数理解析研究所講究録
第 1601 巻 2008 年 1-6

Statistical Theory of Inhomogeneous Turbulence based on the Cross-Independence Closure Hypothesis

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Abstract

Inhomogeneous turbulence in an incompressible viscous fluid is studied statistically using the cross-independence closure hypothesis for the equations of the velocity distributions, which has been proposed by the author and used successfully for homogeneous isotropic turbulence. In the present paper, the velocity field of inhomogeneous turbulence is decomposed into the mean velocity ad the fluctuation velocity around the mean, and the hypothesis is applied to the distributions of the fluctuation velocities. Closed equations are obtained for the mean velocity and the distributions of the one- and two-point velocities, the latter being expressed in terms of the distributions of the sum and difference of the two-point velocities. Like in homogeneous isotropic turbulence, the velocity distributions exhibit clear inertial normality in almost whole space, except for the longitudinal velocity-difference distribution which is non-normal in the local range of Kolmogorov's scale. The general statistical characters of these equations are discussed.

1. Introduction

Complete statistical description of the turbulent velocity field is provided by an infinite set of the joint velocity distributions at arbitrary numbers of spatial and temporal points. The equations governing these distributions have been obtained by Lundgren (1967) and Monin (1967) using the Navier-Stokes equations of motion and the probability conservation law. In practice, however, we have to deal with a finite subset of such equations and then encounter the difficulty of unclosedness of the subset since the equations for a finite number of distributions always includes a new higher-order distribution according the nonlinearity of the equations of motion. Thus, we need a physical hypothesis for making the set of equations closed, and this problem has been one of the principal difficulties in the study of turbulence.

For this purpose, the *cross-independence closure hypothesis* has been proposed by Tatsumi (2001) and applied successfully to homogeneous isotropic turbulence by Tatsumi and Yoshimura (2004, 2007) (see also Tatsumi (2007)). In the present paper, this approach is extended to more general inhomogeneous turbulence with a good prospect of application to various problems of practical importance.

2. Equations of motion

The velocity $\mathbf{u}(\mathbf{x},t)$ at a space-time point $(\mathbf{x},t)$ in an incompressible viscous fluid is governed by the Navier-Stokes equation of motion,

$$\frac{\partial \mathbf{u}}{\partial t}+\left(\mathbf{u}\cdot\frac{\partial}{\partial \mathbf{x}}\right)\mathbf{u}-\nu\left|\frac{\partial}{\partial \mathbf{x}}\right|^2\mathbf{u}=\frac{\partial}{\partial \mathbf{x}}\left(\frac{p}{\rho}\right), \tag{1}$$

and the equation of continuity,

$$\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{u} = 0, \tag{2}$$

where $p(\mathbf{x},t)$ represents the pressure of turbulence and ρ and ν denote the density and the kinetic viscosity of the fluid respectively. The following expression for the pressure p is obtained from Eqs.(1) and (2):

$$\frac{p}{\rho} = \frac{1}{4\pi} \int |\mathbf{x}-\mathbf{x}'|^{-1} \left\{ \frac{\partial}{\partial \mathbf{x}'} \cdot \left(\mathbf{u}' \cdot \frac{\partial}{\partial \mathbf{x}'} \right) \mathbf{u}' \right\} d\mathbf{x}'. \tag{3}$$

For dealing with inhomogeneous turbulence, it is convenient to decompose the velocity $\mathbf{u}(\mathbf{x},t)$ into the mean velocity $\overline{\mathbf{u}}(\mathbf{x},t)$ and the fluctuation velocity $\widehat{\mathbf{u}}(\mathbf{x},t)$ around the mean,

$$\overline{\mathbf{u}}(\mathbf{x},t) = \langle \mathbf{u}(\mathbf{x},t) \rangle, \quad \widehat{\mathbf{u}}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) - \overline{\mathbf{u}}(\mathbf{x},t), \quad \langle \widehat{\mathbf{u}}(\mathbf{x},t) \rangle = 0, \tag{4}$$

where the symbol $\langle\ \rangle$ denotes the mean value with respect to an appropriate intial probability distribution.

Then, Eqs.(1) and (3) are decomposed into those for the mean velocity $\overline{\mathbf{u}}(\mathbf{x},t)$ as

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \left(\overline{\mathbf{u}} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \overline{\mathbf{u}} + \left\langle \left(\widehat{\mathbf{u}} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \widehat{\mathbf{u}} \right\rangle - \nu \left| \frac{\partial}{\partial \mathbf{x}} \right|^2 \mathbf{u} = \frac{\partial}{\partial \mathbf{x}} \left(\frac{\overline{p}}{\rho} \right), \tag{5}$$

$$\frac{\overline{p}}{\rho} = \frac{1}{4\pi} \int |\mathbf{x}-\mathbf{x}'|^{-1} \frac{\partial}{\partial \mathbf{x}'} \cdot \left\{ \left(\overline{\mathbf{u}}' \cdot \frac{\partial}{\partial \mathbf{x}'} \right) \overline{\mathbf{u}}' + \left\langle \left(\widehat{\mathbf{u}}' \cdot \frac{\partial}{\partial \mathbf{x}'} \right) \widehat{\mathbf{u}}' \right\rangle \right\} d\mathbf{x}', \tag{6}$$

and those for the fluctuation velocity $\widehat{\mathbf{u}}(\mathbf{x},t)$ as

$$\begin{aligned} & \frac{\partial \widehat{\mathbf{u}}}{\partial t} + \left((\overline{\mathbf{u}} + \widehat{\mathbf{u}}) \cdot \frac{\partial}{\partial \mathbf{x}} \right) (\overline{\mathbf{u}} + \widehat{\mathbf{u}}) - \left(\overline{\mathbf{u}} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \overline{\mathbf{u}} - \left\langle \left(\widehat{\mathbf{u}} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \widehat{\mathbf{u}} \right\rangle - \nu \left| \frac{\partial}{\partial \mathbf{x}} \right|^2 \widehat{\mathbf{u}} \\ = & \frac{\partial}{\partial \mathbf{x}} \left(\frac{\widehat{p}}{\rho} \right), \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{\widehat{p}}{\rho} = & \frac{1}{4\pi} \int |\mathbf{x}-\mathbf{x}'|^{-1} \times \\ & \times \frac{\partial}{\partial \mathbf{x}'} \cdot \left\{ \left((\overline{\mathbf{u}}' + \widehat{\mathbf{u}}') \cdot \frac{\partial}{\partial \mathbf{x}'} \right) (\overline{\mathbf{u}}' + \widehat{\mathbf{u}}') - \left(\overline{\mathbf{u}}' \cdot \frac{\partial}{\partial \mathbf{x}'} \right) \overline{\mathbf{u}}' - \left\langle \left(\widehat{\mathbf{u}}' \cdot \frac{\partial}{\partial \mathbf{x}'} \right) \widehat{\mathbf{u}}' \right\rangle \right\} d\mathbf{x}'. \end{aligned} \tag{8}$$

3. Velocity distributions

The joint distributions of the multi-point fluctuation velocities $\widehat{\mathbf{u}}_n = \widehat{\mathbf{u}}(\mathbf{x}_n,t)$ $(n \geq 1)$ are defined as

$$f^{(n)}(\mathbf{v}_1, ..., \mathbf{v}_n; \mathbf{x}_1, ..., \mathbf{x}_n; t) = \left\langle \prod_{m=1}^{n} \delta \left(\widehat{\mathbf{u}}_m - \mathbf{v}_m \right) \right\rangle, \tag{9}$$

where $\mathbf{v}_n$ $(n \geq 1)$ denote the probability variables corresponding to $\widehat{\mathbf{u}}_n(\mathbf{x}_n, t)$ $(n \geq 1)$ respectively.

The set of equations for the velocity distributions $f^{(n)}$ are obtained according to the way of Lundgren (1967) and Monin (1967) but they are not closed. Hence a closure hypothesis connecting the distribution $f^{(n)}$ with those of lower orders has to be employed.

The simplest closure hypothesis may be expressed for the distributions f and $f^{(2)}$as

$$f^{(2)}(\mathbf{v}_1, \mathbf{v}_2; \mathbf{x}_1, \mathbf{x}_2; t) = f(\mathbf{v}_1, \mathbf{x}_1, t) f(\mathbf{v}_2, \mathbf{x}_2, t). \tag{10}$$

The relation (10) is exactly valid for the normal distribution of f, so that it is usually called the "quasi-normal approximation". However, if we take it as a relation for an arbitrary f, it is shown to be valid for large distance $r = |\mathbf{x}_2 - \mathbf{x}_1|$ between the two points but definitely not for small r. Actually, this character of the relation (10) has been a serious weakness of those theories which utilize this sort of approximation.

4. Cross-independence closure hypothesis

A new idea has been introduced by Tatsumi (2001) by taking the cross-velocities, or the sum and difference of the velocities $\widehat{\mathbf{u}}_1$and $\widehat{\mathbf{u}}_2$, as

$$\widehat{\mathbf{u}}_+(\mathbf{x}_1, \mathbf{x}_2; t) = \frac{1}{2}(\widehat{\mathbf{u}}_1 + \widehat{\mathbf{u}}_2), \qquad \widehat{\mathbf{u}}_-(\mathbf{x}_1, \mathbf{x}_2; t) = \frac{1}{2}(\widehat{\mathbf{u}}_2 - \widehat{\mathbf{u}}_1) \tag{11}$$

and considering the one- and two-body distributions of the cross-velocities $\widehat{\mathbf{u}}_+$ and $\widehat{\mathbf{u}}_-$ as follows:

$$\begin{aligned} g_+(\mathbf{v}_+; \mathbf{x}_1, \mathbf{x}_2; t) &= \langle \delta\left(\widehat{\mathbf{u}}_+ - \mathbf{v}_+\right) \rangle, \\ g_-(\mathbf{v}_-; \mathbf{x}_1, \mathbf{x}_2; t) &= \langle \delta\left(\widehat{\mathbf{u}}_- - \mathbf{v}_-\right) \rangle, \\ g^{(2)}(\mathbf{v}_+, \mathbf{v}_-; \mathbf{x}_1, \mathbf{x}_2; t) &= \langle \delta\left(\widehat{\mathbf{u}}_+ - \mathbf{v}_+\right) \delta\left(\widehat{\mathbf{u}}_- - \mathbf{v}_-\right) \rangle, \end{aligned} \tag{12}$$

where $\mathbf{v}_+$ and $\mathbf{v}_-$ denote the probability variables corresponding to the cross-velocities $\widehat{\mathbf{u}}_+$ and $\widehat{\mathbf{u}}_-$ respectively. It may be seen from Eq.(11) that the distributions $g^{(2)}$ is nothing but another expression of the distribution $f^{(2)}$,

$$f^{(2)}(\mathbf{v}_1, \mathbf{v}_2; \mathbf{x}_1, \mathbf{x}_2; t) = 2^{-3} g^{(2)}(\mathbf{v}_+, \mathbf{v}_-; \mathbf{x}_1, \mathbf{x}_2; t). \tag{13}$$

Like the independence relation (10) of the two f's, we can assume the independence between g_+ and g_- as

$$g^{(2)}(\mathbf{v}_+, \mathbf{v}_-; \mathbf{x}_1, \mathbf{x}_2; t) = g_+(\mathbf{v}_+; \mathbf{x}_1, \mathbf{x}_2; t) g_-(\mathbf{v}_-; \mathbf{x}_1, \mathbf{x}_2; t). \tag{14}$$

This relation (14) together with the identity (13) provides us with another closure hypothesis for the distribution $f^{(2)}$ or the *cross-independence closure hypothesis.* Unlike the ordinary independence closure (10), the cross-independence closure (14) is shown to be valid for both large and small values of the distance r (see Tatsumi (2001)). This property is particularly important for the present closure since the equation for the distribution $f^{(n)}$ includes the higher-order distribution $f^{(n+1)}$ only in its degenerate form with vanishing distance between $\mathbf{x}_n$.and $\mathbf{x}_{n+1}$. Concerning the physical meaning of this closure hypothesis and its outcome in homogeneous isotropic turbulence, reference may be made to Tatsumi and Yoshimura (2004, 2007) and Tatsumi (2007).

5. Closed equations for velocity distributions

Applying Eqs.(13) and (14) to the equations for the fluctuation velocity distributions derived from Eqs.(7) and (8) according to the method of Lundgren (1967) and Monin (1967), we obtain the closed equations for the one-point velocity distribution f and the two-point velocity distribution $f^{(2)}$, the latter of which being expressed in terms of the velocity-sum distribution g_+ and the velocity-difference distribution g_-.

5.1. Equation for one-point velocity distribuion

The closed equation for the one-point velocity distribution f is expressed as follows:

$$\begin{aligned} & \frac{\partial f}{\partial t} + (\bar{\mathbf{u}} + \mathbf{v}) \cdot \frac{\partial f}{\partial \mathbf{x}} - \left(\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}}\right) \bar{\mathbf{u}} \cdot \frac{\partial f}{\partial \mathbf{v}} - \nu \left|\frac{\partial}{\partial \mathbf{x}}\right|^2 f + \alpha(\mathbf{x}, t) \left|\frac{\partial}{\partial \mathbf{v}}\right|^2 f \\ = & \frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{v}} \{\beta(\mathbf{v}, \mathbf{x}, t) + \gamma(\mathbf{v}, \mathbf{x}, t)\} f, \end{aligned} \tag{15}$$

with

$$\begin{aligned} \alpha(\mathbf{x}, t) &= \frac{1}{3}\varepsilon(\mathbf{x}, t) = \frac{2}{3}\nu \lim_{|\mathbf{r}'| \to 0} \left|\frac{\partial}{\partial \mathbf{r}'}\right| \int |\mathbf{v}'_-|^2 g_- (\mathbf{v}'_-; \mathbf{x}, \mathbf{r}'; t)\, d\mathbf{v}'_-, \\ \beta(\mathbf{v}, \mathbf{x}, t) &= \frac{1}{4\pi} \int\!\!\int |\mathbf{r}'|^{-1} \left((\mathbf{v} + 2\mathbf{v}'_-) \cdot \frac{\partial}{\partial \mathbf{r}'}\right)^2 g_- (\mathbf{v}'_-; \mathbf{x}, \mathbf{r}'; t)\, d\mathbf{r}' d\mathbf{v}'_-, \\ \gamma(\mathbf{v}, \mathbf{x}, t) &= \frac{1}{4\pi} \int\!\!\int |\mathbf{r}'|^{-1} \left((\mathbf{v} + 2\mathbf{v}'_-) \cdot \frac{\partial}{\partial \mathbf{r}'}\right)^2 \left(\mathbf{v}'_- \cdot \frac{\partial}{\partial \mathbf{v}}\right) g_- (\mathbf{v}'_-; \mathbf{x}, \mathbf{r}'; t)\, d\mathbf{r}' d\mathbf{v}'_-. \end{aligned} \tag{16}$$

where $\varepsilon(\mathbf{x}, t) = 3\alpha(\mathbf{x}, t)$ represents the energy dissipation rate at the point $(\mathbf{x}, t)$, and $\beta(\mathbf{v}, \mathbf{x}, t)$ and $\gamma(\mathbf{v}, \mathbf{x}, t)$ have the dimension of the energy at $(\mathbf{x}, t)$.

5.2. Equation for velocity-sum distribution

The closed equation for the velocity-sum distribution g_+ is expressed as follows:

$$\begin{aligned} & \frac{\partial g_+}{\partial t} + \sum_{i=1,2} \left\{ (\bar{\mathbf{u}}_i + \mathbf{v}_+) \cdot \frac{\partial g_+}{\partial \mathbf{x}_i} - \frac{1}{2}\left(\mathbf{v}_+ \cdot \frac{\partial}{\partial \mathbf{x}_i}\right) \bar{\mathbf{u}}_i \cdot \frac{\partial g_+}{\partial \mathbf{v}_+} \right\} \\ & + \sum_{i=1,2} \left\{ -\nu \left|\frac{\partial}{\partial \mathbf{x}_i}\right|^2 + \frac{1}{4}\alpha(\mathbf{x}_i, t) \left|\frac{\partial}{\partial \mathbf{v}_+}\right|^2 \right\} g_+ \\ = & \frac{1}{2}\frac{\partial}{\partial \mathbf{v}_+} \cdot \sum_{i=1,2} \frac{\partial}{\partial \mathbf{x}_i} \left\{ \beta(\mathbf{v}_+, \mathbf{x}_i, t) + \frac{1}{2}\gamma(\mathbf{v}_+, \mathbf{x}_i, t) \right\} g_+. \end{aligned} \tag{17}$$

5.3. Equation for velocity-difference distribution

The closed equation for the velocity-difference distribution g_- is expressed as follows:

$$\frac{\partial g_-}{\partial t} + \sum_{i=1,2} \left\{ \left(\overline{\mathbf{u}}_i + (-1)^i \mathbf{v}_- \right) \cdot \frac{\partial g_-}{\partial \mathbf{x}_i} - \frac{1}{2} \left(\mathbf{v}_- \cdot \frac{\partial}{\partial \mathbf{x}_i} \right) \overline{\mathbf{u}}_i \cdot \frac{\partial g_-}{\partial \mathbf{v}_-} \right\}$$
$$+ \sum_{i=1,2} \left\{ -\nu \left| \frac{\partial}{\partial \mathbf{x}_i} \right|^2 + \frac{1}{4} \alpha(\mathbf{x}_i, t) \left| \frac{\partial}{\partial \mathbf{v}_-} \right|^2 \right\} g_-$$
$$= \frac{1}{2} \frac{\partial}{\partial \mathbf{v}_-} \cdot \sum_{i=1,2} (-1)^i \frac{\partial}{\partial \mathbf{x}_i} \left\{ \beta(\mathbf{v}_-, \mathbf{x}_i, t) + \frac{1}{2} \gamma(\mathbf{v}_-, \mathbf{x}_i, t) \right\} g_-. \tag{18}$$

6. Discussions

Now, Eq.(15) for the one-point distribution f, Eq.(17) for the velocity-sum distribution g_+ and Eq.(18) for the velocoty-dofference distribution g_- constitute, together with Eqs.(5) and (6) for the mean velocity $\overline{\mathbf{u}}$, a closed set of the governing equations for the present theory of turbulence. All statistical knowledges on turbulence related with these velocity distributions can be obtained as the solutions of this quartet of equations without resort to any other ad hoc asssumption. Before proceeding to practical application of the present theory, let us make some comparative discussions of the theory with Kolmogorov's and other existing theories of turbulence.

6.1. Kolmogorov's local isotropic turbulence

It has already been pointed out that the cross-independence hypothesis assumed in the present theory is analogous to Kolmogorov's hypothesis of local isotropic turbulence, which assumes the independence of small eddies represented by the velocity-difference $\triangle \mathbf{u} = 2\mathbf{u}_- = \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$ from large eddies represented by the velocities $\mathbf{u}(\mathbf{x})$ and $\mathbf{u}(\mathbf{x}+\mathbf{r})$ (Kolmogorov (1941)). This analogy is quite true although the large eddies are represented by the velocity-sum $2\mathbf{u}_+ = \mathbf{u}(\mathbf{x}+\mathbf{r}) + \mathbf{u}(\mathbf{x})$ in the present theory and subsequent analysis is made of the probabity distributions rather than the dimensional analyses of the statistical means in the latter. Thus, it may be natural to expect that the two theories have basically similar consequences with each other.

On the other hand, a significant discrepancy exists between the two theories since the steadiness in time is assumed in the notion of the local isotropy of turbulence while no such limitation is made in the present theory. Actually, this discrepancy is the worst in homogeneous isotropic turbulence which is definitely decaying in time. We expect, however, to be able to resolve this problem by dealing with the steady quasi-homogeneous turbulence as an example in the present study.

Except for this discrepancy, Kolmogorov's hypothesis of the local isotropy is totally satisfied in the fundamental equations of the present theory. It may easily be seen from Eqs.(15), (17) and (18) that these equations include the mean velocity $\overline{\mathbf{u}}$ only in the transfer terms and all other dissipation and pressure terms are not infuenced by the mean flow. This property is expected to simplify considrably the study of inhomogeneous turbulence..

6.2. Turbulent energy dissipation

It may also be observed from Eqs.(15), (17) and (18) that the dissipation terms of these equations are composed of the one representing the diffusion in the physical space due to the molecular viscosity ν and the other expressing the counter-diffusion in the velocity space due to the energy dissipation $\alpha = \varepsilon/3$. The former means the viscous dissipation equivalent to that in laminar flows and the latter the inertial dissipation equivalent to that in homogeneous turbulence. This result clearly shows that there exists no such turbulent dissipation term that can be expressed in terms of the spatial diffusion due to the "turbulent viscosity".

6.3. Energy balance equation

The equation for the balance of the turbulent energy,

$$E(\mathbf{x}, t) = \frac{1}{2} \left\langle \widehat{u}_i(\mathbf{x}, t)^2 \right\rangle, \tag{19}$$

where the summation convention being used, is immediately derived from the equation of the fluctuation velocity (7) as

$$\left[\frac{\partial}{\partial t} + \overline{u}_k \frac{\partial}{\partial x_k} - \nu \left(\frac{\partial}{\partial x_k} \right)^2 \right] E(\mathbf{x}, t) = - \langle \widehat{u}_i \widehat{u}_k \rangle \frac{\partial \overline{u}_i}{\partial x_k} - \frac{1}{6} \frac{\partial}{\partial x_k} \langle \widehat{u}_k \widehat{u}_i^2 \rangle - \nu \left\langle \left(\frac{\partial \widehat{u}_i}{\partial x_k} \right)^2 \right\rangle. \tag{20}$$

This equation has important roles in the practical theories of turbulent flows, being conveniently used as the basis for various turbulence models such as the K-ε model and the large-eddy simulation (see Pope (2000)). In those theories, the terms on the right-hand side are dealt with as separate unknowns and have to be evaluated using various approximations. On the other hand, Eq.(20) is derived in the present theory from Eq.(15) using the identity,

$$\varepsilon(\mathbf{x}, t) = \nu \left\langle \left(\frac{\partial \widehat{u}_i}{\partial x_k} \right)^2 \right\rangle, \tag{21}$$

and hence all terms in the equation can be evaluated within the framework of the theory. At this stage, we know that these equations are rather complicated for mathematical analysis and may be simplified reasonably for practical purposes.

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