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$\mathrm{XOR}^2 = 90$

— Graded Algebra Structure of the Boolean Algebra of Local Transition Rules —

Kyushu Institute of Technology
Mitsuhiko FUJIO
fujio@ces.kyutech.ac.jp

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Abstract

Notion of composition of cellular automata (CA for short) on a group G is introduced by means of transition functions and translated in terms of local transition rules. Although the Boolean algebra $\mathcal{T}_V$ (resp. $\mathcal{L}_V$) of transition functions (resp. local transition rules) of the same locality (resp. support) $V \subseteq G$ is not closed under the composition, any families $\{\mathcal{L}_V\}$ of Boolean algebras indexed by product closed increasing families of subsets of G are shown to carry filtration structures compatible with the composition. As a corollary, graded algebra structures on the Boolean algebra of local transiton rules are induced. Similar filtrations for transition functions are considered, however, grading is not defined in this case.

Composition of CAs can be used to reduce a complex behaved dynamics into simpler ones when the CA under consideratoin is composed of others. As an example, the rule 90 CA is shown to be factorized into the square of XOR (exclusive or).

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1 Cellular automata on groups

In what follows, we denote by **2** the Boolean algebra consisting of two constants 0 and 1.

In this work, for the sake of generality, we formulate CA as dynalmical systems on groups by their features of locality and homogenuity. Notion of CA on groups was first treated as special cases for CA on graphs named *Caylay graphs* which represent groups ([1], [2], [3]), The following rather direct definition of CA on groups is a slight modification of that found in [3] but we do not assume any finiteness of gnerators of groups.

Let G be a group. By regarding its power set $\mathscr{C} = \mathbf{2}^G$ as the *configuration space*, a *local transition rule* with the support $V \subseteq G$ is defined as a family $\mathfrak{L}$ of subsets of V. The role of $\mathfrak{L}$ is explained as follows. Let c' be the evolution of a configuration $c \in \mathscr{C}$. Then the state at any site $g \in c'$ is determined according to whether the state pattern of c around g tranlated onto V around the unit element e by g^{-1} is in $\mathfrak{L}$ or not. More explicitly,

$$c \quad \mapsto \quad c' = \{g \in G \mid g^{-1}c \cap V \in \mathfrak{L}\}.$$

Such a *transition function* is characterized as a tranformation $T : \mathscr{C} \to \mathscr{C}$ commutes with the G-action:

$$T(ac) \quad = \quad a(T(c)), \qquad (a \in G, c \in \mathscr{C}),$$

where the action of $a \in G$ on a configuration $c \in \mathscr{C}$ is given by $ac = \{ag \mid g \in c\}$. The usual infinite (resp. periodic with the size N) 1-dimensional CA is obtained by considering a symmetric interval $V = [-r, r]$ in the Abelian group $G = \mathbf{Z}$ (resp. $\mathbf{Z}_N$: the residue ring modulo N).

2 Algebraic structure

For a given set X and a Boolean algebra B we consider the point-wize Boolean algebra structure on the set B^X of all B-valued functions defined on X. Namely, it is defined by

$$(f \vee g)(x) := f(x) \vee g(x), \qquad (f \wedge g)(x) := f(x) \wedge g(x), \qquad (\neg f)(x) := \neg(f(x)).$$

for $f, g \in B^X$ and $x \in X$. Then we have

$$f \leq g \quad \Leftrightarrow \quad f(x) \leq g(x) \quad (\forall x \in X).$$

For $B = \mathbf{2}$, the Boolean algebra structure of B^X coincide with that of set lattice and $\vee$, $\wedge$, $\neg$ and $\leq$, respectively become union, intersection, complement and inclusion. The set $\mathcal{T}_G$ of all transition functions on G is a subalgebra of the Boolean algebra $[\mathbf{2}^G \to \mathbf{2}^G] = (\mathbf{2}^G)^{(\mathbf{2}^G)}$. On the other hand, the set $\mathcal{L}_V = [\mathbf{2}^V \to \mathbf{2}]$ of all local transition rules with the support V is regarded as a Boolean algebra as $\mathbf{2}^{(\mathbf{2}^V)}$. All of X^B, $\mathcal{T}_G$ and $\mathcal{L}_V$ are complete.

For any pair of subsets $V \subseteq W \subseteq G$, by virture of $\mathbf{2}^V \subseteq \mathbf{2}^W$, $\mathcal{L}_V$ can be naturally regarded as a sublattice of $\mathcal{L}_W$. We remark that the complementation depends on the algebra to which the element belongs. When the support should be explicitely indicated, we write as $\neg_V$.

Now suppose that an increasing sequence $\mathfrak{F} = \{V_i\}$ $(i \in \mathbb{N})$ of subsets of G is given. Then we have a *filtration* of the space $\mathcal{L}_G$ of all local transition rules on G. Here the word filtration means that

$$i \leq j \quad \Rightarrow \quad \mathcal{L}_{V_i} \subseteq \mathcal{L}_{V_j}. \tag{1}$$

As it will be seen in the next section (Theorem1), $\mathcal{L}_V$ is isomorphic to a subalgebra $\mathcal{T}_V$ of the Boolean algebra $\mathcal{T}_G$ of all transition functions. Thus $\mathcal{T}_G$ also carries a siilar filtration.

3 Correspondence between local transition rules and transition functions

For a transition rules $\mathfrak{L} \in \mathcal{L}_V$ with the support V, we define a transition function on G by

$$T_{\mathfrak{L}}(\mathbf{c}) \quad = \quad \left\{ g \in G \mid g^{-1}\mathbf{c} \cap V \in \mathfrak{L} \right\} \quad (\mathbf{c} \in \mathscr{C}).$$

It can be easily verified that $T_{\mathfrak{L}}$ commutes with G-action.

To consider a converse correspondence, we introduce for transition functions a notion corresponding to supports of local transition rules. Let $T : \mathscr{C} \to \mathscr{C}$ be a transition function and $V \subseteq G$. We say that T is *local* at a site $g \in G$ on V if

$$\mathbf{c} \cap V = \mathbf{c}' \cap V \quad \Rightarrow \quad g \in T(\mathbf{c}) \text{ iff } g \in T(\mathbf{c}')$$

is satisfied for any configurations $\mathbf{c}$, $\mathbf{c}' \in \mathscr{C}$. We call V a *domain of locality* of T at g. We denote the set of all transition functions local at a site g on V by $\mathcal{T}_{g,V}$. $\mathcal{T}_{g,V}$ is a complete Boolean subalgebra of $\mathcal{T}_G$.

For simplicity, we denote by $\mathcal{T}_V$ instead of $\mathcal{T}_{e,V}$ for the unit element e. Then it can be easily shown that $\mathcal{T}_V = \mathcal{T}_{g,gV}$.

Now we assign a local transition function $\mathfrak{L}_T \in \mathcal{L}_V$ with the support V to each transition function $T \in \mathcal{T}_V$ by

$$\mathfrak{L}_T \quad = \quad \left\{ \mathbf{c} \in \mathbf{2}^V \mid e \in T(\mathbf{c}) \right\}.$$

By virtue of commutativity of T with G-action,

$$\mathfrak{L}_T \quad = \quad \left\{ g^{-1}\mathbf{c} \in \mathscr{C} \mid \mathbf{c} \in \mathbf{2}^{gV}, g \in T(\mathbf{c}) \right\}$$

holds for an arbitrary $g \in G$.

Theorem 1 Both of the mappings $\mathcal{T}_V \ni T \mapsto \mathfrak{L}_T \in \mathcal{L}_V$ and $\mathcal{L}_V \ni \mathfrak{L} \mapsto T_{\mathfrak{L}} \in \mathcal{T}_V$ are isomorphisms of Boolean algebrals and each of them is the inverse of each other.

Proof. It is straightforwardly veryfied that these two mappings are Boolean algebra homomorphisms. Thus we show the latter part. Let $T \in \mathcal{T}_V$. Then by defintion,

$$T_{\mathfrak{L}_T}(\mathbf{c}) = \{g \in G \mid g^{-1}\mathbf{c} \cap V \in \mathfrak{L}_T\} = \{g \in G \mid e \in T(g^{-1}\mathbf{c} \cap V)\}$$

since T is local at e on V,

$$= \{g \in G \mid e \in T(g^{-1}\mathbf{c})\}$$

furthermore, since T commutes with the G-action,

$$= \{g \in G \mid g \in T(\mathbf{c})\} = T(\mathbf{c}).$$

On the other hand, for $\mathfrak{L} \in \mathcal{L}_V$,

$$\mathfrak{L}_{T_{\mathfrak{L}}} = \{\mathbf{c} \in \mathbf{2}^V \mid e \in T_{\mathfrak{L}}(\mathbf{c})\} = \{\mathbf{c} \in \mathbf{2}^V \mid \mathbf{c} \cap V \in \mathfrak{L}\} = \mathfrak{L}.$$

Thus $T \mapsto \mathfrak{L}_T$, $\mathfrak{L} \mapsto T_{\mathfrak{L}}$ are mutually inverse of each other. *q.e.d.*

As mentioned at the end of the previous section, $\mathcal{L}_V$ is a sublattice of $\mathcal{L}_W$ for $V \subseteq W$. Similarly, $\mathcal{T}_V$ is a sublattice of $\mathcal{T}_W$. More precisely, we have the following.

Proposition 2 Let $V \subseteq W \subseteq G$.

(1) $\mathcal{L}_V$ is an ideal of $\mathcal{L}_W$, *i.e.*, $\mathcal{L}_V$ is a sublattice of $\mathcal{L}_W$ and if $\mathfrak{M} \subseteq \mathfrak{L}$ ($\mathfrak{L} \in \mathcal{L}_V$, $\mathfrak{M} \in \mathcal{L}_W$) then $\mathfrak{M} \in \mathcal{L}_W$.

(2) $\mathcal{T}_V$ is an Boolean subalgebra of $\mathcal{T}_W$, *i.e.*, $\mathcal{L}_V$ is a sublattice of $\mathcal{L}_V$ and if $T \in \mathcal{T}_V$ then $\neg_V T = \neg_W T$.

It follows from this proposition that the two embeddings $\mathcal{L}_V \subseteq \mathcal{L}_W$ and $\mathcal{T}_V \subseteq \mathcal{T}_W$ are not compatible with the isomorphisms $\mathcal{L}_V \cong \mathcal{T}_V$ and $\mathcal{L}_W \cong \mathcal{T}_V$ obtained by Theorem1. In fact, if we denote by $T_{\mathfrak{L}}^W$ the transition function corresponding to $\mathfrak{L} \in \mathcal{L}_V$ regarded as an element of $\mathcal{L}_W$, then $T_{\mathfrak{L}}^W \leq T_{\mathfrak{L}}$. Siilarly, denoting by $\mathfrak{L}_T^W$ the local transition rule corresponding to $T \in \mathcal{T}_V$ regarded as in $\mathcal{T}_W$, we have $\mathfrak{L}_T \subseteq \mathfrak{L}_T^W$. Consequently, though a filtration

$$i \leq j \quad \Rightarrow \quad \mathcal{T}_{V_i} \subseteq \mathcal{T}_{V_j}. \tag{2}$$

of $\mathcal{T}_G$ for an increasing sequence of subsets $\mathfrak{F} = \{V_i\}_{i \in \mathbf{Z}}$ of G is obtained as well as the filtration (1) of $\mathcal{L}_G$, (2) is a filtration by Boolean subalgebras while (1) is a filtration by ideals. One can go on to the grading from the former always but from the latter may not.

4 Composition of local transition rules

We define the product of V, $W \subseteq G$ by

$$V \otimes W \quad = \quad \{vw \in G \mid v \in V, w \in W\}.$$

This is a non-commutative version of Minkowski addition $\oplus$ for Abelian groups ([4]). Then we define the ***local composition*** of $\mathfrak{L} \in \mathcal{L}_V$, $\mathfrak{M} \in \mathcal{L}_W$ by

$$\mathfrak{L} \diamond \mathfrak{M} \ = \ \{\mathbf{c} \in \mathbf{2}^{V \otimes W} \mid {\sigma_{\mathbf{c}}}^{-1}(\mathfrak{M}) \in \mathfrak{L}\}.$$

Here $\sigma_{\mathbf{c}} : V \to \mathbf{2}^W$ is a function defined by

$$\sigma_{\mathbf{c}}(v) \quad = \quad v^{-1}\mathbf{c} \cap W$$

for $\mathbf{c} \in \mathbf{2}^{V \otimes W}$ and ${\sigma_{\mathbf{c}}}^{-1}(\mathfrak{M})$ denotes the inverse image of the set $\mathfrak{M} \subseteq \mathbf{2}^W$ with respect to this function. Finally, $\mathfrak{L} \diamond \mathfrak{M}$ is the set of all configurations $\mathbf{c}$ such that this inverse image coincides with a member of $\mathfrak{L}$. By definition, $\mathfrak{L} \diamond \mathfrak{M} \in \mathcal{L}_{V \otimes W}$.

The following theorem ensures us to call this as local composition:

Theorem 3 Let $\mathfrak{L} \in \mathcal{L}_V$, $\mathfrak{M} \in \mathcal{L}_W$. Then the composition $T_{\mathfrak{L}} \circ T_{\mathfrak{M}}$ of $T_{\mathfrak{L}} \in \mathcal{T}_V$ and $T_{\mathfrak{M}} \in \mathcal{T}_W$ is a unique transition function that is local at e on $V \otimes W$ satisfying

$$T_{\mathfrak{L}} \circ T_{\mathfrak{M}} \quad = \quad T_{\mathfrak{L} \diamond \mathfrak{M}}.$$

Proof. To begin with, we show that $T_{\mathfrak{L}} \circ T_{\mathfrak{M}} \in \mathcal{T}_{V \otimes W}$. It is clear that the composition commutes with the G-action. Thus it suffices to show that it is local at e on $V \otimes W$. We remark that by the definitions of $T_{\mathfrak{M}}(\mathbf{c})$ and $\sigma_{\mathbf{c}}$,

$$T_{\mathfrak{M}}(\mathbf{c}) \cap V \quad = \quad \{v \in V \mid \sigma_{\mathbf{c}}(v) \in \mathfrak{M}\} \tag{3}$$

for any configuration $\mathbf{c}$, Suppose that two configurations $\mathbf{c}$, $\mathbf{c}'$ satisfies $\mathbf{c} \cap (V \otimes W) = \mathbf{c}' \cap (V \otimes W)$. Then for any $v \in V$, since $vW \subseteq V \otimes W$, it follows in particular that $\mathbf{c} \cap vW = \mathbf{c}' \cap vW$. This means that $\sigma_{\mathbf{c}}(v) = \sigma_{\mathbf{c}'}(v)$. Thus from the above remark we have

$$T_{\mathfrak{M}}(\mathbf{c}) \cap V \quad = \quad T_{\mathfrak{M}}(\mathbf{c}') \cap V.$$

On the other hand, since

$$e \in T_{\mathfrak{L}}(T_{\mathfrak{M}}(\mathbf{c})) \quad \text{iff} \quad T_{\mathfrak{M}}(\mathbf{c}) \cap V \in \mathfrak{L} \tag{4}$$

and the similar equivalence holds for $\mathbf{c}'$,

$$e \in T_{\mathfrak{L}}(T_{\mathfrak{M}}(\mathbf{c})) \quad \text{iff} \quad e \in T_{\mathfrak{L}}(T_{\mathfrak{M}}(\mathbf{c}'))$$

holds. This implies that $T_{\mathfrak{L}} \circ T_{\mathfrak{M}}$ is local at e on $V \otimes W$.

Now put $\mathfrak{N} = \mathfrak{L}_{T_{\mathfrak{L}} \circ T_{\mathfrak{M}}}$. Then by virtue of Theorem 1, $\mathfrak{N}$ is the unique local transition rule with the support $V \otimes W$ satisfyin $T_{\mathfrak{N}} = T_{\mathfrak{L}} \circ T_{\mathfrak{M}}$. Thus all we have to show is that $\mathfrak{N}$ coincides with $\mathfrak{L} \diamond \mathfrak{M}$. For

$$\begin{aligned}
\mathfrak{N} &= \{\mathbf{c} \in \mathbf{2}^{V \otimes W} \mid e \in (T_{\mathfrak{L}}(T_{\mathfrak{M}}(\mathbf{c}))\} \\
&= \{\mathbf{c} \in \mathbf{2}^{V \otimes W} \mid T_{\mathfrak{M}}(\mathbf{c}) \cap V \in \mathfrak{L}\} && \text{(by (4))} \\
&= \{\mathbf{c} \in \mathbf{2}^{V \otimes W} \mid \{v \in V \mid \sigma_{\mathbf{c}}(v) \in \mathfrak{M}\} \in \mathfrak{L}\} && \text{(by (3))} \\
&= \{\mathbf{c} \in \mathbf{2}^{V \otimes W} \mid \sigma_{\mathbf{c}}^{-1}(\mathfrak{M}) \in \mathfrak{L}\} = \mathfrak{L} \diamond \mathfrak{M}.
\end{aligned}$$

q.e.d.

5 Filtration and gradings

Suppose that an increasing sequence $\mathfrak{F} = \{V_i\}_{i \in \mathbf{Z}}$ of subsets of G satisfies

$$V_i \otimes V_j \quad \subseteq \quad V_{i+j}. \tag{5}$$

Then the filtrations (1) of $\mathcal{L}_G$ and (2) of $\mathcal{T}_G$, respectively satisfies

$$\mathcal{L}_{V_i} \diamond \mathcal{L}_{V_j} \quad \subseteq \quad \mathcal{L}_{V_{i+j}} \tag{6}$$

and

$$\mathcal{T}_{V_i} \circ \mathcal{T}_{V_j} \quad \subseteq \mathcal{T}_{V_{i+j}}. \tag{7}$$

As remarked at the end of section 3, since the filtration of the Boolean algebra $\mathcal{L}_G$ is the one comsisiting of ideals (by Propsition2), we obtain the following graded Boolean algebra:

$$\mathscr{L} \quad = \quad \oplus_i \mathscr{L}_i,$$

where $\mathscr{L}_i$ denotes the quotient Boolean algebra defined by

$$\mathscr{L}_i \quad = \quad \mathcal{L}_{V_i} / \mathcal{L}_{V_{i-1}}$$

which can be regarded as the set of local transition rules that is essentially supported by V_i. $\mathscr{L}$ is accompanied with the product

$$\mathscr{L}_i \diamond \mathscr{L}_j \quad \subseteq \quad \mathscr{L}_{i+j}$$

induced from the local composition. We call $\mathscr{L}$ the *graded Boolean algebra of local transition rules* (with respect to $\mathfrak{F}$). At this moment, it is not clear how much the algebraic structure of $\mathscr{L}$ depends on the choice of $\mathfrak{F}$.

6 Examples

Consider usual 1-dimensional CAs. Then $G = \mathbb{Z}$ (for infinite case) or $\mathbb{Z}_N$ (for N-periodic case). If one wants to consider the symmetric transition rules, by taking

$$V_i = \begin{cases} \emptyset & (i < 0) \\ \{-i, \ldots, 0, \ldots, i\} & (i \geq 0) \end{cases}$$

(for N-periodic case, when $i > N/2$ put $V_i = G$), the sequence $\mathfrak{F} = \{V_i\}$ becomes an increasing sequence satisfying (5). In fact, in this case equalities

$$V_i \otimes V_j = V_{i+j}$$

are satisfied.

Since by the above example, only domains of odd length appear, we consider the following domains of one-sided type:

$$V_i = \begin{cases} \emptyset & (i < 0) \\ \{0, \ldots, i-1, i\} & (i \geq 0) \end{cases}$$

Here and in what follows, we put $V_i = G$ when $N \geq i$ for the N-periodic case. Again, the sequence $\mathfrak{F} = \{V_i\}$ is increasing and satisfies (5) by equatities. By identifying $\mathbf{2}^{V_i}$ with $\mathbf{2}^{i+1}$, we represent each local transition rule $\mathfrak{L} \in \mathcal{L}_{V_i}$ as the $(i+1)$-variabe function $\mathfrak{L}(x_0, \ldots, x_i)$ defined by

$$\mathfrak{L}(x_0, \ldots, x_i) = \begin{cases} 1 & ((x_0, \ldots, x_i) \in \mathfrak{L}) \\ 0 & ((x_0, \ldots, x_i) \notin \mathfrak{L}) \end{cases}.$$

Then the corresponding transition function for $\mathfrak{L}$ is

$$T_{\mathfrak{L}} : (c_n) \mapsto (c'_n), \quad c'_n = \mathfrak{L}(c_n, \ldots, c_{n+i}).$$

Here and in what follows, the indices are taken to be modulo N for the N-periodic case.

The *Wolfram number* for $\mathfrak{L} \in \mathcal{L}_{V_i}$ is defiend by the number whose binary expression with the length 2^{i+1} is given by

$$\mathfrak{L}(1, \ldots, 1) \cdots \mathfrak{L}(0, \ldots, 0)$$

and is calculated by

$$\sum_{a_0, \ldots, a_i \in \mathbf{2}} \mathfrak{L}(a_0, \ldots, a_i) \cdot 2^{\sum_{k=0}^{i} a_j 2^{i-k}}.$$

Since XOR in the title is the element of $\mathcal{L}_{V_1}$ such that

$$(0,0) \mapsto 0, \qquad (0,1) \mapsto 1, \qquad (1,0) \mapsto 1, \qquad (1,1) \mapsto 0,$$

its Worfram number is 0110 in binary expression, that is, 6. Similarly, the rule 90 indicates the local rule in $\mathcal{L}_{V_2}$ with the function form

$$\begin{array}{llll} (0,0,0) \mapsto 0, & (0,0,1) \mapsto 1, & (0,1,0) \mapsto 0, & (0,1,1) \mapsto 1, \\ (1,0,0) \mapsto 1, & (1,0,1) \mapsto 0, & (1,1,0) \mapsto 1, & (1,1,1) \mapsto 0. \end{array}$$

We note that every local transition rule must be reffered to by its Wolfram number with the degree i.

The local compositon $\mathfrak{L} \diamond \mathfrak{M} \in \mathcal{L}_{V_{i+j}}$ of $\mathfrak{L} \in \mathcal{L}_{V_i}$ and $\mathfrak{M} \in \mathcal{L}_{V_j}$ is now simply expressed in terms of functions as

$$(\mathfrak{L} \diamond \mathfrak{M})(x_0, \ldots, x_{i+j}) = \mathfrak{L}(\mathfrak{M}(x_0, \ldots, x_j), \ldots, \mathfrak{M}(x_i, \ldots, x_{i+j})).$$

Let $\mathfrak{L} = \mathrm{XOR}^2 = \mathrm{XOR} \diamond \mathrm{XOR} \in \mathcal{L}_{V_2}$. Then by $\mathfrak{L}(x, y, z) = \mathrm{XOR}(\mathrm{XOR}(x, y), \mathrm{XOR}(y, z))$,

$$\begin{array}{ll} \mathfrak{L}(0,0,0) = \mathrm{XOR}(0,0) = 0, & \mathfrak{L}(0,0,1) = \mathrm{XOR}(0,1) = 1, \\ \mathfrak{L}(0,1,0) = \mathrm{XOR}(1,1) = 0, & \mathfrak{L}(0,1,1) = \mathrm{XOR}(1,0) = 1, \\ \mathfrak{L}(1,0,0) = \mathrm{XOR}(1,0) = 0, & \mathfrak{L}(1,0,1) = \mathrm{XOR}(1,1) = 0, \\ \mathfrak{L}(1,1,0) = \mathrm{XOR}(0,1) = 1, & \mathfrak{L}(1,1,1) = \mathrm{XOR}(0,0) = 0. \end{array}$$

Hence $\mathfrak{L}$ has the Wolfram number 90 of degree 2. Thus we have established the equality in the title.

7 Concluding remarks

This work was first motivated by seaching a Normal Form of (local transition rules) of CAs. As Boolean functions, Conjunctive and Dijunctive Normal Forms are well-known. But they are redundant. For example, since the local transiton rule with the Wolfram number 51 of degree 2 can be represented as

$$\begin{array}{llll} (0,0,0) \mapsto 1, & (0,0,1) \mapsto 1, & (0,1,0) \mapsto 0, & (0,1,1) \mapsto 0, \\ (1,0,0) \mapsto 1, & (1,0,1) \mapsto 1, & (1,1,0) \mapsto 0, & (1,1,1) \mapsto 0 \end{array}$$

as a function, it becomes

$$(\neg x_0 \wedge \neg x_1 \wedge \neg x_2) \vee (\neg x_0 \wedge \neg x_1 \wedge x_2) \vee (x_0 \wedge \neg x_1 \wedge \neg x_2) \vee (x_0 \wedge \neg x_1 \wedge x_2) \tag{8}$$

in Conjunctive Normal Form. By factorizing,

$$\begin{aligned} (8) &= (\neg x_0 \wedge \neg x_1) \wedge (\neg x_1 \vee x_2) \vee (x_0 \wedge \neg x_1) \wedge (x_1 \vee x_2) \\ &= (\neg x_0 \wedge \neg x_1) \vee (x_0 \wedge \neg x_1) = (\neg x_0 \vee x_0) \wedge \neg x_1 \\ &= \neg x_1. \end{aligned}$$

Namely, the variables x_0 and x_2 are eliminated and it happens to depend only on one variable x_0. It is of degree 0 in our term. Such a phenomenon motivates us to investigate the degree of Boolean functions as the number of essetial variables. Hence our aim is to find simplest represetation of Boolean functions.

The independency on the latter variables are made clear by moving onto the quotient Boolean algebra. That is, if $\mathfrak{L}(x_0, \ldots, x_i) \in \mathcal{L}_{V_i}$ is independent of x_i, it is contained in $\mathcal{L}_{V_{i-1}}$ and is mapped to $\mathbf{0}$ in $\mathscr{L}_i = \mathcal{L}_{V_i}/\mathcal{L}_{V_{i-1}}$. Unfortunately, with the increasing sequences in the examples we can not describe the independency on the former variablesf.

Although the rule 90 with degree 2 is simplest in the sense that the number of its variables cannot be reduced, it can be factorized into the composition of double XORs, that is the rules 6 with degree 1. The fact that XOR can not be factraized any more is observed as follows. The possibilities for making local rules with degree 1 by local composition are only twe cases of the composition of ones with degree -1 with degree 2 and of the composition of ones with degree 0 with degree 1. For the former case, since for the local transiiton rules of degree -1 only two trivial constants $\mathbf{0}$ and $\mathbf{1}$ are available. On the other hand, for the latter case, there are 4 functions for degree 1. Two of them are reduced to the trivial constants and the other two are the identity transformation and its negation. So XOR has no non-trivial factorization and thus we can conclude that it is one of the simplest local transition rules of complex behavior.

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