数理解析研究所講究録
第 1602 巻 2008 年 33-36

Independence in generic structures

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Abstract

Wagner [W] proved that in generic structures forking independence and independence defined by dimension function are essentially the same. He proved the result under the assumption that the closure of a finite set is also finite. Verbovskiy and Yoneda [VY] provided some notions for studying generic structures without this finiteness condition and eliminated the finiteness assumption from the result. Here we give a very short proof of the result.

1 Introduction

Let $L = \{R_i : i \in \omega\}$ and for each $i \in \omega$ let $\alpha_i > 0$ be given. δ is the function assigning to each finite L-structure the value $|A| - \sum \alpha_i |R_i^A|$. Let K be the class of all finite L-structures A such that $\delta(A_0) \geq 0$ for every substructure A_0 of A. K_0 is a subclass of K and M is a stable structure all of whose finite substructures belong to K_0. $\mathcal{M}$ is a big model of $T = Th(M)$. The following proposition is proved by Wagner [W] under the finite closure assumption. Later Verbovskiy and Yoneda [VY] eliminated the finiteness assumption from the result. Here we give a direct proof. We do not assume the finiteness condition.

Proposition 1 *Let B, C be closed sets in $\mathcal{M}$. Suppose that $A = B \cap C$ is algebraically closed. Suppose also that B and C are independent over A. Then (1) B and C are free over A and (2) BC is closed.*

In section 1, we recall some definitions and state basic lemmas on generic structures. In section 2, we prove the above proposition by a straightforwad method. We assume that the reader has some knowledge of stability theory. In particular, the reader is supposed to know the notion Morley sequence.

2 Preliminaries

Definition 2 1. Let $A \subset B \in K$. We say that A is closed in B (in symbol $A \leq B$) if whenever $X \subset B - A$ then $\delta(X/A)(= \delta(XA) - \delta(A)) \geq 0$.

2. Let $A \subset N$, where $N \models T$.

   (a) We say that A is closed in N if whenever B is a finite subset of N then $A \cap B \leq B$.

   (b) The closure of A (in N) is the minimum closed set containing A. (The closure always exists.) The closre of A is written as $cl(A)$.

Lemma 3 *For every A, $cl(A) \subset \mathrm{acl}(A)$.*

Proof. Let $N \prec \mathcal{M}$ be a small model with $N \supset A$ and choose the closure C of A in N. Then, by $N \prec \mathcal{M}$, C is the closure of A in $\mathcal{M}$. Suppos that there is $c \in C$ which is nonalgebraic over A. Then we can choose an element $d \in \mathcal{M} - N$ with $\mathrm{tp}(c/A) = \mathrm{tp}(d/A)$. Let σ be an A-automorphism sending c to d. Then we would have two different closures C and $\sigma(C)$. A contradiction.

Lemma 4 *Let $A \subset B_0 \leq B_1$ and $A \subset C_0 \leq C_1$. Suppose that B_1 and C_1 are free over A. If B_1C_1 is closed then B_0C_0 is also closed.*

Proof. We assume B_1C_1 is closed. Let $X \subset \mathcal{M} - B_0C_0$ be a finite set and put $X_B = X \cap B_1$, $X_C = X \cap C_1$ and $\hat{X} = X - B_1C_1$. Then we have the following inequalities:

$$\begin{aligned} \delta(X/B_0C_0) &= \delta(\hat{X}/B_0C_0X_BX_C) + \delta(X_BX_C/B_0C_0) \\ &\geq \delta(\hat{X}/B_1C_1) + \delta(X_BX_C/B_0C_0) \\ &\geq \delta(X_BX_C/B_0C_0) \\ &= \delta(X_B/X_CB_0C_0) + \delta(X_B/B_0C_0). \end{aligned}$$

By the freeness and $B_0 \leq B_1$, $\delta(X_B/X_CB_0C_0) = \delta(X_B/B_0) \geq 0$. Similarly, $\delta(X_B/B_0C_0) \geq 0$. So we have $\delta(X/B_0C_0) \geq 0$.

3 Proof of the Proposition

Let $B' = \mathrm{acl}(B)$ and $C' = \mathrm{acl}(C)$. If we prove $B'C' = B' \otimes_A C' \leq \mathcal{M}$, then $BC = B \otimes_A C \leq \mathcal{M}$ follows from lemma. So we can assume that B and C are algebraically closed. By $B \perp_A C$, we can choose sequences $\{B_i : i \in \omega\}$ and $\{C_i : i \in \omega\}$ satisfying the following conditions:

1. $\{B_i : i \in \omega\}$ is a Morley sequence of $\mathrm{tp}(B/A)$;
2. $\{C_i : i \in \omega\}$ is a Morley sequence of $\mathrm{tp}(C/A)$;
3. $\{B_i : i \in \omega\}$ and $\{C_i : i \in \omega\}$ are independent over A, so the set $\{B_i : i \in \omega\} \cup \{C_i : i \in \omega\}$ is an independent set over A.
4. $\mathrm{tp}(B_iC_j/A) = \mathrm{tp}(BC/A)$, for any $i, j \in \omega$.

Such sequences can be found by using an easy compactness argument.

(1) Freeness: By way of a contradiction, we assume there are tuples $\emptyset \neq \bar{b} \in B - A$, $\emptyset \neq \bar{c} \in C - A$ and $\bar{a} \in A$ with $R_i(\bar{b}, \bar{c}, \bar{a})$. By condition 4, we can find $\bar{b}_i \in B$ and $\bar{c}_i \in C_i$ such that for any $i, j \in \omega$, $\mathrm{tp}(\bar{b}_i\bar{c}_j\bar{a}) = \mathrm{tp}(\bar{b}\bar{c}\bar{a})$. So $R(\bar{b}_i, \bar{c}_j, \bar{a})$ holds for any $(i, j) \in \omega^2$. We fix $n \in \omega$. Then we have the following inequality:

$$\delta(\textstyle\bigcup_{i<n} \bar{b}_i\bar{c}_i\bar{a}) \leq n|\bar{b}\bar{c}\bar{a}| - \alpha_i n^2 \ .$$

This right value is negative for a sufficiently large n. A contradiction.

(2) Suppose that BC is not closed and choose finite tuples $\bar{d} \in \mathrm{acl}(BC) - BC$, $\bar{b} \in B$ and $\bar{c} \in C$ with $\varepsilon := \delta(\bar{d}/\bar{b}\bar{c}) < 0$.

By condition 4 above, for all $i, j \in \omega$, we can choose $\bar{b}_i \in B_i$, $\bar{c}_i \in C_i$ and $\bar{d}_{ij}$ such that $\mathrm{tp}(\bar{b}\bar{c}\bar{d}BC) = \mathrm{tp}(\bar{b}_i\bar{c}_i\bar{d}_{ij}B_iC_j)$.

Claim A $(\bigcup_{(i,j)\in\omega^2} d_{ij}) \cap (\bigcup_{i\in\omega} B_iC_i) = \emptyset$

Suppose otherwise and choose i, j, m and $e \in \bar{d}_{ij} \cap (B_mC_m)$. By symmetry, we may assume $e \in B_m$. So we have $e \in \mathrm{acl}(B_iC_j) \cap B_m$. By choice of $\bar{d}$ (and $\bar{d}_{ij}$), $m \neq i$. So, from $B_iC_j \perp_A B_m$, we have $e \in \mathrm{acl}(A) = A$. So we must have $\bar{d}_{ij} \cap A \neq \emptyset$, a contradiction.

Claim B *$\bar{d}_{ij}$'s are disjoint.*

By way of a contradiction, we assume $e \in \bar{d}_{ij} \cap \bar{d}_{i'j'}$ for some pair $(i,j) \neq (i',j')$. First assume $\{i,j\} \cap \{i',j'\} = \emptyset$. Then, by the independence of B_iC_j and $B_{i'}C_{j'}$ over A, we have $e \in A$, so we have $\bar{d}_{ij} \cap A \neq \emptyset$, a contradiction. Then, since other cases are similar, we can assume $i = i'$ and $j \neq j'$. In this case, we have $e \in \mathrm{acl}B_i = B_i$. Again, this is a contradiction.

So, as in (1), we have

$$\begin{aligned} \delta(\bigcup_{(i,j)\in n^2} \bar{d}_{(i,j)} \cup \bigcup_{i<n} \bar{b}_i\bar{c}_i) &\leq \delta(\bigcup_{(i,j)\in n^2} \bar{d}_{(i,j)} / \bigcup_{i<n} \bar{b}_i\bar{c}_i) + \delta(\bigcup_{i<n} \bar{b}_i\bar{c}_i) \\ &\leq n^2\varepsilon + n\delta(\bar{b}_0\bar{c}_0). \end{aligned}$$

For a sufficiently large n, we get a contradiction.

Remark 5 1. In our proof of Proposition 1, we did not use the "genericity" of the structure M. If we assume the "genericity", the converse of Proposition 1 is true by the following argument. Suppose that $BC = B \otimes_A C \leq \mathcal{M}$. Let $\{C_i : i < \alpha\}$ be a sufficiently long Morley sequence of $\mathrm{tp}(C/A)$. Then, by stability, there is i such that B and C_i are independent over A. By proposition $BC_i = B \otimes_A C_i \leq \mathcal{M}$. Then we have $BC \cong_A BC_i$ and that they are closed. So they have the same type over A, hence $BC = B \otimes_A C \leq \mathcal{M}$. (For details see [W] or [VY].)

2. The assumption that A is algebraically closed is necessary in general. But Ikeda [I] showed that the algebraicity assumption can be eliminated if ($L = \{R(*,*)\}$ and) K_0 is closed under subgraphs.

References.

[K] Koichiro Ikeda, Algebraic types of generic graphs, RIMS meeting, November 2006.

[VY] Viktor Verbovskiy and Ikuo Yoneda, CM-triviality and relational structures, Annals of Pure and Applied Logic 122 (2003) pp. 175–194.

[W] Frank O. Wagner, Relational structures and dimensions, in Automorphism of First-Order Structures, The Clarendon Press, Oxford University Press, New York 1994, pp.153–180.