A SHORT PROOF OF NUBLING’S RESULT

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ABSTRACT. Nubling shows that CM-triviality (=non-2-ampleness) is preserved under reducts in finite U-rank theories. We give a short proof.

1. REDUCTION AND INDEPENDENCE

Let \( T^- \) be a reduct of \( T \). Let \( \mathcal{M} \models T \), \( \mathcal{M}^- \models T^- \) be big models. \( a, b, c, \ldots \overline{a}, \overline{b}, \overline{c}, \ldots \) denote finite tuples, and \( A, B, C, \ldots \) denote small sets. Let \( A \subset \mathcal{M}^eq \). ACL\(^{eq}\)(\( A \)) denotes the algebraically closure of \( A \) in \( T \), and acl\(^{eq}\)(\( A \)) denotes the algebraically closure of \( A \cap (\mathcal{M}^-)^eq \). Let \( \overline{a} \in (\mathcal{M}^-)^eq \). TP(\( \overline{a}/A \)) denotes the type of \( \overline{a} \) over \( A \) in \( T \), and \( \text{tp}(\overline{a}/A) \) denotes the type of \( \overline{a} \) over \( A \) in \( T^- \). SU denotes Lascar rank in \( T \), and su denotes Lascar rank in \( T^- \). We show the following fact in the last section.

**Fact 1.1.** Let \( T \) be a simple theory having EHI such that \( T^- \) also has EHI, where \( T^- \) be a reduct of \( T \). Let \( a, C \subset (\mathcal{M}^-)^eq \) and \( B \subset \mathcal{M}^eq \). If \( a \not\models B \), then \( a \not\models B^- \), where \( B^- = \text{ACL}^{eq}(B) \cap (\mathcal{M}^-)^eq \) and \( \models \) is the non-forking relation in \( T^- \).

**Proposition 1.2.** If \( \text{SU}(T) < \omega \), then \( \text{su}(T^-) < \omega \).

**Proof.** Let \( a \in (\mathcal{M}^-)^eq \), \( A \subset \mathcal{M}^eq \). Put \( A^- = \text{ACL}^{eq}(A) \cap \mathcal{M}^eq \). We will show that there exists \( a' \models \text{tp}(a/A^-) \) such that \( \text{SU}(a'/A) \geq \text{su}(a'/A^-) \) by induction on \( n = \text{su}(a/A^-) \).

If \( n = 0 \), it is clear. Let \( \text{su}(a/A^-) = n + 1 \). So, there exists \( A^- \subset B \subset (\mathcal{M}^-)^eq \) such that \( \text{su}(a/B) = n \). So, \( a \not\models B^- \). Put \( B^- = \text{ACL}^{eq}(B) \cap (\mathcal{M}^-)^eq \). So, we have \( A^- \subset B \subseteq B^- \). Take \( a_1 \models \text{tp}(a/B) \) such that \( \overline{a}_1 \not\models B^- \). As \( \text{su}(a_1/B^-) = n \), by induction hypothesis, there exists \( a'_1 \models \text{tp}(a_1/B^-) \) such that \( \text{SU}(a'_1/B) \geq \text{su}(a'_1/B^-) = n \). As \( \text{tp}(a'_1/A^-) = \text{tp}(a/A^-) \), we see \( a'_1 \not\models A^- \). As \( B^- \subseteq B \) and \( a \not\models B^- \), by Fact 1.1, we see \( a'_1 \not\models B \). Therefore we have \( \text{SU}(a'_1/A) \geq \text{SU}(a'_1/B) + 1 \geq \text{su}(a'_1/B^-) + 1 = n + 1 = \text{su}(a/A^-) = \text{su}(a'_1/A^-) \), as desired.

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Lemma 1.3. Suppose that $U(T) < \omega$. Let $T^-$ be a reduct of $T$. $u$ denotes the Lascar rank in $T^-$. (Then $u(T^-) < \omega$.) Let $a, b, c \in (M^-)^{eq}$ be algebraically independent in $T^-$ such that $u(a/b) = 1$ (So, $a \not\in b^-$, because $a \not\in acl^{eq}(bc)$.) Then there exist $a', b', c' \in M^{eq}$ such that $a', b', c'$ are algebraically independent in $T$, a realization of $tp(abc)$ with $a' \not\in b'c'$.

Proof. Let $a'b'c' \models tp(abc)$ be such that $U(a'b'c')$ is maximal.

Claim. $a', b', c'$ are algebraically independent in $T$.

As $a' \not\in acl^{eq}(b'c')$, we can find $a'' \models tp(a'/b'c')$ such that $a'' \not\in acl^{eq}(b'c')$. So, if $a' \in acl^{eq}(b'c')$, then $SU(a''b'c') > SU(a'b'c')$, a contradiction. Similarly, we see $b' \not\in acl^{eq}(a'c')$ and $c' \not\in acl^{eq}(a'b')$.

Claim. $a' \not\in b'c'$.

By way of contradiction, suppose that $a' \not\in b'c'$. Let $a_0' \models tp(a'/ACL^{eq}(b'))$ such that $a_0' \not\in b'c'$. As $1 = u(a'/b')$, $stp(a'/b') = stp(a'/b')$ and $a_0' \not\in acl^{eq}(b'c') \supset acl^{eq}(b'c')$, we see $1 = u(a_0'/b') \geq u(a_0'/b'c') \geq 1$. So we see $a_0' \not\in b'c'$. By STATIONARITY of strong types, we see $stp(a_0'/b'c') = stp(a'/b'c')$. In particular, $a_0'b'c' \models tp(a'b'c')$. Now, we have

\[
U(a_0'b'c') = U(a_0'/b'c') + U(b'c') = U(a_0'/b') + U(b'c') = U(a'/b') + U(b'c') > U(a'/b'c') + U(b'c') = U(a'b'c')
\]

\[\square\]

2. A short proof

We begin with basics of supersimple theories.

Fact 2.1. Let $T$ be a supersimple theory.

1. Let $a \in M^{eq}, A \subseteq M^{eq}$. Then there exists finite tuple $\bar{b} \subseteq M^{eq}$ such that $acl^{eq}(Cb(a/A)) = acl^{eq}(\bar{b}) = acl^{eq}(Cb(a/\bar{b}))$.

2. Let $A \subseteq M$ be finitely generated algebraically closed set, and $B = acl(B) \subseteq A$. Then $B$ is finitely generated algebraically closed.

3. Let $SU(T) < \omega$ and $p$ be a non-algebraic type. Then there exists a minimal type, non-orthogonal to $p$. (Coordination Theorem)

Proof. (1): Let $B = Cb(a/A)$. Take a finite tuple $\bar{b} \subseteq B \subseteq M^{eq}$ such that $a \not\in b. B$. Then $B = Cb(a/A) = Cb(a/\bar{b})$ and $acl^{eq}(\bar{b}) = acl^{eq}(B)$.

(2): By way of contradiction, suppose that there exist $C_0 \subseteq C_1 \subseteq \cdots C_n \subseteq \cdots B \subseteq A = acl(\bar{a})$, where $C_i$ are f.g. algebraically closed. Let $\bar{a}_n$ be such that $\bar{a}_n \equiv C_n \bar{a}$ and $\bar{a}_n \not\in C_n \bar{a}$. As $C_n \subseteq acl(\bar{a})$, we see that $C_n = acl(\bar{a}_n) \cap acl(\bar{a})$.****
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As $C_n \subset C_{n+1}$, so $\bar{a}_{n+1} \not\subset_{C_n} \bar{a}$. So, $\bar{a} \not\subset_{C_n} C_{n+1}$, because $\bar{a} \not\subset_{C_n} C_{n+1}$ and $\bar{a} \not\subset_{C_{n+1}} \bar{a}_{n+1}$ imply $\bar{a} \not\subset_{C_n} \bar{a}_{n+1}$. This contradicts supersimplicity.

(3): We may assume that $p = tp(a)$. Let $n = SU(p)$. Take $B$ such that $SU(a/B) = n - 1$. Let $b \in M^eq$ be such that $acl^eq(Cb(a/B)) = acl^eq(b) = acl^eq(Cb(a/b))$ by (1). As $a \not\subset b$, $b \not\in acl^eq(\emptyset)$. Take $C$ be such that $SU(b/C) = 1$. We may assume $C \not\subset b$. Then we have $a \not\subset_{C} b$, otherwise $Cb(a/bC) = Cb(a/b) \subset acl(C)$, so $b \in acl(C)$ would follow. On the other hand, as $n = SU(a) \geq SU(a/C) > SU(a/Cb) = SU(a/b) = n - 1$, we see $\bar{a} \not\subset C$.

□

Notation 2.2. $A \wedge B$ denotes $acl^eq(A) \cap acl^eq(B)$. $a \vdash A$ denotes $a \in acl^eq(A)$.

Definition 2.3. (1) We say that a sequence $(a_0, a_1, a_2)$ is 2-ample over $A$, if $a_0 A \wedge a_1 A = A$, $a_0 a_1 A \wedge a_0 a_1 A = A$, $a_2 \downarrow_{a_1 A} a_0$ and $a_2 \not\subset_{a_1 A} a_0$.

(2) We say that a sequence $(a_0, a_1, a_2)$ is weakly 2-ample over $A$, if $a_2 \not\subset_{a_1 A} a_0$ and $a_2 \not\subset_{a_1 A} a_0$.

(3) A complete simple theory $T$ with EHI is (weakly) 2-ample, if there exist (weakly) 2-ample sequence over some parameters.

Remark 2.4. (1) $T$ is 2-ample if and only if $T$ is weak 2-ample.

(2) If $(a_0, a_1, a_2)$ is weakly 2-ample, then so is $(a_2, a_1, a_0)$.

(3) If $(a_0, a_1, a_2)$ is weakly 2-ample, then $(a_0, a_1, a_2)$ are algebraically independent.

Proof. (1): Clearly, any 2-ample sequence is weakly 2-ample. Let $(a_0, a_1, a_2)$ be weakly 2-ample and let $a'_0$ be such that $acl^eq(a'_0) = a_0 a_1 \wedge a_0 a_2$. Then we have $a'_0 a_1 \wedge a'_0 a_2 = acl^eq(a'_0)$ and $a'_0 \wedge a_1 = a_1 \wedge a_0 a_2$. Then we see that $(a'_0, a_1, a_2)$ is 2-ample over $a_1 \wedge a_0 a_2$. (2): Clear. (3): If $a_0$ or $a_2$ were algebraic over $a_1$, then it would be algebraic over $a_1 \wedge a_0 a_2$. If $a_1$ were algebraic over $a_0 a_2$, then $acl^eq(a_1) = a_1 \wedge a_0 a_2$ would follow. As $a_2 \not\subset a_1 a_0$, we see $a_0, a_1, a_2$ are algebraically independent. □

From now on, we work in a finite SU-rank theory.

Lemma 2.5. Let $(a_0, a_1, a_2)$ be weakly 2-ample.

(1) There exist $a'_0$ and $B$ such that $a'_0 \vdash_{a_0 B} SU(a'_0/B) = 1$ and $(a_0, a_1, a_2)$ is weakly 2-ample over $B$.

(2) Fixing $a_1$, after adding some parameters, we can retake $a_0, a_2$ such that $SU(a_0/a_1) = SU(a_2/a_1) = 1$.

Proof. (1): By coordinatization theorem, there exist $a'_0$ and $B$ such that $a'_0 \not\subset_B a_0$, $SU(a'_0/B) = 1$ and $a_0 \not\subset B$. We may assume $a_1 a_2 \not\subset a_0 B a'_0$. Since $a_0 a_1 a_2 \not\subset B$, $(a_0, a_1, a_2)$ is weakly 2-ample over $B$, as desired.

(2): By remark 2.4 (2), we have only to retake $a_0$ such that $SU(a_0/a_1) = 1$.
Let $a_0$ be minimal of SU-rank such that $(a_0, a_1, a_2)$ is weakly 2-ample. Suppose that $SU(a_0/a_1) > 1$. By (1) take $a'_0$ such that $a'_0 \leftarrow a_0, SU(a'_0) = 1$. By Fact 2.1 (2), take $a$ be such that $acl^{eq}(a) = a_0 \land a'_0 a_1$. Then $SU(a_0) > SU(a), SU(a_0/a)$, because $SU(a_0) = SU(a_0/a) + SU(a)$ and $SU(a), SU(a_0/a) \geq 1$. (If $a_0 \leftarrow a$, then $a_0, a'_0$ are interalgebraic over $a_1$, a contradiction.) If $a_0 \not\subseteq a_0 \wedge a_0 a_2$, then $(a_0, a_1, a_2)$ is weakly 2-ample over $a$, which contradicts the minimality of $SU(a_1)$. If $a_0 \downarrow a_1 \wedge a_0 a_2$, then $a_0 \not\subseteq a_0 a_2$, so we see $(a, a_1, a_2)$ is weakly 2-ample over $a_1 \wedge a_0 a_2$, a contradiction.

Proposition 2.6. Let $(a_0, a_1, a_2)$ be weakly 2-ample. Then, after adding some parameters, we can retake $a_0, a_1, a_2$ such that

$$SU(a_0/a_1) = SU(a_2/a_1) = SU(a_1/a_0 a_2) = 1.$$  

Proof. By Lemma 2.5, take $a_0$ to be minimal of SU-rank such that $(a_0, a_1, a_2)$ is weakly 2-ample and $SU(a_0/a_1) = SU(a_2/a_1) = 1$. Suppose that $SU(a_1/a_0 a_2) > 1$. Take $a'_1 \leftarrow a_1$ be such that $SU(a'_1) = 1$ after possibly adding parameters. Let $a, b$ be such that $acl^{eq}(a) = a_0 a_1 \land a_0 a'_1 a_2$ and $acl^{eq}(b) = a \land a_1$. Then $SU(a_1) > SU(b), SU(a_1/b)$. (If $a_1 \leftarrow a_0 a'_1 a_2$, then $a_1, a'_1$ would be interalgebraic over $a_0 a_2$. So we see $SU(a_1/b) \geq 1$. Clearly $SU(b) \geq 1$. The above follows from $SU(a_1) = SU(a_1/b) + SU(b)$.) If $a \not\subseteq b a_2$, then $(a, a_1, a_2)$ is weakly 2-ample over $b$, because $b \subseteq (a_1 \land a_0 a_2) b \subseteq a_1 \land a_0 a'_1 a_2 = b$. As $a \leftarrow a_0 a_1$ and $b \leftarrow a_1$, we have $SU(a/a_1 b) = SU(a_2/a_1 b) = 1$. This contradicts the minimality of $SU(a_1)$. If $a \downarrow b a_2$, then $a_0 \downarrow b a_2$. Then $(a_0, b, a_2)$ is weakly 2-ample over $a_1 \land a_0 a_2$. By Lemma 2.5, we may assume $SU(a_0/b) = SU(a_2/b) = 1$. This also contradicts the minimality of $SU(a_1)$.  

Now, we prove the Nubling's theorem.

Theorem 2.7. Suppose that $U(T) < \omega$. If a reduct $T^-$ of $T$ is 2-ample, then so is $T$.

Proof. By Proposition 2.6, let $(a_0, a_1, a_2)$ be weakly 2-ample such that $u(a_0/a_1) = u(a_2/a_1) = u(a_1/a_0 a_2) = 1$. As $a_0, a_1, a_2$ are algebraically independent in $T^-$, by Lemma 1.3, there exist $abc \models tp(a_0 a_2 a_3)$ such that $a, b, c$ are algebraically independent in $T$ and $a \downarrow b c$.

Claim. $a \not\subseteq acl^{eq}(b) \cap acl^{eq}(ac)$. So, $(a, b, c)$ is weakly 2-ample.

Put $A = acl^{eq}(b) \cap acl^{eq}(ac)$, and $A^- = A \cap (M^-)^{eq}$. By way of contradiction, suppose that $a \downarrow_A c$. Then we have $a \downarrow_A c$ by Fact 1.1. As $a \not\subseteq acl^{eq}(b) = acl^{eq}(b a) \supseteq acl^{eq}(b A^-)$, we see $1 = u(a/b) \geq u(a/b A^-)$, so $a \downarrow A^-$ follows. Moreover, as $c \not\subseteq acl^{eq}(ab) = acl^{eq}(ab A) \supseteq acl^{eq}(a b A^-)$,
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We see $1 = u(c/b) \geq u(c/abA^{-}) \geq 1$, so $c \downarrow_{b}^{-}aA^{-}$ holds. So, we have $A^{-} \downarrow_{b}^{-}ac$. On the other hand, $b \not\in \text{ACL}^{eq}(ac) = \text{ACL}^{eq}(acA) \supseteq \text{acl}^{eq}(acA^{-})$, we see $1 = u(b/ac) \geq u(b/acA^{-}) \geq 1$, we have $b \downarrow_{ac}^{-}A^{-}$. So, we have $\text{Cb}(\text{tp}(A^{-}/abc)) \subseteq b \wedge_{b}^{-}ac := \text{acl}^{eq}(b) \cap \text{acl}^{eq}(ac), A^{-} \downarrow_{b}^{-}ac \text{abc holds. Since}$ $a \downarrow_{b}^{-}ac$ and $a \downarrow_{b}^{-}ac A^{-}$, so $a \downarrow_{b}^{-}ac$ $A^{-},$ $(a, b, c)$ is not weakly 2-ample in $T^{-}$, a contradiction. □

Remark 2.8. There is a modular O-minimal theory which has a non-CM-trivial reduct [Y]. Nubling theorem can not be extended to finite $U^{*}$-rank theories.

3. INDISCERNIBLE SEQUENCES AND THE PROOF OF FACT 1.1

We work in a complete theory and consider imaginary elements.
Let $(a_{i} : i \in I)$ be a sequence and $I_{0} \subseteq I$. $a_{i_{0}}$ denotes $(a_{i} : i \in I_{0})$. When $I$ is an partially ordered set, $a_{<i}$ denotes $(a_{j} : j < i)$. Similarly for $a_{>i}$. We write $I_{0} < I_{1}$, if $I_{0}, I_{1} \subseteq I$ and $i_{1} < i_{2}$ holds for any $i_{1} \in I_{1}, i_{2} \in I_{2}$.

Definition 3.1. Let $X = (a_{i} : i \in I)$ be a B-indiscernible sequence and $A \subseteq B$.

(1) Put $\ker_{A}(X) := \bigcup_{|I_{0}|=|I_{1}|=|J_{0}|=|J_{1}|}(\text{acl}^{eq}(a_{I_{0}}) \cap \text{acl}^{eq}(a_{J_{0}}))$. We call it the kernel of $X$ over $A$.

(2) We say that $X$ is algebraically independent over $A$, if $\text{acl}^{eq}(Aa_{I_{0}}) \cap \text{acl}^{eq}(A_{I_{1}}) = \text{acl}^{eq}(A)$ for any $I_{0} < I_{1} \subseteq I$.

Lemma 3.2. Let $X = (a_{i} : i \in I)$ be a B-indiscernible sequence.

(1) For infinite subsets $I_{1} < I_{2}$, $\ker_{B}(X) = \text{acl}^{eq}(a_{I_{1}}) \cap \text{acl}^{eq}(a_{I_{2}})$.

(2) ker$_{B}(X)$ is the smallest algebraically closed set (containing $B$) over which $X$ is algebraically independent.

(3) $X$ is indiscernible over ker$_{B}(X)$.

(4) ker$_{B}(X)$ is the biggest subset (containing $B$) of acl$_{eq}(XB)$ over which $X$ is indiscernible.

Proof. For ease of notation, we assume $B = \emptyset$.

(1): Suppose that $I_{0}, I_{1}, J$ are finite with the same size and $I_{0}, I_{1} < J$. As $a_{I_{0}} \equiv_{\text{acl}^{eq}(a_{J})} a_{I_{1}}$, we see

$$\text{acl}^{eq}(a_{I_{0}}) \cap \text{acl}^{eq}(a_{J}) = \text{acl}^{eq}(a_{I_{1}}) \cap \text{acl}^{eq}(a_{J}).$$

By the same argument, we see that

$$\text{acl}^{eq}(a_{I_{0}}) \cap \text{acl}^{eq}(a_{J_{0}}) = \text{acl}^{eq}(a_{I_{1}}) \cap \text{acl}^{eq}(a_{J_{1}}).$$

for any $I_{0} < J_{0}, I_{1} < J_{1}, |I_{0}| = |I_{1}| = |J_{0}| = |J_{1}|$. Therefore, we see ker$(X) \subseteq \text{acl}^{eq}(a_{I_{1}}) \cap \text{acl}^{eq}(a_{I_{2}})$ for any infinite $I_{1} < I_{2}$. We show the converse inclusion. Let $a \in \text{acl}^{eq}(a_{I_{1}}) \cap \text{acl}^{eq}(a_{I_{2}})$. Then there exist $J_{1} \subset I_{1}, J_{2} \subset I_{2}$ such that
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$|J_1| = |J_2| < \omega$ such that $a \in \text{acl}^{eq}(a_{J_1}) \cap \text{acl}^{eq}(a_{J_2})$. By the above argument, we see $\text{acl}^{eq}(a_{J_1}) \cap \text{acl}^{eq}(a_{J_2}) \subseteq \ker(X)$.

(2): Let $C$ be such that $X$ is algebraically independent over $C$. Then, for any infinite $I_0 < J_0$, $\ker(X) = \text{acl}^{eq}(a_{J_1}) \cap \text{acl}^{eq}(a_{J_2}) \subseteq \text{acl}^{eq}(Ca_{I_0}) \cap \text{acl}^{eq}(Ca_{J_0}) = \text{acl}^{eq}(C)$, as desired.

(3): By (1), we see that if $X'$ is an extended indiscernible sequence of $X$, then $\ker(X) = \ker(X')$. It suffices to show that, if $I_0, J_0$ are finite sets with the same size, then $a_{I_0} \equiv_{\ker(X)} a_{J_0}$. Take an infinite set $J \subseteq I$ such that $I_0, J_0 < J$, if necessarily, extend $X$. As $a_{I_0} \equiv_{\text{acl}^{eq}(a_{J_1})} a_{J_0}$, we see the conclusion.

(4): Let $C \subseteq \text{acl}(X)$ be such that $X$ is indiscernible over $C$. Let $c \in C$. Then there exists a finite $I_1$ such that $c \in \text{acl}^{eq}(a_{I_1})$. For any $I_0 < I_1, |I_0| = |I_1|$, we have $c \in \text{acl}^{eq}(a_{I_0}) \cap \text{acl}^{eq}(a_{I_1})$, since $a_{I_0} \equiv c a_{I_1}$. Now, we see that $C \subseteq \ker(X)$. \hfill \Box

From now on, we work in a simple theory $T$ with EHI.

**Lemma 3.3.** Let $X = (a_i : i \in I)$ be a $B$-indiscernible sequence and $A \subseteq B$.

1. If $X$ is sufficiently long and independent over $A$, then $X \downarrow_{A} B$.
2. If $X$ is sufficiently long, then $\text{Cb}(B/(a_i : i \in I)A) \subseteq \ker_{A}(X)$.
3. If $X$ is a Morley sequence over $B$, then $\ker_{A}(X) \subseteq \text{acl}(B)$.

**Proof.**

(1): By simplicity, take $B_0 \subseteq a_{<|T|^{+}}$ such that $B \downarrow_{B_0} a_{<|T|^{+}}$ and $|B_0| \leq |T|$. So there exists $\lambda < |T|^{+}$ such that $B_0 \subseteq a_{<\lambda}$. We have $a_{<|T|^{+}} \downarrow_{a_{<\lambda}} B$. By $B$-indiscernibility of $X$, we have $a_{\geq \lambda} \downarrow_{a_{<\lambda}} B$. So, $a_{\geq \lambda} \downarrow_{a_{<\lambda}} A$. As $X$ is independent over $A$, $a_{\geq \lambda} \downarrow_{A} a_{<\lambda} B$ follows. By $A$-independence of $X$ again, we see the conclusion.

(2): Let $I_0 \subseteq I$ be such that $|I_0| = |T|^{+}$. Then there exists $B_0 \subseteq a_{I_0}$ such that $B \downarrow_{B_0} a_{I_0}$ and $|B_0| \leq |T|$. As there exists $\lambda \in I_0$ such that $B_0 \subseteq a_{<\lambda}$, we see $a_{I_0} \downarrow_{a_{<\lambda}} B$. By $B$-indiscernibility and finite character, we have $a_{\geq \lambda} \downarrow_{a_{<\lambda}} B$. Therefore we have $a_{I} \downarrow_{a_{I_0}A} B$. As we assume EHI, $\text{Cb}(B/Aa_{I}) \subseteq \text{acl}(a_{I_0}A)$.

Let $I_1$ be such that $I_0 < I_1$ and $|I_1| = |T|^{+}$. By the same argument, we see $\text{Cb}(B/Aa_{I_1}) \subseteq \text{acl}(a_{I_0}A) \cap \text{acl}(a_{I_1}A) = \ker_{A}(X)$.

(3): By our statement, we may assume $X$ is sufficiently long. By 3.2 (4), we have $\ker_{A}(X)B \subseteq \ker_{B}(X)$. So, $X$ is $\ker_{A}(X)B$-indiscernible and independent over $B$. By (1), we see $X \downarrow_{B} \ker_{A}(X)$. Since $\ker_{A}(X) \subseteq \text{acl}(BX)$, we see $\ker_{A}(X) \subseteq \text{acl}(B)$. \hfill \Box

**Proposition 3.4.** Let $X = (a_i : i \in I)$ be an $A$-indiscernible sequence.

1. If $X$ is algebraically independent over $A$, then $X$ is a Morley sequence over $A$.
2. If $X$ is a Morley sequence over $A$, then $\ker(X) = \text{acl}(\text{Cb}(a_{0}/A))$. 

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Proof. (1): By our assumption, we may assume $X$ is sufficiently long. Let $a_{\infty}$ be such that $a_{I}, a_{\infty}$ is an extended $A$-indiscernible sequence, algebraically independent over $A$. As $X$ is $Aa_{\infty}$-indiscernible and algebraically independent over $A$, by Lemma 3.3 (1), $\text{Cb}(Aa_{\infty}/AX) \subseteq \ker_{A}(X) = \text{acl}^{eq}(A)$. Therefore $a_{\infty} \downarrow_{A} a_{I}$. By $A$-indiscernibility of $a_{I}a_{\infty}$, we see $X$ is independent over $A$.

(2): As $X$ is algebraically independent over $\text{Cb}(a_{0}/A)$, we see $\ker(X) \subseteq \text{acl}^{eq}(\text{Cb}(a_{0}/A))$ by Lemma 3.2 (2). By Lemma 3.3 (3), $\ker(X) \subseteq \text{acl}^{eq}(A)$. As $X$ is $\ker(X)$-indiscernible and algebraically independent over $\ker(X)$, $X$ is a Morley sequence over $\ker(X)$ by (1). Now, by Lemma 3.3 (1), we have $X \downarrow_{\ker(X)} A$. In particular, $a_{0} \downarrow_{\ker(X)} A$ holds. So, we see $\text{Cb}(a_{0}/A) \subseteq \text{acl}^{eq}(\ker(X))$.

FACT 1.1: Let $T$ be a simple theory having EHI such that $T^{-}$ also has EHI, where $T^{-}$ be a reduct of $T$. Let $a, C \subset (\mathcal{M}^{-})^{eq}$ and $B \subset \mathcal{M}^{eq}$. If $a \downarrow_{B} C$, then $a \downarrow_{B^{-}} C$, where $B^{-} = \text{ACL}^{eq}(B) \cap (\mathcal{M}^{-})^{eq}$ and $\downarrow_{-}$ is the non-forking relation in $T^{-}$.

Proof. Let $X = (a_{i} : i \in \mathbb{Z})$ be a Morley sequence of $\text{TP}(a/BC)$. Then, by Proposition 3.4 (2) and our assumption, we have $\text{ACL}^{eq}(a_{<0}) \cap \text{ACL}^{eq}(a_{>0}) = \ker(X) = \text{ACL}^{eq}(\text{Cb}(a/BC)) \subseteq \text{ACL}^{eq}(B)$. So, $\text{acl}^{eq}(a_{<0}) \cap \text{acl}^{eq}(a_{>0}) \subseteq B^{-}$. As $X$ is algebraically independent over $BC$, so is over $B^{-}C$ in $T^{-}$. Since $X$ is $B^{-}C$-indiscernible in $T^{-}$, by Proposition 3.4 (1), $X$ is a Morley sequence of $\text{tp}(a/B^{-}C)$. By $\ker^{-}(X) \subseteq B^{-}$ and Proposition 3.4 (2), we see $a \downarrow_{B^{-}} C$. □

REFERENCES


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