Nonstandard arguments and the stability of generic structures

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Abstract

Generic 構造の研究に超準的手法を導入する。本稿では特に、Wagner
が行った generic 構造の安定性の強さについての研究 [4] に、弱冠の新
しい結果を付け足す。Wagner は saturated な generic 構造が安定にな
る為の十分条件 DS と ω-安定になる為の十分条件 DW を定義した。本
稿では DS を簡略化し、DS と DW の間の関係を調べる。

1 Preliminaries

Let $L$ be a countable relational language. Let $\mathbb{K}$ be a nonempty class of finite
$L$-structures closed under isomorphisms and substructures (we consider the
emptyset as an $L$-structure). Suppose $A \leq B$ is a reflexive and transitive
relation on elements $A \subseteq B$ of $\mathbb{K}$, which is invariant under isomorphisms.
If $A \leq B$ holds, we say that $A$ is closed in $B$. We also assume that $(\mathbb{K}, \leq)$
satisfies the following properties:

1. $\emptyset \leq A$,
2. $A \subseteq B \subseteq C, A \leq C \Rightarrow A \leq B$,
3. $A \leq B \Rightarrow A \cap C \leq B \cap C$.

Let $(\mathbb{K}, \leq)$ be as above. Let $N$ be an $L$-structure whose any finite sub-
structure belongs to $\mathbb{K}$. Note that for any $A \subseteq N$, there is a unique smallest
closed superset of $A$ in $N$. We call this set the closure of $A$.

Definition 1 Let $A \subseteq B$. We say that $B$ is a minimal extension of $A$ if
the following conditions are satisfied

- $A \not\leq B$
\begin{itemize}
\item $A \leq B'$ for any $A \subseteq B' \subset B$.
\end{itemize}

\textbf{Definition 2} Let $\leq$ be a closed relation on $\mathbb{K}$. Then we say that $(\mathbb{K}, \leq)$ satisfies finite closure axiom if there is no infinite chain $(A_i)_{i<\omega}$ of elements of $\mathbb{K}$ such that $A_{i+1}$ is a minimal extension of $A_i$ for each $i < \omega$.

We assume that $(\mathbb{K}, \leq)$ satisfies the finite closure axiom in this paper. We say that an $L$-structure $N$ has finite closures if for any finite $A \subseteq N$, the closure of $A$ is also finite. Put $\overline{\mathbb{K}} = \{ N : L$-structure $| A \in \mathbb{K}$ for any $A \subset_{\text{fin}} N \}$.

\textbf{Fact 3} [2] Let $\leq$ be a closed relation on $\mathbb{K}$. Then the following are equivalent:

1. $\mathbb{K}$ satisfies finite closure axiom.
2. Every member of $\overline{\mathbb{K}}$ has finite closures.
3. Every $\omega$-saturated member of $\overline{\mathbb{K}}$ has finite closures.
4. Some $\omega$-saturated member of $\overline{\mathbb{K}}$ has finite closures.

\textbf{Definition 4} Let $M$ be an $L$-structure. We say that $M$ is a $\mathbb{K}$-generic structure if the following conditions are satisfied:

1. $M$ is countable.
2. $\forall A \subset_{\text{fin}} M, A \in \mathbb{K}$ (i.e. $M \in \overline{\mathbb{K}}$).
3. $A \leq M, A \leq B \in \mathbb{K} \Rightarrow \exists B' \leq M$ such that $B' \cong A B$.

\textbf{Fact 5} Suppose that $(\mathbb{K}, \leq)$ satisfies the finite closure axiom. Then a $\mathbb{K}$-generic structure is unique.

\textbf{Definition 6} Let $d$ be a function from $\{ A : A \leq_{\text{fin}} M \}$ to $\mathbb{R}_{\geq 0}$. We say $d$ is a dimension function for $M$ if for all $A, B \leq_{\text{fin}} M$,

1. $A \subset B \Rightarrow d(A) \leq d(B)$
2. (Monotonicity) $d(A \cup B) + d(A \cap B) \leq d(A) + d(B)$
3. $A \cong B \Rightarrow d(A) = d(B)$

For arbitrary $A \subset_{\text{fin}} M$, we put $d(A) = d(\overline{A})$. We define $d(A/B)$ the relative dimension of $A$ over $B$. For finite $A, B$, $d(A/B) = d(AB) - d(B)$. For finite $A$, arbitrary $B$, $d(A/B) = \inf \{ d(A/B_0) : B_0 \subset_{\text{fin}} B \}$. It is easy to check that these two definitions has the same value in the case $A$ and $B$ are finite.
2 Nonstandard arguments

Let $M$ be the $\mathbb{K}$-generic structure and $d$ be a dimension function for $M$. We consider $M$ to be a 3-sorted structure

$$(M \cup P \cup \mathbb{R}; F, \in, d \leq, \cdots)$$

where $P$, $F$, $\in$ are as above, $d$ is the dimension function of $M$, $\leq$ is the closed relation on $P \times P$.

We define the nonstandard model $M^*$ of $M$ by a sufficiently saturated extension of this structure

$$(M \cup P \cup \mathbb{R}, F, \in, d \leq, \cdots) \prec (M^* \cup P^* \cup \mathbb{R}^*, F^*, \in^*, d^*, \leq^*, \cdots)$$

**Definition 7** A set $A \in F^*$ is said to be a hyperfinite set. For $A \subseteq M$, $A^* \in F^*$ is said to be a hyperfinite extension of $A$ if

- $M^* \models a \in^* A^*$ for each $a \in A$, and
- $M^* \models A^* \subseteq^* A$.

write $A \subset_{hf} A^*$, $A^* \supset_{hf} A$

By saturation, a hyperfinite extension of $A$ always exists.

**Lemma 8** For any subseteq $A$ of $M$, there exists a hyperfinite extension of $A$.

**Proof:** It is enough to prove that the following set of formulas is satisfiable:

$$\Gamma(X) = \{a \in^* X|a \in A\} \cup \{X \subseteq^* A\} \cup \{X \in F\}.$$ 

But for any finite subseteq $A_0$ of $A$, $A_0$ realizes the following set of formulas:

$$\{a \in^* X|a \in A_0\} \cup \{X \subseteq^* A\} \cup \{X \in F\}.$$ 

So, by compactness, $\Gamma(X)$ is satisfiable.

Let $x$, $y$ be two nonstandard (or standard) real numbers. We write $x \approx y$ if $|x - y| < 1/n$ for each $n \in \omega$.

**Lemma 9** For $r \in \mathbb{R}$, $\bar{a} \in M$ and $A \subset M$, the following are equivalent.

1. $d(\bar{a}/A) = r;$
2. $d^*(\bar{a}/A^*) \approx r$ for any $A^* \supset hf A$

3. $d^*(\bar{a}/A^*) \approx r$, for some $A^* \supset hf A$.

Proof: $(1 \rightarrow 2)$: By monotonicity of $d$, there are $A_n \subset_{\text{fin}} A$ ($n = 1, 2, \ldots$) such that $\forall X \in F$

$$A_n \subset X \subset A \rightarrow r \leq d(\bar{a}/X) \leq r + 1/n.$$

These statements hold also in $M^*$. So if $A^*$ is a hyperfinite extension of $A$, then we have

$$r \leq d^*(\bar{a}/A^*) \leq r + 1/n \ (n = 1, 2, \ldots)$$

So we have $d^*(\bar{a}/A^*) \approx r$.

$(2 \rightarrow 3)$: trivial.

$(3 \rightarrow 1)$: We assume 3 and choose a witness $A^*$. Then $(d^*(\bar{a}/A^*) \approx r)$.

Suppose 1 is not the case. Then there is $s \neq r$ such that $d(\bar{a}/A) = s$. By 1 $\Rightarrow$ 2, we have $d^*(\bar{a}/A^*) \approx s$. A contradiction.

Note that $M \models \forall A \in P \exists! \overline{A} (A \subseteq \overline{A} \leq M \wedge \forall X A \subseteq X \leq M \rightarrow \overline{A} \subseteq X)$.

This formula holds also in $M^*$. For $X \in P^*$, we write $\overline{X}$ as the "closure" of $X$ in $M^*$. In this paper, $M \models X \in F^* \rightarrow \overline{X} \in F^*$ because $\mathbb{K}$ satisfies the finite closure condition.

3 Main result

Definition 10 ([4])

1. Let $A, B \subset_{\text{fin}} M$ and $C \subset M$. Then we say $A$ and $B$ are $d$-independent over $C$ and write $A \downarrow^d_C B$ if the following conditions are satisfied:

- $d(A/BC) = d(A/C)$, and
- $\overline{AC} \cap \overline{BC} = \overline{C}$.

2. For arbitrary $A, B, C \subset M$, we say $A$ and $B$ are $d$-independent over $C$ if for each $A_0 \subset_{\text{fin}} A, B_0 \subset_{\text{fin}} B$, $A_0 \downarrow^d_C B_0$

Note that for closed sets $A, B$, $A$ and $B$ are $d$-independent over $A \cap B$ if and only if for each $A_0 \subset_{\text{fin}} A, B_0 \subset_{\text{fin}} B$, $d(A_0/B_0(A \cap B)) = d(A_0/A \cap B)$. 
Definition 11 Let $A$ and $B$ be closed subsets of $M$. Then we say $A$ and $B$ are $d^*$-independent over $A \cap B$ if the following conditions are satisfied: there exist a hyperfinite extension $A^*$ of $A$ and a hyperfinite extension $B^*$ of $B$ such that

- $A^*$ and $B^*$ are both closed
- $d(A^*/B^*) = d(A^*/A^* \cap B^*)$

Wagner’s definition of DS (a sufficient condition for saturated $M$ to be stable) is as follows:

For any closed $A, B$, if $\forall n \in \omega, \forall A_0 \subset \text{fin} \ A, \forall B_0 \subset \text{fin} \ B$, $A_0 \subset \exists A' \leq \text{fin}$ $A, B_0 \subset \exists B' \leq \text{fin} \ B$ such that

$$d(A') + d(B') \leq d(A'B') + d(A' \cap B') + 1/n,$$

then $A$ and $B$ are free over $A \cap B$ and $AB$ is closed.

On the other hands, Wagner’s definition of DW (a sufficient condition for saturated $M$ to be $\omega$-stable) is as follows:

- for any closed $A, B$, if $A \downarrow_{A \cap B}^d B$, then $A$ and $B$ are free over $A \cap B$ and $AB$ is closed and
- for any $\bar{a}$ and $X$, there exists finite $X_0 \subseteq X$ such that $d(\bar{a}/X_0) = d(\bar{a}/X)$.

Theorem 12 For arbitrary closed $A, B$, the following are equivalent:

1. $\forall n \in \omega, \forall A_0 \subset \text{fin} \ A, \forall B_0 \subset \text{fin} \ B$, $A_0 \subset \exists A' \leq \text{fin} \ A, B_0 \subset \exists B' \leq \text{fin} \ B$ such that $d(A') + d(B') \leq d(A'B') + d(A' \cap B') + 1/n$

2. $A \downarrow_{A \cap B}^{d^*} B$

3. $A \downarrow_{A \cap B}^{d} B$

Proof: $(1 \rightarrow 2)$: Assume 1. Then by saturatedness, There exist a closed hyperfinite extension $A^*$ of $A$ and a closed hyperfinite extension $B^*$ of $B$ such that for all $n \in \omega$,

$$d^*(A^*) + d^*(B^*) \leq d^*(A^*B^*) + d^*(A^* \cap B^*) + 1/n.$$  

The other direction

$$d^*(A^*) + d^*(B^*) \geq d^*(A^*B^*) + d^*(A^* \cap B^*)$$
is clear by monotonicity.

So we have

\[ d^*(A^*) + d^*(B^*) \approx d^*(A^*B^*) + d^*(A^* \cap B^*), \]
equivalently,

\[ d^*(A^*/B^*) \approx d^*(A^*/A^* \cap B^*). \]

(2 → 1): Fix any \( n \in \omega, A_0 \subset \text{fin} A, \) and \( B_0 \subset \text{fin} B. \) Let \( A^* \supset \text{hf} A \) and \( B^* \supset \text{hf} B \) be a witness of \( d^* \)-independent. By the finite closure condition, we can take \( A^* \) and \( B^* \) to be both closed. Then \( A^* \) and \( B^* \) satisfy the following formula:

- \( A_0 \subset \exists A^* \leq \text{fin} A, \) \( B_0 \subset \exists B^* \leq \text{fin} B, \) and
- \( d(A^*) + d(B^*) \leq d(A^*B^*) + d(A^* \cap B^*) + 1/n. \)

Because \( M \) is an elementary substructure of \( M^* \), we can take expected sets.

(2 → 3): Let \( A^* \) and \( B^* \) be witness of \( d^* \)-independence. Take any \( A' \subset \text{fin} A \) and any \( B' \subset \text{fin} B. \) Then \( d(A^*/B^*) \approx d(A^*/A^* \cap B^*). \) By transposition, \( d(B^*/A^*) \approx d(B^*/A^* \cap B^*). \) By monotonicity of \( d, \) \( d(B^*/A'A^* \cap B^*) \approx d(B^*/A^* \cap B^*). \) By transposition, \( d(A'/B^*) \approx d(A'/A^* \cap B^*). \) By Monotonicity, \( d(A'/B'A^* \cap B^*) \approx d(A'/A^* \cap B^*). \) By Lemma 9, \( d(A'/B'A \cap B) = d(A'/A \cap B). \)

(3 → 2): Take a closed hyperfinite extension \( A^* \) of \( A \) and a closed hyperfinite extension \( B^* \) of \( B. \) By compactness, it is enough to prove that for any \( A_0 \subset \text{fin} A, \) the following set of formulas are satisfiable:

1. \( X \in F \)
2. \( X \subseteq A \)
3. \( A_0 \subseteq X \)
4. \( X \) is closed
5. \( d(X/B^*) \approx d(X/X \cap B^*) \)

We show \( A_0 = A_0(A^* \cap B^*) \) is a realization of the above set of formulas. 1, 2, 3, and 4 are clear.

5. First,

\[
\begin{align*}
d(A_0^*/B^*) &= d(A_0^*B^*) - d(B^*) \\
&= d(A_0B^*) - d(B^*) \\
&= d(A_0/B^*) \\
&\approx d(A_0/B).
\end{align*}
\]
Second,
\[ d(A_0^*/A_0^* \cap B^*) = d(A_0^*) - d(A_0^* \cap B^*) \]
\[ = d(A_0(A^* \cap B^*)) - d(A_0^* \cap B^*) \]
\[ \leq d(A_0(A^* \cap B^*)) - d(A^* \cap B^*) \]
\[ = d(A_0/A^* \cap B^*) \]
\[ \approx d(A_0/A \cap B) \]

Finally, by the \(d\)-independence of \(A\) and \(B\), \(d(A_0/B) = d(A_0/A \cap B)\).
Hence, \(d(A_0^*/A_0^* \cap B^*) \leq d(A_0^*/B^*)\). The other direction is clear.

**Consequence**
DS is equivalent to the first condition of DW. In particular, DW is a stronger condition than DS.

**Fact 13** [3] Let \(T\) be stable. Then the following are equivalent:

1. \(T\) is superstable.
2. For any \(B \subset \mathcal{M}\) and \(p \in S(B)\), there is finite \(A \subset B\) such that \(p\) does not fork over \(A\).

So, we have the following corollary.

**Corollary 14** Suppose DS and that for any closed set \(A, B\), \(A \Downarrow_{A \cap B}^d B\) if and only if \(A \Downarrow_{A \cap B} B\). Then \(T = \text{Th}(M)\) is \(\omega\)-stable or merely stable.

This corollary is a partial solution of Baldwin's problem[1].

**References**


