数理解析研究所講究録
第 1604 巻 2008 年 104-109

セルオートマトンの局所構造
Local Structures of CA: Equivalence, Isomorphism and Similarity

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Abstract

Of the definition of a cellular automaton, we focus on the local function and the neighborhood (f,ν) called the *local structure* and consider three kinds of the *sameness* of CA: equivalence, isomorphism and similarity. After giving general descriptions of equivalence, isomorphism and similarity, classification of CA is discussed in the context of changing the neighborhood.

1 Definitions

1.1 Cellular Automaton CA (S,Q,f,ν)

A cellular automaton (CA for short) is defined by a 4-tuple (S,Q,f,ν):

- S: a discrete cellular space such as $\mathbb{Z}^d$, hyperbolic space ...
- Q: a finite set of the states of each cell.
- $f : Q^n \to Q$: a local function in $n \geq 1$ variables.
- ν: an injective map from $\{1,...,n\}$ to S, called a *neighborhood function*, which connects the i-th variable of f to $\nu(i)$. That is, $(\nu(1),...,\nu(n))$ becomes a neighborhood of size n.

In order to study effects of changing the neighborhood (function), we define the pair (f,ν) as a *local structure* of CA and investigate its mathematical properties.

1.2 Local Structure (f,ν)

In this paper we assume that S is a d-dimensional Euclidean grid $\mathbb{Z}^d$ $(d \geq 1)$ with the additive operator $+$.

Definition 1 *[neighborhood]*

For $n \in \mathbb{N}$, a neighborhood (function) is a mapping $\nu : \mathbb{N}_n \to \mathbb{Z}^d$, where $\mathbb{N}_n = \{1,2,\dots,n\}$.

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This can equivalently be seen as a list ν with n components; $(\nu_1, \ldots, \nu_n)$, where $\nu_i = \nu(i), 1 \leq i \leq n$.

The set of all neighborhoods of size n will be denoted as $\mathcal{N}_n$.

Definition 2 *[local structure, reduced]*

A pair (f, ν) of a local function $f : Q^n \to Q$ and a neighborhood $\nu \in \mathcal{N}_n$ is called a local structure. We call n the arity of the local structure.

A local structure is called reduced, if and only if the following conditions are fulfilled:

- *f depends on all arguments.*
- *ν is injective, i.e. $\nu_i \neq \nu_j$ for $i \neq j$ in the list of neighborhood ν.*

Each local structure induces the *global function* $F : Q^{\mathbb{Z}^d} \to Q^{\mathbb{Z}^d}$ or the dynamics of CA. Every element $c \in Q^{\mathbb{Z}^d}$ is called a *(global) configuration.* For any global configuration $c \in Q^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$, let $c(x)$ be the state of cell x in c. Then F is given by $F(c)(x) = f(c(x + \nu_1), c(x + \nu_2), ..., c(x + \nu_n))$.

1.3 Equivalence

Definition 3 *[equivalence]*

Two local structures (f, ν) and (f', ν') are called equivalent, if and only if they induce the same global function. In that case we sometimes write $(f, \nu) \approx (f', \nu')$.

Lemma 1
For each local structure (f, ν) there is an equivalent reduced local structure (f', ν').

Note that for a local structure, the equivalent reduced local structure is *not unique*. As a simple example consider the local function $f(x_1, x_2)$ over $GF(2) : (x_1, x_2) \mapsto x_1 + x_2$ (mod.2). Since the order of the arguments x_i does not matter for the value $f(x_1, x_2)$, the local structures $(f, (0, 1))$ and $(f, (1, 0))$ are equivalent. At the same time both are obviously reduced.

1.4 Permutation of Local Structure

Definition 4 *[permutation of local structure]*
Let π denote a permutation of the numbers in $\mathbb{N}_n$.

- *For a neighborhood ν, denote by ν^π the neighborhood defined by $\nu^\pi_{\pi(i)} = \nu_i$.*
- *For an n-tuple $\ell \in Q^n$, denote by ℓ^π the permutation of ℓ such that $\ell^\pi(i) = \ell(\pi(i))$ for $1 \leq i \leq n$.*

  For a local function $f : Q^n \to Q$, denote by f^π the local function $f^\pi : Q^n \to Q$ such that $f^\pi(\ell) = f(\ell^\pi)$ for all ℓ.

In the first part of the definition we have preferred the given specification to the equally possible $\nu^\pi_i = \nu_{\pi(i)}$, because the former leads to a slightly simpler formulation of the following lemma.

2 Results

Some basic properties of the equivalence of local structures are given without proofs, for which we refer to [1].

Lemma 2
(f,ν) and (f^{π},ν^{π}) are equivalent for any permutation π.

We are now going to show that for *reduced* local structures, the relationship *via a permutation* is the *only* possibility to get equivalence.

Lemma 3
If (f,ν) and (f',ν') are two reduced local structures which are equivalent, then there is a permutation π such that $\nu^{\pi}=\nu'$.

Lemma 4
If (f,ν) and (f',ν') are two reduced local structures which are equivalent, then there is a permutation π such that $(f^{\pi},\nu^{\pi})=(f',\nu')$.

By choosing different neighborhoods which are not permutations of each other, one immediately gets the following corollary, which claims the same thing as Theorem 1 of H.Nishio, MCU2007 [2]: *By changing the neighborhood function ν, infinitely many different global CA functions are induced by any single local function $f_3(x,y,z)$ which is not constant.* Proof was given for 1-dimensional CA by concretely showing biinfinite words which correspond to different neighborhoods.

Corollary 1
For each reduced non-constant local function f, there are infinitely many reduced neighborhoods ν, such that the local structures (f,ν) induce pairwise different global CA functions.

3 Isomorphism and similarity

There could be several definitions of the *sameness* of a certain mathematical object like the local structure. In the above sections, we defined and investigated the most naive notion of *equivalence* as such. In the following, based upon those results on the equivalence, we will introduce two other notions of *isomorphism and similarity*, which are *weaker* notions than equivalence.

3.1 Isomorphism

In the same space S, consider two CA A and B having different local structures (f_A,ν_A) and (f_B,ν_B), where f_A and f_B are defined on possibly different domains; $f: Q_A^n \rightarrow Q_A$ and $f_B: Q_B^{n'} \rightarrow Q_B$.

Definition 5
If $|Q_A|=|Q_B|$, then we can consider a bijection $\varphi: Q_A \rightarrow Q_B$. Two CA A and B are called isomorphic under φ denoted by $A \underset{\varphi}{\sim} B$, if and only if the following diagram commutes for all global configurations.

Note that bijection φ naturally extends to $\varphi: Q_A^{\mathbb{Z}^d} \rightarrow Q_B^{\mathbb{Z}^d}$.

$$\begin{array}{ccc} c_A & \xrightarrow{\varphi} & c_B \\ {\scriptstyle F_A}\downarrow & & \downarrow{\scriptstyle F_B} \\ c'_A & \xrightarrow{\varphi} & c'_B \end{array} \qquad (1)$$

where c_A (c_B) is a global configuration of A (B) and c'_A (c'_B) is the next configuration of $c_A(c_B)$.

Both equivalence and isomorphism of local structures are evidently equivalence relations.

From the definitions of equivalence and isomorphism among local structures, we have

Lemma 5
If $(f_A, \nu_A) \approx (f_B, \nu_B)$, *then* $(f_A, \nu_A) \underset{\varphi}{\sim} (f_B, \nu_B)$ *for any* φ. *The converse is not always true.*

Example: We consider 6 Elementary CA which are show to be reversible in page 436 of [3]. Rules 15, 51 and 85 are equivalent (and isomorphic) each other. Rules 170, 204 and 240 are equivalent (and isomorphic). However, rules 15 and 240 (resp. 51 and 204, 85 and 170) are *not equivalent but isomorphic* under $\varphi : 0 \mapsto 1, 1 \mapsto 0$. Summing up those 6 Elementary reversible CA are all isomorphic.

For the isomorphism too, the following lemma is proved in the same manner as Lemma 3.

Lemma 6 (Lemma 3')
If (f_A, ν_A) *and* (f_B, ν_B) *are two reduced local structures which are* φ*-isomorphic under a bijection* $\varphi : Q_A \to Q_B$, *then there is a permutation* π *such that* $\nu_A^\pi = \nu_B$.

Proof:
Assume that there is an x which does not appear in ν_B but does appear in ν_A, say at position i. Since (f_A, ν_A) is reduced, f_A does depend on its i-th argument and there are two configurations c_A and $\overline{c_A}$, which do *only* differ at cell x, such that $F(c_A)(0) \neq F(\overline{c_A})(0)$.

Since ν_B does not contain x, clearly $F_B(\varphi(c_A))(0) = F_B(\overline{\varphi(c_A)})(0)$. It is therefore impossible that $F_A(c_A)(0) = F_B(\varphi(c_A))(0)$ and simultaneously $F_A(\overline{c_A})(0) = F_B(\overline{\varphi(c_A)})(0)$. Hence $F_A(c_A) \neq F_B(\varphi(c_A))$ and $F_A \neq F_B$. ■

3.2 Similarity

Let $\sigma \subset S$ be a finite subset of S such that $|\sigma| = n$ and $0 \in \sigma$, i.e. $\sigma = \{0, j_2, j_3, \ldots, j_n\}, j_i \in S$. If a neighborhood ν consists of the members from σ, then we say that σ is a support of ν and write support$(\nu) = \sigma$. For instance, if $\nu = (-1, 0, 1)$, then $\sigma(\nu) = \{0, -1, 1\}$. Permutation of a neighborhood does not change the support. That is, if $\nu = (-1, 0, 1)$, then $\sigma(\nu^{\pi_i}) = \{0, -1, 1\}, 0 \leq i \leq 5$.

Definition 6 *We say that* f *and* f' *are similar on* σ, *if and only if there are neighborhoods* ν *and* ν' *and an isomorphism* φ *such that* $(f, \nu) \underset{\varphi}{\sim} (f', \nu')$. *If* f *and* f' *are similar on some support* σ, *then we call* f *and* f' *similar and write* $f \asymp f'$.

The relation $\asymp$ is obviously an equivalence relation and the following lemma holds.

Lemma 7 *If* (f, ν) *and* (f', ν') *are equivalent or isomorphic, then* f *and* f' *are similar.*

4 Classification of CA

Classification is a typical problem in the CA study and there are several stand points of classification. For example, CA are classified by the *complexity of dynamical behavior* — fixed points, limit cycles, chaotic and so on, see Chapter 8 of [4] for old references and Wolfram's classification of ECA into four classes according the *complexity of global dynamics* [3].

On the other hand, L. Chua et. al. [5] as well as Guan et.al. [6] focuses on certain geometrical symmetries of the unit cubes corresponding to local functions of ECA and defines the *global equivalence* which classifies 256 ECA into 88 classes. For instance, it classifies rule 15 and 85 as *globally equivalent* but not 51. This is because our isomorphism considers *all* permutations of neighborhoods, while their global equivalence does not.

Note that those past papers assume a standard neighborhood like von Neumann neighborhood and classify the local functions. We will investigate, however, the classification problem from a different point of view — by changing the neighborhood. As discussed above, without loss of generality we shall restrict ourselves to the *reduced* local structures.

The decision problems of equivalence, isomorphism and similarity of local structures are evidently decidable.

4.1 Classification by similarity

We will only state two unsolved problems concerning with similarity. Let denote the set of all local functions of arity $n \geq 1$ by $\mathcal{F}_n$.

Problems 1: Classify $\mathcal{F}_n$ with respect to $\asymp$. Specifically, classify the set of all (256) ELF with respect to $\asymp$. How many classes are there?

Problems 2: Suppose that f and f' are similar on σ. Is there any subset $\sigma' \neq \sigma$ such that f and f' are similar on σ'? Are there local functions f and f' such that they are similar on every subset of S?

4.2 Classification of reversible, injective and surjective CA

First we consider reversible 6 Elementary CA, see page 436 of [3].

	000	001	010	011	100	101	110	111
R15	1	1	1	1	0	0	0	0
R51	1	1	0	0	1	1	0	0
R85	1	0	1	0	1	0	1	0
R170	0	1	0	1	0	1	0	1
R204	0	0	1	1	0	0	1	1
R240	0	0	0	0	1	1	1	1

Rules 15, 51 and 85 are equivalent (and isomorphic) each other. Rules 170, 204 and 240 are equivalent (and isomorphic). However, rules 15 and 240 (resp. 51 and 204, 85 and 170) are *not equivalent but isomorphic* under $\varphi : 0 \mapsto 1, 1 \mapsto 0$. Summing up, all reversible ECA are isomorphic.

If $|Q| = 3$, reversibility is not preserved from changing the neighborhood. See Proposition 9. in [2].

Characterization or classification of local structures which induce not injective and surjective CA is an interesting problem, but we have not yet obtained definite results. As a computational example, we can tell something about rule 30 and 6 permutations of $\pi_0 = (-1, 0, 1)$, see below. By using a Java program *catest*, we see that 6 local structures $\{(30, \pi_i),\ i = 0, .., 5\}$ are all not injective, while $(30, \pi_0)$ and other 3 ones are surjective but $(30, \pi_1)$ and $(30, \pi_3)$ are not.

6 permutations of $(-1, 0, 1)$

$$\pi_0 = (-1,0,1), \pi_1 = (-1,1,0), \pi_2 = (0,-1,1),$$
$$\pi_3 = (0,1,-1), \pi_4 = (1,-1,0), \pi_5 = (1,0,-1).$$

5 Concluding remarks

In this paper we considered three notions of the *sameness* of local structures of CA. Classification of CA was discussed from the point of view of changing the neighborhood. One of other possible definitions of similarity will be *strong similarity*: If f and f' are similar on *every* support, then we call f and f' *strongly similar*. Strong similarity is *stronger* than equivalence, i.e. there could be equivalent functions f and f' on some support, which are not equivalent on other supports. Then we have a problem: Are there nontrivial strongly similar local structures?

References

[1] Nishio, H., Worsch, T.: Local strucure of cellular automata. Technical Report (kokyuroku) vol. 1599, RIMS, Kyoto University (June 2008)

[2] Nishio, H.: Changing the neighborhood of cellular automata. In: Proceedings of MCU2007, eds. J. Durand-Lose and M. Margenstern. Volume LNCS 4664. (2007) 255–266

[3] Wolfram, S.: A New Kind of Science. Wolfram Media, Inc. (2002)

[4] Garzon, M.: Models of Massive Parallelism, Analysis of Cellular Automata and Neural Networks. Springer (1995)

[5] Chua, L., Sbitnev, V., Yoon, S.: A nonlinear dynmics perspective of Wolfram's new kind of science. Part III: Predicting the unpredictable. Int. Journal of Bifurcation and Chaos **14** (2004) 3689–3820

[6] Guan, J., Shen, S., Tang, C., Chen, F.: Extending Chua's global equivalence theorem on Wolfram's New Kind of Science. Int. Journal of Bifurcation and Chaos **17** (2007) 4245 – 4259

manuscript for RIMS kôkyûroku (May 19, 2008)