Independence in generic structures

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Abstract

Wagner [W] proved that in generic structures forking independence and independence defined by dimension function are essentially the same. He proved the result under the assumption that the closure of a finite set is also finite. Verbovskiy and Yoneda [VY] provided some notions for studying generic structures without this finiteness condition and eliminated the finiteness assumption from the result. Here we give a very short proof of the result.

1 Introduction

Let $L = \{R_i : i \in \omega\}$ and for each $i \in \omega$ let $\alpha_i > 0$ be given. $\delta$ is the function assigning to each finite $L$-structure the value $|A| - \sum \alpha_i|R_i^A|$. Let $K$ be the class of all finite $L$-structures $A$ such that $\delta(A_0) \geq 0$ for every substructure $A_0$ of $A$. $K_0$ is a subclass of $K$ and $M$ is a stable structure all of whose finite substructures belong to $K_0$. $\mathcal{M}$ is a big model of $T = Th(M)$. The following proposition is proved by Wagner [W] under the finite closure assumption. Later Verbovskiy and Yoneda [VY] eliminated the finiteness assumption from the result. Here we give a direct proof. We do not assume the finiteness condition.

Proposition 1 Let $B, C$ be closed sets in $\mathcal{M}$. Suppose that $A = B \cap C$ is algebraically closed. Suppose also that $B$ and $C$ are independent over $A$. Then (1) $B$ and $C$ are free over $A$ and (2) $BC$ is closed.

In section 1, we recall some definitions and state basic lemmas on generic structures. In section 2, we prove the above proposition by a straightforward method. We assume that the reader has some knowledge of stability theory. In particular, the reader is supposed to know the notion Morley sequence.
2 Preliminaries

Definition 2 1. Let $A \subset B \in K$. We say that $A$ is closed in $B$ (in symbol $A \leq B$) if whenever $X \subset B - A$ then $\delta(X/A)(= \delta(XA) - \delta(A)) \geq 0$.

2. Let $A \subset N$, where $N \models T$.

(a) We say that $A$ is closed in $N$ if whenever $B$ is a finite subset of $N$ then $A \cap B \leq B$.

(b) The closure of $A$ (in $N$) is the minimum closed set containing $A$. (The closure always exists.) The closure of $A$ is written as $cl(A)$.

Lemma 3 For every $A$, $cl(A) \subset acl(A)$.

Proof. Let $N \prec \mathcal{M}$ be a small model with $N \supset A$ and choose the closure $C$ of $A$ in $N$. Then, by $N \prec \mathcal{M}$, $C$ is the closure of $A$ in $\mathcal{M}$. Suppose that there is $c \in C$ which is nonalgebraic over $A$. Then we can choose an element $d \in \mathcal{M} - N$ with $tp(c/A) = tp(d/A)$. Let $\sigma$ be an $A$-automorphism sending $c$ to $d$. Then we would have two different closures $C$ and $\sigma(C)$. A contradiction.

Lemma 4 Let $A \subset B_0 \leq B_1$ and $A \subset C_0 \leq C_1$. Suppose that $B_1$ and $C_1$ are free over $A$. If $B_1C_1$ is closed then $B_0C_0$ is also closed.

Proof. We assume $B_1C_1$ is closed. Let $X \subset \mathcal{M} - B_0C_0$ be a finite set and put $X_B = X \cap B_1$, $X_C = X \cap C_1$ and $\hat{X} = X - B_1C_1$. Then we have the following inequalities:

$$\delta(X/B_0C_0) = \delta(\hat{X}/B_0C_0X_BX_C) + \delta(X_BX_C/B_0C_0) \geq \delta(\hat{X}/B_0C_1) + \delta(X_BX_C/B_0C_0) \geq \delta(X_BX_C/B_0C_0) = \delta(X_B/X_CB_0C_0) + \delta(X_B/B_0C_0).$$

By the freeness and $B_0 \leq B_1$, $\delta(X_B/X_CB_0C_0) = \delta(X_B/B_0) \geq 0$. Similarly, $\delta(X_B/B_0C_0) \geq 0$. So we have $\delta(X/B_0C_0) \geq 0$. 

3 Proof of the Proposition

Let $B' = \text{acl}(B)$ and $C' = \text{acl}(C)$. If we prove $B'C' = B' \otimes_A C' \leq \mathcal{M}$, then $BC \subseteq B \otimes_A C \leq \mathcal{M}$ follows from lemma. So we can assume that $B$ and $C$ are algebraically closed. By $B \perp_A C$, we can choose sequences $\{B_i : i \in \omega\}$ and $\{C_i : i \in \omega\}$ satisfying the following conditions:

1. $\{B_i : i \in \omega\}$ is a Morley sequence of $\text{tp}(B/A)$;
2. $\{C_i : i \in \omega\}$ is a Morley sequence of $\text{tp}(C/A)$;
3. $\{B_i : i \in \omega\}$ and $\{C_i : i \in \omega\}$ are independent over $A$, so the set $\{B_i : i \in \omega\} \cup \{C_i : i \in \omega\}$ is an independent set over $A$.
4. $\text{tp}(B_i C_j/A) = \text{tp}(BC/A)$, for any $i, j \in \omega$.

Such sequences can be found by using an easy compactness argument.

(1) Freeness: By way of a contradiction, we assume there are tuples $\emptyset \neq \overline{b} \in B - A$, $\emptyset \neq \overline{c} \in C - A$ and $\overline{a} \in A$ with $R_i(\overline{b}, \overline{c}, \overline{a})$. By condition 4, we can find $\overline{b}_i \in B$ and $\overline{c}_i \in C_i$ such that for any $i, j \in \omega$, $\text{tp}(\overline{b}_i \overline{c}_j \overline{a}) = \text{tp}(\overline{b} \overline{c} \overline{a})$. So $R(\overline{b}_i, \overline{c}_j, \overline{a})$ holds for any $(i, j) \in \omega^2$. We fix $n \in \omega$. Then we have the following inequality:

$$\delta(\bigcup_{i<n} \overline{b}_i \overline{c}_i \overline{a}) \leq n|\overline{b} \overline{c} \overline{a}| - \alpha_i n^2 .$$

This right value is negative for a sufficiently large $n$. A contradiction.

(2) Suppose that $BC$ is not closed and choose finite tuples $\overline{d} \in \text{acl}(BC) - BC$, $\overline{b} \in B$ and $\overline{c} \in C$ with $\varepsilon := \delta(\overline{d}/\overline{b} \overline{c}) < 0$.

By condition 4 above, for all $i, j \in \omega$, we can choose $\overline{b}_i \in B_i$, $\overline{c}_i \in C_i$ and $\overline{d}_{ij}$ such that $\text{tp}(\overline{b} \overline{c} \overline{d} BC) = \text{tp}(\overline{b}_i \overline{c}_i \overline{d}_{ij} B_i C_j)$.

Claim A $(\bigcup_{(i,j) \in \omega^2} \overline{d}_{ij}) \cap (\bigcup_{i \in \omega} B_i C_i) = \emptyset$

Suppose otherwise and choose $i, j, m$ and $e \in \overline{d}_{ij} \cap (B_mC_m)$. By symmetry, we may assume $e \in B_m$. So we have $e \in \text{acl}(B_mC_j) \cap B_m$. By choice of $\overline{d}$ (and $\overline{d}_{ij}$), $m \neq i$. So, from $B_mC_j \perp_A B_m$, we have $e \notin \text{acl}(A) = A$. So we must have $\overline{d}_{ij} \cap A \neq \emptyset$, a contradiction.

Claim B $\overline{d}_{ij}$'s are disjoint.
By way of a contradiction, we assume \( e \in \overline{d}_{ij} \cap \overline{d}_{i'j'} \) for some pair \((i, j) \neq (i', j')\). First assume \( \{i, j\} \cap \{i', j'\} = \emptyset \). Then, by the independence of \( B_iC_j \) and \( B_{i'}C_{j'} \) over \( A \), we have \( e \in A \), so we have \( \overline{d}_{ij} \cap A \neq \emptyset \), a contradiction. Then, since other cases are similar, we can assume \( i = i' \) and \( j \neq j' \). In this case, we have \( e \in \text{acl} B_i = B_i \). Again, this is a contradiction.

So, as in (1), we have

\[
\delta(\bigcup_{(i,j) \in \mathcal{V}} \overline{d}_{(i,j)} \cup \bigcup_{i<n} \overline{b}_i \overline{c}_i) \
\leq \
\delta(\bigcup_{(i,j) \in \mathcal{V}} \overline{d}_{(i,j)} / \bigcup_{i<n} \overline{b}_i \overline{c}_i) + \delta(\bigcup_{i<n} \overline{b}_i \overline{c}_i) \
\leq \
n^2 \varepsilon + n \delta(\overline{b}_0 \overline{c}_0).
\]

For a sufficiently large \( n \), we get a contradiction.

**Remark 5**

1. In our proof of Proposition 1, we did not use the "genericity" of the structure \( M \). If we assume the "genericity", the converse of Proposition 1 is true by the following argument. Suppose that \( BC = B \otimes_A C \leq \mathcal{M} \). Let \( \{C_i : i < \alpha\} \) be a sufficiently long Morley sequence of \( \text{tp}(C/A) \). Then, by stability, there is \( i \) such that \( B \) and \( C_i \) are independent over \( A \). By proposition \( BC_i = B \otimes_A C_i \leq \mathcal{M} \). Then we have \( BC \equiv_A BC_i \) and that they are closed. So they have the same type over \( A \), hence \( BC = B \otimes_A C \leq \mathcal{M} \). (For details see [W] or [VY].)

2. The assumption that \( A \) is algebraically closed is necessary in general. But Ikeda [I] showed that the algebraicity assumption can be eliminated if \( (L = \{R(*, *)\}) \) and \( K_0 \) is closed under subgraphs.

**References.**

