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New renormalization for the field operators on constructive QFT by means of the *Hida Product*

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Abstract

A new definition of product between the elements of Hida distributions defined on the sharp time free field is given. By this procedure the space of Hida distributions has the structure of *ring*. This newly defined product is different to the well known $\mathcal{S}$-transform. Its structure is much more complicated than the one of the $\mathcal{S}$-transform, and here we call the newly defined product " the Hida product". Through the *Hida product* we are able to define new operators that contains both the creation and annihilation terms, and then it is shown that the $(\Phi_4)^p$ field theory is ralizable within a framework of Banach space (i.e. by modifying the usual Hilbert space arguments to that of Banach space).

0 Preliminaries

Throughout this paper, we set $d \in \mathbf{N}$, where $\mathbf{N}$ is the set of natural numbers, the space-time dimension, and take that $d-1$ is the space dimension and 1 is the dimension of time. Correspondingly, we use the notations

$$\mathbf{x} \equiv (t, \vec{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}.$$

Let $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^{d-1})$) be the Schwartz space of rapidly decreasing test functions on the d dimensional Euclidean space $\mathbb{R}^d$ (resp. $d-1$ dimensional Euclidean space $\mathbb{R}^{d-1}$), equipped with the usual topology by which it is a Fréchet nuclear space. Let $\mathcal{S}'(\mathbb{R}^d)$ (resp. $\mathcal{S}'(\mathbb{R}^{d-1})$) be the topological dual space of $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^{d-1})$).

In order to simplify the notations, in the sequel, by the symbol D we denote both d and $d-1$. In each discussion we exactly explain the dimension (space-time or space) of the field on which we are working.

Now, suppose that on a complete probability space $(\Omega, \mathcal{F}, P)$ we are given an isonormal Gaussian process $B^D = \{B^D(h), h \in L^2(\mathbb{R}^D; \lambda^D)\}$, where λ^D denotes the Lebesgue measure on $\mathbb{R}^D$ (cf., e.g., [HKPS], [SiSi], [AY1,2] and references therein). Precisely, B^D is a centered Gaussian family of random variables such that

$$E[B^D(h)\, B^D(g)] = \int_{\mathbb{R}^D} h(\mathbf{x})\, g(\mathbf{x}) \lambda^D(d\mathbf{x}), \quad h,\ g \in L^2(\mathbb{R}^D; \lambda^D). \tag{0.1}$$

We write

$$B^D_\omega(h) = \int_{\mathbb{R}^D} h(y) \dot{B}^D_\omega(\mathbf{y})\, dy, \qquad \omega \in \Omega.$$

Namely, $\dot{B}^D_\omega(\cdot)$ is the Gaussian white noise on $\mathbb{R}^D$.

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By the framework of the calculus on the abstract Wiener spaces, $\dot{B}(\mathbf{y})d\mathbf{y}$ is written by $W_\omega^D(d\mathbf{y})$ (cf., [Nu], [AFY]). As far as the discussions for (0.6) and (0.7), which are not singular, we may use the notation of the calculus on the abstract Wiener spaces, but to make the symbols clear and for the discussions on the **Hida distributions** (going into the more singular discussions) we prefer to use the notations of white noise analysis.

We are considering a massive scalar field and suppose that we are given a mass $m > 0$. Let Δ_d and resp. Δ_{d-1} be the d, resp. $d-1$, dimensional Laplace operator, and define the pseudo differential operators $L_{-\frac{1}{2}}$ and $H_{-\frac{1}{4}}$ as follows:

$$L_{-\frac{1}{2}} = (-\Delta_d + m^2)^{-\frac{1}{2}}. \tag{0.2}$$

$$H_{-\frac{1}{4}} = (-\Delta_{d-1} + m^2)^{-\frac{1}{4}}, \tag{0.3}$$

By the same symbols as $L_{-\frac{1}{2}}$ and $H_{-\frac{1}{4}}$, we also denote the integral kernels of the corresponding pseudo differential operators, i.e., the Fourier inverse transforms of the corresponding symbols of the pseudo differential operators.

By making use of stochastic integral expressions, we define two extremely important random fields ϕ_N, the **Nelson's Euclidean free field**, and ϕ_0, the **sharp time free field**, as follows: For $d \geq 2$,

$$\phi_N(\cdot) \equiv \int_{\mathbb{R}^d} L_{-\frac{1}{2}}(\mathbf{x} - \cdot)\dot{B}^d(\mathbf{x})\, d\mathbf{x}, \tag{0.4}$$

$$\phi_0(\cdot) \equiv \int_{\mathbb{R}^{d-1}} H_{-\frac{1}{4}}(\vec{x} - \cdot)\dot{B}^{d-1}(\vec{x})\, d\vec{x}. \tag{0.5}$$

These definitions of ϕ_N and resp. ϕ_0 seems formal, but they are rigorously defined as $\mathcal{S}'(\mathbb{R}^d)$ and resp. $\mathcal{S}'(\mathbb{R}^{d-1})$ valued random variables through a limiting procedure (cf. [AY1,2]), more precisely it has been shown that

$$P(\phi_N(\cdot) \in B_d^{a,b}) = 1, \text{ for } a, b \text{ such that } \min(1, \frac{2a}{d}) + \frac{2}{d} > 1,\ b > d \tag{0.6}$$

$$P(\phi_0 \in B_{d-1}^{a',b'}) = 1, \text{ for } a', b' \text{ such that } \min(1, \frac{2a'}{d-1}) + \frac{1}{d-1} > 1,\ b' > d-1. \tag{0.7}$$

Here for each a, b, $D > 0$, the Hilbert spaces $B_d^{a,b}$, which is a linear subspace of $\mathcal{S}'(\mathbb{R}^D)$, is defined by

$$B_D^{a,b} = \{(|\mathbf{x}|^2 + 1)^{\frac{b}{4}}(-\Delta_D + 1)^{+\frac{a}{2}} f :\ f \in L^2(\mathbb{R}^D; \lambda^D)\}, \tag{0.8}$$

where $\mathbf{x} \in \mathbb{R}^D$ and λ denotes the Lebesgue measure on $\mathbb{R}$, the scalar product of $B_d^{a,b}$ is given by

$$\begin{aligned} < u \,|\, v > \ &= \int_{\mathbb{R}^D} \left\{(-\Delta_D + 1)^{\frac{a}{2}}((1 + |\mathbf{x}|^2)^{-\frac{b}{4}} u(\mathbf{x}))\right\} \\ &\times \left\{(-\Delta_D + 1)^{\frac{a}{2}}((1 + |\mathbf{x}|^2)^{-\frac{b}{4}} v(\mathbf{x}))\right\} d\mathbf{x},\ u, v \in B_D^{a,b}. \end{aligned} \tag{0.9}$$

Let μ_0 be the probability measure on $\mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})$ which is the probability law of the sharp time free field ϕ_0 on $(\Omega, \mathcal{F}, P)$ (cf. (0.7)), and μ_N be the probability measure on $\mathcal{S}'(\mathbb{R}^d \to \mathbb{R})$ which is the probability law of the Nelson's Euclidean free field on $\mathbb{R}^d$ (cf. (0.6)).

We denote

$$\phi_0(\varphi) \equiv < \phi_0, \varphi > \equiv \int_{\mathbb{R}^{d-1}} \left(H_{-\frac{1}{4}}\varphi\right)(\vec{x})\, \dot{B}^{d-1}(\vec{x})\, d\vec{x},$$

and

$$
\begin{aligned}
&:\phi_0(\varphi_1)\cdots\phi_0(\varphi_n): \\
&= \int_{\mathbb{R}^{k(d-1)}} H_{-\frac{1}{4}}\varphi_1(\vec{x}_1)\cdots H_{-\frac{1}{4}}\varphi_1(\vec{x}_k) : \dot{B}^{d-1}(\vec{x}_1)\cdots\dot{B}^{d-1}(\vec{x}_k) : \\
&\times d\vec{x}_1\cdots d\vec{x}_k \in \cap_{q\geq 1} L^q(\mu_0) \\
&\text{for}\quad \varphi,\varphi_j \in \mathcal{S}(\mathbb{R}^{d-1}\to\mathbb{R}),\quad j=1,\cdots,k,\quad k\in\mathbb{N},
\end{aligned}
\tag{0.10}
$$

where (0.10) is the k-th multiple stochastic integral with respect to the Gaussian white noise $\dot{B}^{d-1}$ on $\mathbb{R}^{d-1}$.

Since, $:\phi_0(\varphi_1)\cdots\phi_0(\varphi_n):$ is nothing more than an element of the n-th Wiener chaos of $L^2(\mu_0)$, it also adomits an expression by means of the **Hermite polynomial** of $\phi_0(\varphi_j)$, $j=1,\cdots,k$ (cf., e.g., [AY1,2] and references therein).

Remark 1. From the view point of the notational rigorousness, ϕ_0 and ϕ_N are the distribution valued random variables on the probability space $(\Omega,\mathcal{F},P)$, hence the notation such as

$$
:\phi_0(\varphi_1)\cdots\phi_0(\varphi_n):\in \bigcap_{q\geq 1} L^q(\mu_0)
$$

is incorrect. However in the above and in the sequel, since there is no ambiguity, for the simplicity of the notations we use the notations ϕ_0 and ϕ_N (with an obvious interpretation) to indicate the measurable functions X and resp. Y on the measure spaces $(\mathcal{S}'(\mathbb{R}^{d-1}),\mu_0,\mathcal{B}(\mathcal{S}'(\mathbb{R}^{d-1})))$ and resp. $(\mathcal{S}'(\mathbb{R}^{d}),\mu_N,\mathcal{B}(\mathcal{S}'(\mathbb{R}^{d})))$ such that

$$
P\Big(\{\omega : \phi_0(\omega)\in A\}\Big) = \mu_0\Big(\{\phi : X(\phi)\in A\}\Big),\quad A\in\mathcal{B}(\mathcal{S}'(\mathbb{R}^{d-1})),
$$
$$
P\Big(\{\omega : \phi_N(\omega)\in A'\}\Big) = \mu_N\Big(\{\phi : Y(\phi)\in A'\}\Big),\quad A'\in\mathcal{B}(\mathcal{S}'(\mathbb{R}^{d})),
$$

respectively, where $\mathcal{B}(S)$ denotes the Borel σ-field of the topological space S.

□

Let

$$
H_{\frac{1}{2}} \equiv (-\Delta_{d-1}+m^2)^{\frac{1}{2}},
\tag{0.11}
$$

and define the operator $d\Gamma(H_{\frac{1}{2}})$ on $L^2(\mu_0)$ (for the notations cf. Remark 1.)

$$
\begin{aligned}
d\Gamma(H_{\frac{1}{2}})(:\phi_0(\varphi_1)\cdots\phi_0(\varphi_n):) &=: \phi_0(H_{\frac{1}{2}}\varphi_1)\phi_0(\varphi_2)\cdots\phi_0(\varphi_n): + \cdots \\
&\cdots + :\phi_0(\varphi_1)\cdots\phi_0(\varphi_{n-1})\phi_0(H_{\frac{1}{2}}\varphi_k):
\end{aligned}
\tag{0.12}
$$

We state the well known fundamental structures on the free field with the space time dimension d as follows:

Proposition 0.1 *The operator $d\Gamma(H_{\frac{1}{2}})$ on $L^2(\mu_0)$ with the natural domain is an essentially self adjoint non negative operator, and it is a generator of the Markovian semigroup, denoted by $T_t \equiv e^{-td\Gamma(H_{\frac{1}{2}})}$, (i.e. satisfying the properties of positivity preserving and $L^q(\mu_0)$ contraction $(q\in[0,\infty])$), moreover T_t is* **hypercontractive** *(cf. [G]).*
ii) The operator $T_t\phi_0(\varphi)$ on the sharp time free field $L^2(\mu_0)$ is identified with the Nelson Euclidean free field ϕ_N in such a way that

$$
\begin{aligned}
&\int_{\mathcal{S}'(\mathbb{R}^{d-1}\to\mathbb{R})} T_{t_1}\Big(\Big(T_{t_2} <\cdot,\varphi_2>_{\mathcal{S}',\mathcal{S}}\Big)(\cdot) <\cdot,\varphi_1>_{\mathcal{S}',\mathcal{S}}\Big)(\phi)\ \mu_0(d\phi) \\
&\quad = E^{\mu_N}[<\phi,\varphi_1\times\delta_{\{t_1\}}(\cdot)><\phi,\varphi_2\times\delta_{\{t_1+t_2\}}(\cdot)>],
\end{aligned}
\tag{0.13}
$$

iii) The operator $e^{-itd\Gamma(H_{\frac{1}{2}})}$ *on the sharp time free field* $L^2(\mu_0)$ *is the time translation gropup on the Wightman free field, and*

$$e^{itd\Gamma(H_{\frac{1}{2}})}\phi_0(\varphi)e^{-itd\Gamma(H_{\frac{1}{2}})} \tag{0.14}$$

are the field operators on the free field with the space time dimension d.

□

1 Definition of *Hida product*

Let $d \in \mathbb{N}$ $(d \geq 2)$ be a given space time dimension, and ϕ_0 and ϕ_N be the corresponding sharp time free field and Nelson's Euclidean free field defined by (0.5) and (0.4) respectively, and μ_0 and μ_N be the probability laws of ϕ_0 and ϕ_N respectively.

For real γ, let

$$H_{-\gamma} \equiv (-\Delta_{d-1} + m^2)^{-\gamma}. \tag{1.1}$$

For $r \in \mathbb{N}$, let $\Lambda_{r,d-1} \in C_0^\infty(\mathbb{R}^{d-1} \to \mathbb{R}_+)$ be a given function such that

$$0 \leq \Lambda_{r,d-1}(\vec{x}) \leq 1\ (\vec{x} \in \mathbb{R}^{d-1}),\ \ \Lambda_{r,d-1} \equiv 1\ (|\vec{x}| \leq r),\ \ \Lambda_{r,d-1} \equiv 0\ (|\vec{x}| \geq r+1),$$

for $p \in \mathbb{N}$ define a Hida distribution $<: \phi_0^{2p} :, \Lambda_{r,d-1} >$ on the space the test functions, $\cap_{q\geq 1} L^q(\mu_0)$ as follows:

$$\begin{aligned}
&<: \phi_0^{2p} :, \Lambda_{r,d-1} > \\
&\equiv \int_{(\mathbb{R}^{d-1})^{2p}} \left\{ \int_{\mathbb{R}^{d-1}} \Lambda_{r,d-1}(\vec{x}) \prod_{k=1}^{2p} H_{-\frac{1}{4}}(\vec{x} - \vec{x}_k)\, d\vec{x} \right\} \\
&\times : \dot{B}^{d-1}(\vec{x}_1) \cdots \dot{B}^{d-1}(\vec{x}_{2p}) : d\vec{x}_1 \cdots d\vec{x}_{2p},
\end{aligned} \tag{1.2}$$

here, all the way of using notations follow the rule given by Remark 1.

For $d = 2$ $(d - 1 = 1)$ we know that

$$<: \phi_0^{2p} :, \Lambda_{r,1} > \in \bigcap_{q\geq 1} L^q(\mu_0).$$

But our main interest is concentrated on the case where $d = 4$ $(d - 1 = 3)$, and in this case $<: \phi_0^{2p} :, \Lambda_{r,3} >$ is not a random variable any more, but a Hida distribution.

In the sequel, if there is no indication of the dimension d in each discussion, then we should understand that the consideration is carried out on $d = 4$, $d - 1 = 3$.

Let us define a new multiplication between two Hida distributions $<: \phi_0^{2p} :, \Lambda_{r,3} >$ and $<: \phi_0^{2p} :, \Lambda_{r,3} >$. We denote this new multiplication procedure as *Hida product.* It produces one Hida distribution from another two, and the resulting distributions are different from the ones derived through the well known S-transform and others. Hence, equipping this multiplication, the space of Hida distribution has the structure of *ring.*

We have to stress that the Hida distributions generated through this new production procedure are much more complicated than the ones given by the known multiplication procedures, but they are much more fruitful, in fact by these distributions (operators) we may define the non-trivial interactions on the 4-dimensional space time quantum field.

Definition. (*Hida product*) For the space time dimension $d = 4$, let ϕ_0 be the sharp time free field with $d-1 = 3$. Let the Hida distribution $<: \phi_0^{2p} :, \Lambda_{r,3} >$ be the $2p$-th Wick power of the sharp time free field defined by (1.2). The *Hida product* $<:\phi_0^{2p}:,\Lambda_{r,3}> \times^{\mathcal{H}} <:\phi_0^{2p}:,\Lambda_{r,3}>$ is defined as a distribution on the space of the test functions $\cap_{q\geq 1}L^q(\mu_0)$ as follows:

$$
\begin{aligned}
&<:\phi_0^{2p}:,\Lambda_{r,3}> \times_{\mathcal{H}} <:\phi_0^{2p}:,\Lambda_{r,3}> \\
&\quad = <:\phi_0^{2p}:,\Lambda_{r,3}> \times <:\phi_0^{2p}:,\Lambda_{r,3}> \\
&\quad -(\text{all the terms that are not Hida distributions}),
\end{aligned} \tag{1.3}
$$

explicitly

$$
\begin{aligned}
&<:\phi_0^{2p}:,\Lambda_{r,3}> \times_{\mathcal{H}} <:\phi_0^{2p}:,\Lambda_{r,3}> \\
&\equiv \int_{(\mathbf{R}^3)^{4p}} \left\{ \int_{\mathbf{R}^3} \Lambda_{r,3}(\vec{y}) \prod_{k=1}^{2p} H_{-\frac{1}{4}}(\vec{y}-\vec{x}_k)\, d\vec{y} \right\} \\
&\times \left\{ \int_{\mathbf{R}^3} \Lambda_{r,3}(\vec{y}') \prod_{k=1}^{2p} H_{-\frac{1}{4}}(\vec{y}'-\vec{x}'_k)\, d\vec{y}' \right\} \\
&\times :\dot{B}^3(\vec{x}_1)\cdots\dot{B}^3(\vec{x}_{2p})\dot{B}^3(\vec{x}'_1)\cdots\dot{B}^3(\vec{x}'_{2p}): d\vec{x}_1\cdots d\vec{x}_{2p}d\vec{x}'_1\cdots d\vec{x}'_{2p} \\
&+4p^2 \int_{(\mathbf{R}^3)^{4p-2}} \left[\int_{\mathbf{R}^3\times\mathbf{R}^3} \left\{ \Lambda_{r,3}(\vec{y}) \prod_{k=1}^{2p-1} H_{-\frac{1}{4}}(\vec{y}-\vec{x}_k) \right\} \left\{ \Lambda_{r,3}(\vec{y}') \prod_{k=1}^{2p-1} H_{-\frac{1}{4}}(\vec{y}'-\vec{x}'_k) \right\} \right. \\
&\left. \times H_{-\frac{1}{2}}(\vec{y}-\vec{y}')d\vec{y}d\vec{y}' \right] :\dot{B}^3(\vec{x}_1)\cdots\dot{B}^3(\vec{x}_{2p-1})\dot{B}^3(\vec{x}'_1)\cdots\dot{B}^3(\vec{x}'_{2p-1}): \\
&\times d\vec{x}_1\cdots d\vec{x}_{2p-1}d\vec{x}'_1\cdots d\vec{x}'_{2p-1}.
\end{aligned} \tag{1.4}
$$

□

Remark 2. i) To get a production between two $<:\phi_0^{2p}:,\Lambda_{r,3}>$, if we use the S-transform, then we may only have the first term of (1.4) and do not have the second term of (1.4). In fact, by making use of the S-transform one can reform the operators that only has the **creation** terms and does not have any **annihilation** terms, on the other hand through the *Hida product* we have the new operators that contains both the **creation** and **annihilation** terms.

ii) For example for the 6-th power $(<:\phi_0^{2p}:,\Lambda_{r,3}>)^6$, the corresponding *Hida product*, denoted by $(<:\phi_0^{2p}:,\Lambda_{r,3}>)^6_{\mathcal{H}}$, involves much complicated terms. In particular, it possesses the following important term having a *hexagonal* form:

$$
\begin{aligned}
&\int_{(\mathbf{R}^3)^{6\times 2p-12}} \left[\int_{(\mathbf{R}^3)^6} \left\{ \Lambda_{r,3}(\vec{y}_1) \prod_{k=1}^{2p-2} H_{-\frac{1}{4}}(\vec{y}_1-\vec{x}_{1,k}) \right\} \right. \\
&\times H_{-\frac{1}{2}}(\vec{y}_1-\vec{y}_2) \left\{ \Lambda_{r,3}(\vec{y}_2) \prod_{k=1}^{2p-2} H_{-\frac{1}{4}}(\vec{y}_2-\vec{x}_{2,k}) \right\} \\
&\times H_{-\frac{1}{2}}(\vec{y}_2-\vec{y}_3) \left\{ \Lambda_{r,3}(\vec{y}_3) \prod_{k=1}^{2p-2} H_{-\frac{1}{4}}(\vec{y}_3-\vec{x}_{3,k}) \right\} \times\cdots \\
&\left. \cdots\times H_{-\frac{1}{2}}(\vec{y}_6-\vec{y}_1) \left\{ \Lambda_{r,3}(\vec{y}_6) \prod_{k=1}^{2p-2} H_{-\frac{1}{4}}(\vec{y}_6-\vec{x}_{6,k}) \right\} d\vec{y}_1\cdots d\vec{y}_6 \right] \\
&\times :\dot{B}^3(\vec{x}_{1,1})\cdots\dot{B}^3(\vec{x}_{6,2p-2}): d\vec{x}_{1,1}\cdots d\vec{x}_{6,2p-2}.
\end{aligned} \tag{1.5}
$$

□

The definition of the *Hida product* can be extended to the higher dimensions by a obvious manner. By using the *Hida product* we can define the power series of Hida distributions, in particular we can set

$$(e^{C<:\phi_0^{2p}:,\Lambda_{r,d-1}>})_{\mathcal{H}} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}(C <: \phi_0^{2p} :, \Lambda_{r,d-1} >)^n_{\mathcal{H}}, \qquad C \in \mathbf{C}. \tag{1.6}$$

Remark 3. By modifying the usual multiplication procedures of the random variables on the sharp time field to the *Hida product*, we can define the operator (cf. (0.14))

$$e^{itd\Gamma(H_{\frac{1}{2}})}(e^{it<:\phi_0^{2p}:,\Lambda_{r,d-1}>})_{\mathcal{H}}\times^{\mathcal{H}}\phi_0(\varphi)\times^{\mathcal{H}}e^{-itd\Gamma(H_{\frac{1}{2}})}(e^{-it<:\phi_0^{2p}:,\Lambda_{r,d-1}>})_{\mathcal{H}}\times^{\mathcal{H}} \tag{1.7}$$

with the domain $\cap_{q\geq 1}L^q(\mu_0)$, which has the structure of the *ring*.

[GrotS] defines the operators which are defined by making use of the S-transform, and [AGW] constructs the operators through the convolution of pseudo differential opetators with generalized white noises, and they showed that these operators do not include the non trivial interactions. [NaMu] introduces an anther important approach of the construction of the field by choosing special test functions that are not $\mathcal{S}$.

For $d = 4$, every existing known result on the field operator with non trivial interaction, in particular Φ_4^4, is a statement that the field operator can not be really an operator but a form on a Hilbert space and hence it does not admit an operation of productions.

□

In the next section, we try to construct the framework of the $(\Phi_4)^p$ field theory on a Banach space where the operators defined by (1.7) plays the role of the field operators (cf. Remark 2-i)).

2 Banach space realization of $(\Phi_d)^p$ field theory

For $f(t,\vec{x}) \in \mathcal{S}(\mathbb{R}^d \to \mathbb{C})$ let (cf. (0.14))

$$\phi(f) \equiv \int_{\mathbf{R}} e^{itd\Gamma(H_{\frac{1}{2}})}\phi_0(f(t,\cdot))e^{-itd\Gamma(H_{\frac{1}{2}})}dt. \tag{2.1}$$

Note that in (1.6) if we let $r \to \infty$, then

$$(e^{C<:\phi_0^{2p}:,\mathbb{R}^{d-1}>})_{\mathcal{H}} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}(C <: \phi_0^{2p} :, \mathbb{R}^{d-1} >)^n_{\mathcal{H}}, \qquad C \in \mathbf{C} \tag{2.2}$$

is still well defined as a *Hida distribution*.
Thus for $f(t,\vec{x}) \in \mathcal{S}(\mathbb{R}^d \to \mathbb{C})$ and $\lambda \geq 0$ we define an operator (cf. (1.7))

$$\phi_{\lambda V}(f) \equiv \int_{\mathbf{R}} e^{itd\Gamma(H_{\frac{1}{2}})}(e^{it\lambda<:\phi_0^{2p}:,\mathbb{R}^{d-1}>})_{\mathcal{H}}\times_{\mathcal{H}}\phi_0(f(t,\cdot)\times_{\mathcal{H}}e^{-itd\Gamma(H_{\frac{1}{2}})}(e^{-it\lambda<:\phi_0^{2p}:,\mathbb{R}^{d-1}>})_{\mathcal{H}}\times_{\mathcal{H}}\, dt \tag{2.3}$$

which maps a *Hida distribution* to a *Hida distribution*.

Let E' be the linear space of *Hida distributions* and E be a space of test functions that is the finite linear conbinations of

$$:\phi(g_1)\cdots\phi(g_r):1, \qquad g_j\in\mathcal{S}(\mathbb{R}^d\to\mathbb{C}), \quad j=1,\ldots,r, \quad r\in\mathbb{Z}_+.$$

We define a dualization between E' and E as follows:
for $f_l\in\mathcal{S}(\mathbb{R}^d\to\mathbb{C})$, $(l=1,\ldots,k\ )$ and $g_j\in\mathcal{S}(\mathbb{R}^d\to\mathbb{C})$, $(j=1,\ldots,r)$, $k,r\in\mathbb{Z}_+$

$$\begin{aligned}&<\phi_{\lambda V}(f_1)\cdots\phi_{\lambda V}(f_k)1,\ :\phi(g_1)\cdots\phi(g_r):1>_{E',E}\\ &\quad\equiv E\left[\left\{\phi_{\lambda V}(f_1)\cdots\phi_{\lambda V}(f_k)1\right\}_r\left(:\phi(g_1)\cdots\phi(g_r):1\right)\right],\end{aligned}\tag{2.4}$$

where $E[\cdot]$ denotes the generalized expectation of the *Hida distributions* and

$$\left\{\phi_{\lambda V}(f_1)\cdots\phi_{\lambda V}(f_k)1\right\}_r$$

is the r-th *Hida Chaos* of $\phi_{\lambda V}(f_1)\cdots\phi_{\lambda V}(f_k)1$, i.e.

$$\phi_{\lambda V}(f_1)\cdots\phi_{\lambda V}(f_k)1\equiv\oplus_{r=0}^{\infty}\left\{\phi_{\lambda V}(f_1)\cdots\phi_{\lambda V}(f_k)1\right\}_r.$$

Next, we modify the definition of the space E' as follows:

$$\begin{aligned}E' \ \equiv\ &\text{the finite linear combinations of } \ \phi_{\lambda V}(f_1)\cdots\phi_{\lambda V}(f_k)1,\\ &f_1,\ldots,f_k\in\mathcal{S}(\mathbb{R}^d\to\mathbb{C}),\quad k\in\mathbb{Z}_+.\end{aligned}\tag{2.5}$$

Finally, we define a norm $\|\cdot\|_{E'}$ on E' to make its completion a Banach space:

$$\begin{aligned}&\|\phi_{\lambda V}(f_1)\cdots\phi_{\lambda V}(f_k)1\|_{E'}\\ &\quad\equiv\sup_{g_1,\ldots,g_r\in\mathcal{S},r\in\mathbb{Z}_+}\frac{|<\phi_{\lambda V}(f_1)\cdots\phi_{\lambda V}(f_k)1,\ :\phi(g_1)\cdots\phi(g_r):1>_{E',E}|}{\|:\phi(g_1)\cdots\phi(g_r):1\|_k},\end{aligned}\tag{2.6}$$

where

$$\begin{aligned}&\|:\phi(g_1)\cdots\phi(g_r):1\|_k\\ &\quad\equiv\sum_{j=0}^{\infty}|A_j^{(2^k)}|^{\frac{1}{2^k}}+\cdots+\sum_{j=0}^{\infty}|A_j^{(2^l)}|^{\frac{1}{2^l}}+\cdots+\sum_{j=0}^{\infty}|A_j^{(1)}|^{\frac{1}{2}}+1,\end{aligned}\tag{2.7}$$

with $A_j^{(2^l)}$ being the each term appareing in the explicit expression of the generalized expectation

$$E\left[(e^{-\lambda<:\phi_0^{2p}:,\mathbb{R}^{d-1}>})_{\mathcal{H}}\left\{:\phi(g_1)\cdots\phi(g_r):1\right\}^{2^l}\right],$$

namely

$$\begin{aligned}\sum_{j=0}^{\infty}A_j^{(2^l)} &\equiv \left\{(e^{-\lambda<:\phi_0^{2p}:,\mathbb{R}^{d-1}>})_{\mathcal{H}}\times_{\mathcal{H}}\left\{:\phi(g_1)\cdots\phi(g_r):1\right\}^{2^l}\right\}_0\\ &= \text{0-th } \textit{Hida chaos} \text{ of } \ (e^{-\lambda<:\phi_0^{2p}:,\mathbb{R}^{d-1}>})_{\mathcal{H}}\left\{:\phi(g_1)\cdots\phi(g_r):1\right\}^{2^l}.\end{aligned}\tag{2.8}$$

The following is the main result:

Proposition 2.1 *By completing E' with respect to the norm defined by (2.6), the completion $\overline{E'}$ becomes a Banach space. On the Banach space $\overline{E'}$ the operators $\phi_{\lambda V}(f)$ defined by (2.3) resp. and the time translation operator resp. play the corresponding role of the field operator resp. and time translation operator resp. in the Wightman $(\Phi_d)^p$ field theory with a Hilbert space framework. In fact, $\phi_{\lambda V}(f)$ $(f \in \mathcal{S}(\mathbb{R}^d \to \mathbb{C}))$ satisfy the property of local commutativity in the spacelike separated region and admit the invariance property with respect to the Poincaré group.*

□

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