数理解析研究所講究録
第 1602 巻 2008 年 22-25

A note on independence in generic structures

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Abstract

We show that if $\mathbf{K}$ is closed under quasi-substructures then $\mathrm{tp}(B/C)$ does not fork over $B \cap C$ if and only if B and C are free over $B \cap C$ and BC is closed for any closed $B, C \in \mathbf{K}$.

Our notations and definitions are standard (see [1], [5] for reference).

Let $L = \{R_0, R_1, ...\}$ be a countable relational language, where each R_i is symmetric and irreflexive, i.e., if $\models R_i(\bar{a})$ then the elements of $\bar{a}$ are without repetition and $\models R_i(\sigma(\bar{a}))$ for any permutation σ. Thus, for any L-structure A and R_i with arity n, R_i^A can be thought of as a set of n-element subsets of A.

For a finite L-structure A, a *predimension* of A is defined by $\delta(A) = |A| - \sum_i \alpha_i |R_i^A|$, where $0 < \alpha_i \leq 1$. Let $\delta(B/A)$ denote $\delta(BA) - \delta(A)$.

For $A \subset B$, A is *closed* in B (write $A \leq B$), if $\delta(X/A \cap X) \geq 0$ for any finite $X \subset B$. The *closure* A in B is defined by $\mathrm{cl}_B(A) = \bigcap\{C : A \leq C \leq B\}$.

Let $\mathbf{K}^*$ be the class of all finite L-structures A with $\delta(B) \geq 0$ for any $B \subset A$. Fix a subclass $\mathbf{K}$ of $\mathbf{K}^*$ that is closed under substructures. A countable L-structure M is $\mathbf{K}$-*generic*, if (i) any finite $A \subset M$ belongs to $\mathbf{K}$; (ii) for any $A \leq B \in \mathbf{K}$ with $A \leq M$ there is $B' \cong_A B$ with $B' \leq M$.

Let $\mathcal{M}$ be a big model of a $\mathbf{K}$-generic structure. Note that if $\mathbf{K}$-generic structure M is saturated then $\mathcal{M}$ also satisfies (i) and (ii). We abbreviate $\mathrm{cl}_{\mathcal{M}}(*)$ to $\mathrm{cl}(*)$. $\mathbf{K}$ has *finite closures*, if there is no chain $A_0 \subset A_1 \subset \cdots$ of $A_i \in \mathbf{K}$ with $\delta(A_{i+1}/A_i) < 0$ for each $i \in \omega$. Note that $\mathbf{K}$ has finite closures if and only if $\mathrm{cl}(A)$ is finite for any finite $A \subset \mathcal{M}$.

*Research partially supported by Grants-in-Aid for Scientific Research (no.19540150), Ministry of Education, Science and Culture.

For A, B, C with $B \cap C \subset A$, B and C are *free* over A (write $B \perp_A C$), if $R^{ABC} = R^{AB} \cup R^{AC}$ for every $R \in L$. Note that $B \perp_A C$ if and only if $\delta(\bar{b}/\bar{c}\bar{a}) = \delta(\bar{b}/\bar{a})$ for any $\bar{b} \in B - A, \bar{c} \in C - A$ and $\bar{a} \in A$.

Assumption L is a countable relational language. $\mathbf{K}$ is a class of finite L-structure A with $\delta(B) \geq 0$ for any $B \subset A$. Moreover $\mathbf{K}$ is closed under substructures and has finite closures. $\mathcal{M}$ is a big model of a saturated $\mathbf{K}$-generic structure.

Definition Let A and B be L-structures. Then A is said to be a *quasi-substructure* of B, if the universe of A is contained in that of B, and R^A is contained in R^B for every $R \in L$. If L is a language of a graph, then the notion of quasi-substructures coincides with that of subgraphs.

For $A, B \subset \mathcal{M}$, we denote $B^{\mathrm{Aut}(\mathcal{M}/A)} = \{\sigma(b) : b \in B, \sigma \in \mathrm{Aut}(\mathcal{M}/A)\}$.

Lemma 1 Let $B, C \leq \mathcal{M}$ with $A = B \cap C$. Then $B^{\mathrm{Aut}(\mathcal{M}/A)} = B$ implies $B \perp_A C$ and $BC \leq \mathcal{M}$.

Proof Since $\mathbf{K}$ is closed under quasi-substructures, there is $B' \cong_A B$ with $B' \perp_A C$ and $B'C \in \mathbf{K}$. By genericity, we can assume that $(B' \leq) B'C \leq \mathcal{M}$. So we have $\mathrm{tp}(B/A) = \mathrm{tp}(B'/A)$. By $B^{\mathrm{Aut}(\mathcal{M}/A)} = B$, we have $B = B'$ as a set. Hence $B \perp_A C$ and $BC \leq \mathcal{M}$.

For $A \leq B$, B is said to be *minimal* over A, if $C = A$ or $C = B$ for any $A \subset C \leq B$.

Lemma 2 Let $B, C \leq \mathcal{M}$ with $A = B \cap C$. If $\mathrm{tp}(B/A)$ is algebraic then $B \perp_A C$ and $BC \leq \mathcal{M}$.

Proof We can assume that B is minimal over A. Since $\mathrm{tp}(B/A)$ is algebraic, we can take a maximal set $(B =) B_1, , , , .B_n$ of conjugates of B over A satisfying $B_i \neq B_j$ as a set for each i, j with $1 \leq i < j \leq n$. By minimality, we have $B_i \cap B_j = A$.
Claim: $\perp \{B_i\}_i$ and $B_1 ... B_n \leq \mathcal{M}$.
Proof: Since $\mathbf{K}$ is closed under quasi-substructures, for each i there is a copy B_i' of B_i over A with $\perp \{B_i'\}_i$ and $(A \leq) B_1' ... B_n' \in \mathbf{K}$. We can assume that

$B'_1...B'_n \leq \mathcal{M}$. By maximality of n, we have $B_1...B_n = B'_1...B'_n$ as a set. Hence $\perp\{B_i\}_i$ and $B_1...B_n \leq \mathcal{M}$.

We devide into two cases.
Case: There is i with $B_i \subset C$. By claim, $BB_i \leq \mathcal{M}$. By induction hypothesis, we have $B \perp_{B_i} C$ and $BB_iC = BC \leq \mathcal{M}$. Again, by claim, $B \perp_A B_i$, and hence $B \perp_A C$.
Case: For any i, $B_i \not\subset C$. By minimality, we have $B_i \cap C = A$. Let $B^* = B_1...B_n$. Then $B^{*\mathrm{Aut}(\mathcal{M}/A)} = B^*$. By lemma, $B \perp_A B_i$, and hence $B \perp_A C$.

The following fact is due to Wagner [5]. Recently, Tsuboi [4] gave a short proof of this fact.

Fact 3 Let $B, C \leq \mathcal{M}$ with $A = B \cap C$ algebraically closed. Then $B \downarrow_A C$ iff $B \perp_A C$ and $BC \leq \mathcal{M}$.

The following theorem is a generalization of results obtained in [1] and [3].

Theorem Let $\mathbf{K}$ be closed under quasi-substructures. Let $B, C \leq \mathcal{M}$ with $A = B \cap C$. Then $B \downarrow_A C$ if and only if $B \perp_A C$ and $BC \leq \mathcal{M}$.

Proof ($\Rightarrow$) Suppose that $B \downarrow_A C$. First we show $B \perp_A C$. Let $B' = \mathrm{acl}(A) \cap B$ and $C' = \mathrm{acl}(A) \cap C$. By lemma 2, $B \cup \mathrm{acl}(A), C \cup \mathrm{acl}(A) \leq \mathcal{M}$. So, by fact 3, $B \perp_{\mathrm{acl}(A)} C$. By lemma 2, $B \perp_{B'} \mathrm{acl}(C)$. So $B \perp_{B'} C$. Again, by lemma 2, $B' \perp_A C$. Hence $B \perp_A C$. Next we show $BC \leq \mathcal{M}$. By lemma 2, $BC \cup \mathrm{acl}(A) \leq \mathcal{M}$. So it is enough to show that $BC \cap X \leq X$ for any finite $X \leq BC \cup \mathrm{acl}(A)$. For $D \subset \mathcal{M}$ let X_D denote $X \cap D$. Take any $\bar{e} \in X - X_B X_C$. By lemma 2, we have $B'C \leq \mathcal{M}$, and so $X_{B'} X_C \leq \mathcal{M}$. By lemma 2 and fact 3, we have $B \perp_{B'} \mathrm{acl}(A)$ and $B \perp_{\mathrm{acl}(A)} C$, and so $X_B \perp_{X_{B'}} \bar{e} X_{C'}$ and $X_B \perp_{\bar{e} X_{B'} X_{C'}} X_C$. Therefore

$$\begin{aligned} \delta(\bar{e}/X_B X_C) &= \delta(\bar{e}/X_{B'} X_C) + \delta(X_B/\bar{e} X_{B'} X_C) - \delta(X_B/X_{B'} X_C) \\ &\geq \delta(X_B/\bar{e} X_{B'} X_C) - \delta(X_B/X_{B'} X_C) \\ &= \delta(X_B/\bar{e} X_{B'} X_{C'}) - \delta(X_B/X_{B'} X_{C'}) \\ &= \delta(X_B/X_{B'} X_{C'}) - \delta(X_B/X_{B'} X_{C'}) = 0. \end{aligned}$$

Hence $X_B X_C \leq X$.
($\Leftarrow$) Suppose that $B \perp_A C$ and $BC \leq \mathcal{M}$. Take B' with $B' \downarrow_A C$ and $\mathrm{tp}(B'/A) =$

tp(B/A). By the only-if part of this theorem, we have $B' \perp_A C$ and $B'C \leq \mathcal{M}$. Thus we have tp$(B'/C) =$ tp(B/C) and hence $B \downarrow_A C$.

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