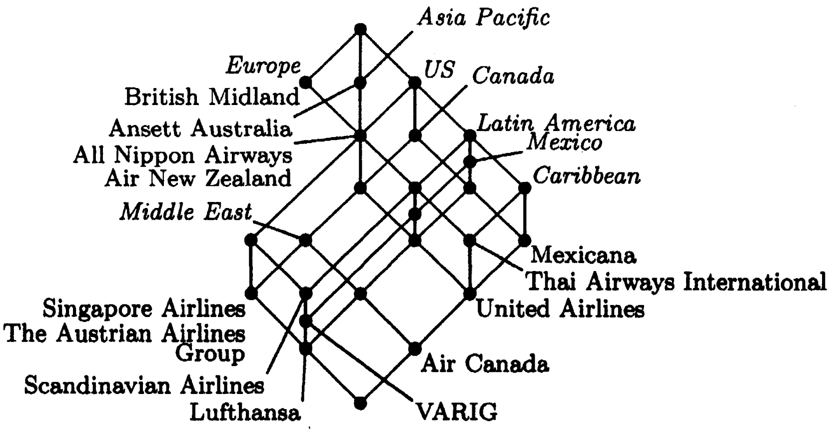

FIGURE 2. Concept lattice of the context of Figure 1

We often write cx, kx, cP and kP for $c(x), k(x), c(P)$ and $k(P)$ respectively. A map c is a closure operator iff $x \le cx$, $ccx = cx$ and $x \le y \implies cx \le cy$. Dually, a map k is a kernel operator iff $x \ge kx$, $kkx = kx$ and $x \le y \implies kx \le ky$. The maps $A \mapsto A''$ and $B \mapsto B''$ are a closure operator on $(\mathcal{P}(G), \subseteq)$ and a kernel operator on $(\mathcal{P}(M), \supseteq)$ respectively. They arise naturally from a Galois connection.

Definition 1.2. A **Galois connection** between $(P, \le)$ and $(Q, \le)$ is a pair (α, β) of maps $\alpha\colon P \to Q$ and $\beta\colon Q \to P$ such that $x \le \beta(y) \iff y \le \alpha(x)$.

The operations $A \mapsto A'$ and $B \mapsto B'$ form a Galois connection between $\mathcal{P}(G)$ and $\mathcal{P}(M)$. $A \mapsto A''$ and $B \mapsto B''$ are the corresponding closure operators.

(i) If (α, β) is a Galois connection then $\alpha \circ \beta$ and $\beta \circ \alpha$ are closure operators.
(ii) If c is a closure operator on P, then c and its inclusion map $\beta : cP \to P$ form a Galois connection between $(P, \le)$ and $(cP, \ge)$ with $\beta \circ c = c$.

1.4. **Concept lattices and implications.** A **closure system** on a set M is set of subsets of M, closed under intersection. The set of extents (resp. intents) of (G, M, I) is a closure system on G (resp. M). Each complete lattice is (a copy of) a closure system, and vice-versa.

Definition 1.3. Let M be a set of properties or attributes. An **implication** between attributes in M is a pair (A, B), denoted by $A \to B$. A is the premise and B the conclusion of $A \to B$. An implication $A \to B$ **holds** in a context (G, M, I) if every object having all the attributes in A also has all the attributes in B. A subset T of M **respects** $A \to B$ if $A \not\subseteq T$ or $B \subseteq T$[3]; we say that T is a model of $A \to B$ and write $T \models A \to B$. T respects a set $\mathcal{L}$ of implications if T respects every implication in $\mathcal{L}$. An implication $A \to B$ holds in a family $\mathcal{T}$ if every $T \in \mathcal{T}$ respects $A \to B$.

Implications can be read off from the lattice diagram. The rule is given by: $B \subseteq A''$ iff $(G, M, I) \models A \to B$ iff $\bigwedge\{\mu a \mid a \in A\} \le \mu m$ for all $m \in B$. For example `{Mexico, Asia Pacific}` $\to$ `{Latin America, US, Canada, Europe}` can be read from the lattice in Figure 2. If $\mathcal{L}$ is a set of implications in M, then $\mathrm{Mod}\mathcal{L} := \{T \subseteq M \mid T \models \mathcal{L}\}$ is a closure system on M. The corresponding closure operator, denoted by $X \mapsto \mathcal{L}(X)$, is obtained by setting $X^{\mathcal{L}^0} := X$,

$$X^{\mathcal{L}^{n+1}} := X^{\mathcal{L}^n} \cup \bigcup\{B \mid A \to B \in \mathcal{L}, A \subseteq X^{\mathcal{L}^n}\} \text{ and } \mathcal{L}(X) := \bigcup_{n\ge 0} X^{\mathcal{L}^n}.$$

[3]This is equivalent to $A \subseteq T \implies B \subseteq T$, and can be interpreted as *the conclusion B is "valid" in T whenever the premise A is "valid" in T.*

01 : Caribbean $\underset{4}{\rightarrow}$ Latin America, US;
02 : Mexico $\underset{5}{\rightarrow}$ Latin America, US;
03 : Africa $\underset{5}{\rightarrow}$ Europe, Asia Pacific, US;
04 : Middle East $\underset{4}{\rightarrow}$ Europe, Canada, Asia Pacific, US;
05 : Asia Pacific, US $\underset{10}{\longrightarrow}$ Europe;
06 : Canada $\underset{6}{\rightarrow}$ US;
07 : Europe, US $\underset{10}{\longrightarrow}$ Asia Pacific;
08 : Europe, Asia Pacific $\underset{10}{\longrightarrow}$ US;
09 : Europe, Canada, Asia Pacific, Africa US $\underset{3}{\rightarrow}$ Middle East;
10 : Latin America $\underset{7}{\rightarrow}$ US;
11 : Latin America, Mexico, Caribbean, US $\underset{3}{\rightarrow}$ Canada;
12 : Latin America, Canada, US $\underset{4}{\rightarrow}$ Mexico;
13 : Latin America, Europe, Asia Pacific, Africa, Caribbean, US $\underset{0}{\rightarrow}$ Canada, Middle East, Mexico.

FIGURE 3. Duquenne-Guigues implication basis of Star Alliance. To each arrow, we attach the number of object supporting this implication.

If $\mathcal{L}$ is the set of all implications of (G, M, I) then Mod$\mathcal{L}$ is the set of all concept intents of (G, M, I). Quite often, the implication list is long[4] and contains trivial implications. So it is desirable to get a minimal list of implications that hold in a context (G, M, I) and that generate all implications valid in (G, M, I).

Definition 1.4. An implication $A \to B$ **follows** from a set $\mathcal{L}$ of implications in M if each subset of M respecting $\mathcal{L}$ also respects $A \to B$. We write $\mathcal{L} \vdash A \to B$. $\mathcal{L}$ is **closed** if every implication following from $\mathcal{L}$ is in $\mathcal{L}$. $\mathcal{L}$ is **non redundant** if no implication in $\mathcal{L}$ follows from other implications of $\mathcal{L}$. A set $\mathcal{L}$ of implications of (G, M, I) is **complete** if every implication that holds in (G, M, I) follows from $\mathcal{L}$. A set $\mathcal{L}$ of implications of (G, M, I) is **sound** if every implication of $\mathcal{L}$ holds in (G, M, I). An **implication basis** of (G, M, I) is a set $\mathcal{L}$ that is sound, complete and non redundant.

A set $\mathcal{L}$ of implications is closed iff for all $X, Y, Z, W \subseteq M$, $X \to X \in \mathcal{L}$, $X \to Y \in \mathcal{L}$ implies $X \cup Z \to Y \in \mathcal{L}$, and $X \to Y \in \mathcal{L}, Y \cup Z \to W \in \mathcal{L}$ imply $X \cup Z \to W \in \mathcal{L}$ (**Armstrong rules**). In addition $\mathcal{L} \vdash A \to B$ iff $B \subseteq \mathcal{L}(A)$. For finite contexts implication bases can be computed. Jean Louis Guigues and Vincent Duquenne shown that there is a natural choice [GD86]. Figure 3 gives the Guigues-Duquenne implication basis of the context in Figure 1, with the number of objects really supporting them[5].

2. APPROXIMATION

The need of an approximation theory for concept lattices is motivated by many reasons: given a context $\mathbb{K}$, its concept lattice can be of huge size and have a complex structure; its implications list can long even if restricted only to an implication basis and contains some non "relevant" implications; we might not get some rules, although they are relevant (just because few exceptions violate them). Therefore the approximation problem can be formulated as follow:
Given a concept lattice L, can we replace L with a lattice $\tilde{L}$, that is nicer

[4]$2^M \times 2^M$ is the set of possible implications in M.
[5]No object supports implication #13; no member flies to Africa and Caribbean.

and easy to handle without loosing meaningful information? How can we reduce the number of concepts without loosing meaningful information? How can we express that two lattices carry almost the same information? Three approaches suggest themselves: approximate a given lattice by a lattice of smaller size (using clusters) such that meaningful information are preserved; relax definitions of *concepts* or *implications* to get some similar *patterns*; or approximate a given lattice by some lattices from a well-known and *easy to handle* class of lattices.

2.1. **Approximation via closure and kernel operators.**

Lemma 2.1. *Let c be a closure operator and k a kernel operator on $(P, \leq)$.*

(i) *If $(P, \leq)$ is a (complete) lattice then $c(cx \vee cy) = c(x \vee y)$, and $c(cx \wedge cy) = cx \wedge cy$, for all $x, y \in P$, and $(cP, \leq)$ is a (complete) lattice.*
(ii) *If $(P, \leq)$ is a (complete) lattice then $k(kx \wedge ky) = k(x \wedge y)$, and $k(kx \vee ky) = kx \vee ky$, for all $x, y \in P$, and $(kP, \leq)$ is a (complete) lattice.*

Lemma 2.1 states that closure and kernel operators are valuable tools for lattice approximation. They keep lattice sizes smaller. However the structure is not always easy to handle. How can we define interesting closure operators?

2.2. **Approximation via association rules.**

Definition 2.1. A **data mining context** is a finite context (G, M, I). The elements of M are called **items** and its subsets **itemsets. Closed itemsets** are intents. An **association rule** is a pair (B_1, B_2) of itemsets, denoted by $B_1 \rightarrow B_2$. Let *minsupp*, *minconf* $\in [0, 1]$, B an itemset and $r := B_1 \rightarrow B_2$ a rule. The support and confidence are defined by:

$$supp(B) := \frac{|B'|}{|G|},\ supp(r) := \frac{|(B_1 \cup B_2)'|}{|G|} \text{ and } conf(r) := \frac{supp(B_1 \cup B_2)}{supp(B_1)}.$$

B is **frequent** if $supp(B) \geq minsupp$. If $conf(r) = 1$, r is called an exact rule.

Here we are interested in frequent itemsets and associations between them. For a rule $A \rightarrow B$, $conf(A \rightarrow B)$ also denoted by $p_A(B)$ or $p(B|A)$ is the probability of B given A. The goal is to extract those that are frequent and have a confidence greater than the *minconf*. The Luxenburger basis [Lu91] of the partial implications of Star Alliance with *minsupp* = 30% and *minconf* = 90% is given by Figure 4 below. We get three new rules: #11, #12 and #13. The implications #09, #11 and #13 of Fig. 3 are not rule anymore, since their support is less than 90%.

2.3. **Approximation via pseudocomplementation.** L denotes a finite lattice. The *pseudocomplement* of $x \in L$ (if it exists) is an element x^* such that $x \wedge y = 0 \iff y \leq x^*$. L is *pseudocomplemented* if x^* exists for every $x \in L$; in this case $x \mapsto x^*$ is a unary operation on L, with $(\bigvee_{k \in K} x_k)^* = \bigwedge_{k \in K} x_k^*$. Moreover $c : x \mapsto x^{**}$ is a closure operator on L such that cL is a Boolean lattice[6], called **skeleton**. For concept lattices the Boolean structure depicts a total independence between (reduced) attributes, meaning that each possible combination is an intent. The structure is also easy to handle. Let $\mathcal{E}$ be a finite closure system on G with $A \mapsto A''$ the corresponding closure operator. For simplicity we assume $\emptyset'' = \emptyset$ and $g'' = h'' \implies g = h$. We set $G_{\min} := \{g \in G \mid g'' = \{g\}\}$, the labels of atomic concepts. Then $\mathcal{E}$ is pseudocomplemented iff all g'', $g \in G_{\min}$ have pseudocomplements. To

[6]A Boolean lattice is distributive ($x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$) and complemented lattice $x \vee x^* = 1$ and $x \wedge x* = 0$).

01 : Latin America, Canada, US 4 $\xrightarrow{100\%}$ 4 Mexico
02 : Latin America 7 $\xrightarrow{100\%}$ 7 US
03 : Europe, Asia Pacific 10 $\xrightarrow{100\%}$ 10 US
04 : Europe, US 10 $\xrightarrow{100\%}$ 10 Asia Pacific
05 : Canada 6 $\xrightarrow{100\%}$ 6 US
06 : Asia Pacific, US 10 $\xrightarrow{100\%}$ 10 Europe
07 : Middle East 4 $\xrightarrow{100\%}$ 4 Europe, Canada, Asia Pacific, US
08 : Africa 5 $\xrightarrow{100\%}$ 5 Europe, Asia Pacific, US
09 : Mexico 5 $\xrightarrow{100\%}$ 5 Latin America, US
10 : Carribean 4 $\xrightarrow{100\%}$ 4 Latin America, US
11 : US 11 $\xrightarrow{91\%}$ 10 Europe, Asia Pacific
12 : Asia Pacific 11 $\xrightarrow{91\%}$ 10 Europe, US
13 : Europe 11 $\xrightarrow{91\%}$ 10 Asia Pacific, US

FIGURE 4. Luxenburger basis of Star Alliance. Each arrow carries the confidence of the described rule (above), the number of object satisfying its premises (left) and the number of object satisfying its conclusion (right).

express the pseudocomplementation we use a projection s and its inverse $[\cdot]$ defined by:

$$s(A) := \bigcup_{g\in A} g'' \cap G_{\min} \text{ and } [A] := \{g \in G \mid s(g) \subseteq s(A)\} \text{ with } s(g) := s(\{g\}).$$

Lemma 2.2. [GK05] *The operator $[\cdot]$ defines a closure operator on G. An element $A \in \mathcal{E}$ has a pseudocomplement iff $[G_{\min} \setminus A] \in \mathcal{E}$. $\mathcal{E}$ is a pseudocomplemented closure system iff $[G_{\min} \setminus \{a\}] \in \mathcal{E}$ with for all $a \in G_{\min}$.*

Thus if $\mathcal{E}$ is not pseudocomplemented, then $[G_{\min} \setminus \{a\}] \notin \mathcal{E}$ for some $a \in G_{\min}$. We collect these $[G_{\min} \setminus \{a\}]$s and generate a new closure system.

Theorem 2.3. [Kw06] *Let $\mathcal{E}$ be a closure system on G. The closure system $\tilde{\mathcal{E}}$ generated by $\mathcal{E} \cup \{[G_{\min} \setminus \{a\}] \mid a \in G_{\min}\}$ is pseudocomplemented. Meets and existing pseudocomplements in $\mathcal{E}$ are preserved in $\tilde{\mathcal{E}}$. $\tilde{\mathcal{E}}$ is the coarsest pseudocomplemented refinement of $\mathcal{E}$.*

Corollary 2.4. [Kw06] *Each (finite) concept lattice L can be $\bigwedge$-embedded into a smallest pseudocomplemented concept lattice $\tilde{L}$.*

The process described in Theorem 2.3 can be performed on the context level. We should first check, by means of arrow-relations, whether a given context has a pseudocomplemented concept lattice. The **arrow-relations** are defined by:

$$\begin{aligned} g \swarrow m &:\iff m \notin g' \text{ and } g' \subsetneq h' \text{ implies } m \in h', \\ g \nearrow m &:\iff g \notin m' \text{ and } m' \subsetneq n' \text{ implies } g \in n', \\ g \mathrel{\swarrow\!\!\!\!\!\nearrow} m &:\iff g \swarrow m \text{ and } g \nearrow m, \text{ for } g,h \in G \text{ and } m,n \in M. \end{aligned}$$

The contextual arrow characterization of pseudocomplementation is given by:

Theorem 2.5 ([GK05]). *The concept lattice of a finite context (G, M, I) is pseudocomplemented iff the following condition holds for all $g \in G$:*

If $g \swarrow n$ for all $n \notin g'$ and $g \nearrow m$ then

if $h \swarrow m$ then $g' = h'$, and if $g \nearrow n$ then $n' = m'$.

Now set $\mathcal{E} := Ext(G, M, I)$, the closure system of extents. For $a \in G_{\min}$, if $[G_{\min} \setminus \{a\}] \in \mathcal{E}$ then there is $m_a \in M$ such that $m_a' = [G_{\min} \setminus \{a\}]$ ([GK05]). Therefore generating a new closure system with $[G_{\min} \setminus \{a\}]$, $a \in G_{\min}$ is equivalent to adding new attributes m_a in the context whenever $[G_{\min} \setminus \{a\}]$ is not an extent. These attributes have exactly $[G_{\min} \setminus \{a\}]$ as extent. Theorem 2.3 says that the so obtained lattice is pseudocomplemented and has $(\mathfrak{B}(G, M, I), \wedge)$ as subsemilattice.

The arrow configuration described in Theorem 2.5 is displayed in Figure 5. The

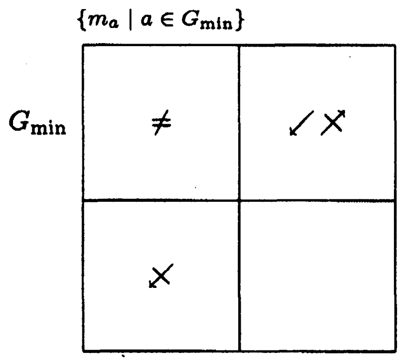

FIGURE 5. Arrow configuration in the context of atomic pseudo-complemented concept lattices.

subcontext $(G_{\min}, \{m_a \mid a \in G_{\min}\})$ is a copy of $(G_{\min}, G_{\min}, \neq)$, with exactly one double arrow in each row and column and crosses elsewhere. The rows of the atoms $G_{\min}$ contain no empty cells (arrowless non-incidences) and no upward arrows except for the double arrows mentioned. The columns corresponding to the attributes $\{m_a \mid a \in G_{\min}\}$ have no other downward arrows. What Theorem 2.5 expresses is that the configuration displayed in Figure 5 is characteristic for p-algebras.

In practice what one has to do is to first enter the arrow relations and check if one can obtain the configuration of Figure 5. If this is not the case one should add new attributes m_a for the atoms a whose inverse images are not extents and compute the new concept lattice.

3. What next?

Investigations should be carried out to compare the implication theory of the pcs-completion with that of the initial lattice. This approach should be compared with other, namely the alpha Galois lattices, fault-tolerance patterns, association rules. In a series of papers, V. Ventos and co-authors ([VPS, VST, VS05]) discussed the use of partitions on the set of objects and introduced the so called **alpha Galois lattices**. More precisely, given a context (G, M, I) and $\alpha \in [0, 1]$, a subset S of G is called an α-model of $T \subseteq M$ (denoted by $S \models_\alpha T$) if $\frac{|T' \cap S|}{|S|} \geq \alpha$. For a partition π on G, we write $g \models_\alpha^\pi T$ to mean that $[g]_\pi \models_\alpha T$, where $[g]_\pi$ denotes the class of g w.r.t. π. This defines a relation $I_\alpha^\pi \subseteq G \times M$ by $gI_\alpha^\pi m :\iff g \models_\alpha^\pi \{m\}$. For $|\pi| = |G|$ and $\alpha = \frac{1}{|G|}$ we have $(G, M, I) = (G, M, I_\alpha^\pi)$. Unfortunately the size of $\mathfrak{B}(G, M, I_\alpha^\pi)$ is not always smaller as announced. A small example can be found in [MK]. In [PB05] the authors proposed a generalization of concepts to fault-tolerant patterns. In fact concepts are maximal rectangle full of crosses. A **fault-tolerant pattern** can be interpreted as a concept with some crosses missing. In [Du96] and [Wh96] the authors considered taking the statistics in account to handle exceptions. As already mentioned closure and kernel operators are approximation means for concept lattices. In [KS08] we gave a correspondence between closure/kernel operators and lower/upper modular valuations. This provides another mean to approximate concept lattices, and will be of great use in bringing together the quantitative and qualitative methods. A general framework for lattice approximation is an urgent need, and will strengthen the use of FCA in huge data.

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