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A note on Schmitt–Vogel lemma

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INTRODUCTION

Let S be a polynomial ring over an infinite field k, and I a squarefree monomial ideal of S. The *arithmetical rank* of I is defined by

$$\operatorname{ara} I := \min\left\{r \;:\; \text{there exist } a_1, \ldots, a_r \in I \text{ such that } \sqrt{(a_1, \ldots, a_r)} = \sqrt{I}\right\}.$$

For ideals $J \subset I \subset S$, J is said to be a *reduction* of I if there exists some $s \in \mathbb{N}$ such that

$$I^{s+1} = JI^s.$$

Note that when this is the case, $\sqrt{J} = \sqrt{I}$ holds. The *analytic spread* of I is defined by

$$l(I) := \min\{\mu(J) \;:\; J \text{ is a reduction of } I\},$$

where $\mu(J)$ denotes the minimal number of generators of J. The existence of the minimal reduction shows $\operatorname{ara} I \leq l(I)$. On the other hand, it is known by Lyubeznik [3] that $\operatorname{pd}_S S/I \leq \operatorname{ara} I$, where $\operatorname{pd}_S S/I$ denotes the projective dimension of S/I. Therefore we have the following inequalities:

$$\operatorname{pd}_S S/I \leq \operatorname{ara} I \leq l(I).$$

In the study of the arithmetical rank, Schmitt–Vogel lemma [5, Lemma, pp. 249] is an important and useful tool, because it gives a sufficient condition for ideals $J \subset I$ to hold $\sqrt{J} = \sqrt{I}$. In this report, we give a sufficient condition for an ideal J with $J \subset I$ to be a reduction of I by refining Schmitt–Vogel lemma. As an application of our theorem, we prove $l(I) = \operatorname{pd}_S S/I$ for the ideal

$$I = (x_{11}, \ldots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \ldots, x_{qi_q}),$$

where $x_{11}, \ldots, x_{qi_q}$ are variables in S pairwise distinct. Schmitt and Vogel [5] proved $\operatorname{ara} I = \operatorname{pd}_S S/I$ for this ideal I using their lemma.

1. Main Theorem

In this section, we consider an arbitrary commutative ring R with unitary. Our main result of this report is the following:

Theorem 1.1. *Let R be a commutative ring with unitary. Let $P_0, P_1, \ldots, P_r \subset R$ be finite subsets, and we set*

$$P = \bigcup_{\ell=0}^{r} P_\ell,$$
$$g_\ell = \sum_{a \in P_\ell} a, \quad \ell = 0, 1, \ldots, r.$$

Assume that

(C1) $\sharp P_0 = 1$.
(C2) *For all $\ell > 0$ and $a, a'' \in P_\ell$ ($a \neq a''$), there exist some ℓ' ($0 \leq \ell' < \ell$), $a' \in P_{\ell'}$, and $b \in (P)$ such that $aa'' = a'b$.*

Then we have $(g_0, g_1, \ldots, g_r)$ is a reduction of (P).

On the other hand, Schmitt–Vogel lemma is the following:

Proposition 1.2 (Schmitt–Vogel [5, Lemma, pp. 249]). *Let R be a commutative ring with unitary. Let $P_0, P_1, \ldots, P_r \subset R$ be finite subsets, and we set*

$$P = \bigcup_{\ell=0}^{r} P_\ell,$$
$$g_\ell = \sum_{a \in P_\ell} a, \quad \ell = 0, 1, \ldots, r.$$

Assume that

(C1) $\sharp P_0 = 1$.
(C2)' *For all $\ell > 0$ and $a, a'' \in P_\ell$ ($a \neq a''$), there exist some ℓ' ($0 \leq \ell' < \ell$) and $a' \in P_{\ell'}$ such that $aa'' \in (a')$.*

Then we have $\sqrt{(g_0, g_1, \ldots, g_r)} = \sqrt{(P)}$.

Second condition of Theorem 1.1 is stronger than that of Schmitt–Vogel lemma, but Theorem 1.1 has a stronger conclusion than Schmitt–Vogel lemma.

Remark 1.3. Schmitt–Vogel lemma allows us to add some exponent $e(a)$ for each $a \in P_\ell$ in the sum g_ℓ, i.e., we may put

$$g_\ell = \sum_{a \in P_\ell} a^{e(a)}.$$

In particular, we can take g_ℓ as homogeneous if R is graded. But a similar statement does not hold for our theorem.

Instead of proving Theorem 1.1, we will give a detailed explanation of an example in Section 3, which illustrates the outline of the proof of the theorem. See also [2].

2. An application

In this section, we apply Theorem 1.1 to some ideals and calculate the analytic spread of them.

Consider the ideal

$$I = (x_{11}, \dots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \dots, x_{qi_q}), \tag{2.1}$$

where $x_{11}, \dots, x_{qi_q}$ are variables in S pairwise distinct.

Lemma 2.1. *For the above ideal I,*

$$\operatorname{pd}_S S/I = \sum_{s=1}^{q} i_s - q + 1.$$

Proof. For an integer $q \geq 1$, we set

$$I_q = (x_{11}, \dots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \dots, x_{qi_q}),$$

$$r_q = \sum_{s=1}^{q} i_s - q + 1.$$

We prove the lemma by induction on q. The case $q = 1$ is clear. Suppose that $q \geq 2$. If we put $P = (x_{q1}, \dots, x_{qi_q})$, then $I_q = I_{q-1} \cap P$ and $r_q = r_{q-1} + \operatorname{height} P - 1 = r_{q-1} + \operatorname{pd}_S S/P - 1$. Consider Mayer–Vietoris sequence

$$0 \to S/I_q \to S/I_{q-1} \oplus S/P \to S/(I_{q-1} + P) \to 0.$$

Since $\operatorname{pd}_S S/I = \max\{i : \operatorname{Tor}_i^S(k, S/I) \neq 0\}$, the long exact sequence

$$\cdots \to 0 = \operatorname{Tor}^S_{r_q+1}(k, S/I_{q-1}) \oplus \operatorname{Tor}^S_{r_q+1}(k, S/P) \to \operatorname{Tor}^S_{r_q+1}(k, S/(I_{q-1} + P))$$
$$\to \operatorname{Tor}^S_{r_q}(k, S/I_q) \to \operatorname{Tor}^S_{r_q}(k, S/I_{q-1}) \oplus \operatorname{Tor}^S_{r_q}(k, S/P) = 0 \to \cdots$$

implies $r_q = \operatorname{pd}_S S/I_q$. □

Schmitt–Vogel [5] proved $\operatorname{ara} I = \operatorname{pd}_S S/I$ (see also Schenzel–Vogel [4]). They proved it by applying Schmitt–Vogel lemma to

$$P_\ell = \{x_{1\ell_1} x_{2\ell_2} \cdots x_{q\ell_q} \ : \ \ell_1 + \cdots + \ell_q = \ell + q\}, \qquad \ell = 0, 1, \dots, r,$$

where $r = \sum_{s=1}^{q} i_s - q$. These $P_0, P_1, \dots, P_r$ also satisfy the assumption of Theorem 1.1. Thus $J = (g_0, g_1, \dots, g_r)$ is a reduction of I. Since

$$r + 1 = \operatorname{pd}_S S/I = \operatorname{ara} I \leq l(I) \leq r + 1,$$

we have $l(I) = \operatorname{pd}_S S/I$. Therefore we have the following corollary:

Corollary 2.2. *Let $I = (x_{11}, \dots, x_{1i_1}) \cap \cdots \cap (x_{q1}, \dots, x_{qi_q})$. Then we have*

$$l(I) = \operatorname{pd}_S S/I.$$

In particular, $(g_0, g_1, \dots, g_r)$ is a minimal reduction of I.

Note that we have a minimal reduction of I explicitly.

Remark 2.3. In general, $l(I) \neq \operatorname{pd}_S S/I$ for a squarefree monomial ideal I. For example, if $\mu(I) - \operatorname{height}(I) = 1$ and S/I is Cohen–Macaulay, then $\operatorname{height}(I) = \operatorname{pd}_S S/I = \operatorname{ara} I < l(I) = \mu(I)$; see [1].

3. An example

In this section, we give one example to illustrate the outline of the proof of Theorem 1.1.

Let us consider the ideal

$$I = (x_1, x_2, x_3) \cap (y_1, y_2, y_3).$$

This is a special form of the ideal (2.1). The minimal graded resolution of S/I is

$$0 \to S(-6) \to S(-5)^6 \to S(-4)^{15} \to S(-3)^{18} \to S(-2)^9 \to S \to S/I \to 0$$

and $\mathrm{pd}_S\, S/I (= 3 + 3 - 2 + 1) = 5$. Then

$$\begin{aligned} P_0 &= \{x_1y_1\}, \\ P_1 &= \{x_1y_2, x_2y_1\}, \\ P_2 &= \{x_1y_3, x_2y_2, x_3y_1\}, \\ P_3 &= \{x_2y_3, x_3y_2\}, \\ P_4 &= \{x_3y_3\}. \end{aligned}$$

Let us see conditions of Theorem 1.1. Since $\sharp P_0 = 1$, (C1) is satisfied. For the assumption (C2), we have the following equations:

$$P_1 : \quad x_1y_2 \cdot x_2y_1 = x_1y_1 \cdot x_2y_2 \in (P_0)(P_2), \tag{3.1}$$

$$P_2 : \begin{cases} x_1y_3 \cdot x_2y_2 = x_1y_2 \cdot x_2y_3 \in (P_1)(P_3), \\ x_1y_3 \cdot x_3y_1 = x_1y_1 \cdot x_3y_3 \in (P_0)(P_4), \\ x_2y_2 \cdot x_3y_1 = x_2y_1 \cdot x_3y_2 \in (P_1)(P_3), \end{cases} \tag{3.2}$$

$$P_3 : \quad x_2y_3 \cdot x_3y_2 = x_2y_2 \cdot x_3y_3 \in (P_2)(P_4). \tag{3.3}$$

Thus (C2) is also satisfied.

Now we shall see $J = (g_0, g_1, g_2, g_3, g_4)$ is a reduction of I, where

$$\begin{aligned} g_0 &= x_1y_1, \\ g_1 &= x_1y_2 + x_2y_1, \\ g_2 &= x_1y_3 + x_2y_2 + x_3y_1, \\ g_3 &= x_2y_3 + x_3y_2, \\ g_4 &= x_3y_3. \end{aligned}$$

We put

$$I_\ell = \left(\bigcup_{j=0}^{\ell} P_j \right), \qquad \ell = 0, 1, 2, 3, 4.$$

Note that $I_4 = I$. It is enough to show

$$I_\ell^{2^\ell} \subset J I^{2^\ell - 1}, \qquad \ell = 0, 1, 2, 3, 4$$

in order to see that J is a reduction of I. We show this by induction on ℓ. In fact, we show

$$I_\ell^{2^\ell} \subset I_{\ell-1}^{2^{\ell-1}} I^{2^\ell - 2^{\ell-1}} + J I^{2^\ell - 1}, \qquad \ell = 0, 1, 2, 3, 4.$$

Step 1: The case $\ell = 0$. In this case, $I_0 = (P_0) = (x_1y_1) = (g_0) \subset J$.

Step 2: The case $\ell = 1$. We want to show $I_1^2 \subset I_0 I + JI$. To see this, it is enough to show that $a_1a_2 \in I_0I + JI$ for all $a_1, a_2 \in P_1$ (we do not assume $a_1 \neq a_2$). When $a_1 \neq a_2$, (3.1) shows $a_1a_2 \in I_0I$. When $a_1 = a_2 = a$, we use g_1. For example,

$$(x_1y_2)^2 = (g_1 - x_2y_1)x_1y_2 = g_1x_1y_2 - x_2y_1 \cdot x_1y_2 = g_1x_1y_2 - x_1y_1 \cdot x_2y_2 \in JI + I_0I.$$

Therefore $I_1^2 \subset I_0I + JI$ holds.

Step 3: The case $\ell = 2$. We want to show $I_2^4 \subset I_1^2I^2 + JI^3$. To see this, we only check $a_1a_2a_3a_4 \in I_1^2I^2 + JI^3$ for all $a_1, a_2, a_3, a_4 \in P_0 \cup P_1 \cup P_2$. There are two cases:

(i) $a_1, a_2, a_3, a_4 \in P_2$;
(ii) for some i, $a_i \in P_0 \cup P_1$.

In case (i), there are two cases dividing large. The first one is that $a_1 \neq a_2$ and $a_3 \neq a_4$ by renumbering a_1, a_2, a_3, a_4. In this case, it is easy to check $a_1a_2a_3a_4 \in I_1^2I^2$ because of (3.2). For example,

$$(x_1y_3)^2x_2y_2 \cdot x_3y_1 = (x_1y_3 \cdot x_2y_2)(x_1y_3 \cdot x_3y_1) = (x_1y_2 \cdot x_1y_1)(x_2y_3 \cdot x_3y_3) \in I_1^2I^2.$$

The second one is that there are no such a renumbering on a_1, a_2, a_3, a_4. In this case, we use g_2 as in the case $\ell = 1$. For example,

$$\begin{aligned}(x_1y_3)^3x_2y_2 &= (x_1y_3)^2(g_2 - x_2y_2 - x_3y_1)x_2y_2 \\ &= g_2(x_1y_3)^2x_2y_2 - (x_1y_3)^2(x_2y_2)^2 - (x_1y_3)^2x_3y_1 \cdot x_2y_2 \\ &= g_2(x_1y_3)^2x_2y_2 - (x_1y_3 \cdot x_2y_2)^2 - (x_1y_3 \cdot x_3y_1)(x_1y_3 \cdot x_2y_2) \\ &\in JI^3 + I_1^2I^2.\end{aligned}$$

In case (ii), if there are two indices i (say, i_1, i_2) such that $a_i \in P_0 \cup P_1$, then $a_{i_1}a_{i_2} \in I_1^2$ and $a_1a_2a_3a_4 \in I_1^2I^2$ hold. Next, we consider the case that there is only one i such that $a_i \in P_0 \cup P_1$. We may assume $a_1 \in P_0 \cup P_1$ and $a_2, a_3, a_4 \in P_2$. Then we need to make only one pair of distinct elements from a_2, a_3, a_4. It is weaker requirement than that of case (i). In fact, to make one pair of distinct elements, we only need two of a_2, a_3, a_4. For example,

$$\begin{aligned}(x_1y_3)^2 &= x_1y_3(g_2 - x_2y_2 - x_3y_1) \\ &= g_2x_1y_3 - x_1y_3 \cdot x_2y_2 - x_1y_3 \cdot x_3y_1 \\ &\in JI + I_1I.\end{aligned}$$

Step 4: The case $\ell = 3$. We want to show $I_3^8 \subset I_2^4I^4 + JI^7$. In this case, the same argument as in Step 3 is also usable. We omit here.

Step 5: The case $\ell = 4$. It is clear that $I_4^{16} \subset I_3^8I^8 + JI^{15}$ since $\sharp P_4 = 1$. Therefore we obtain that J is a reduction of I.

Remark 3.1. The *reduction number* $r_J(I)$ is defined by

$$r_J(I) := \min\{s : I^{s+1} = JI^s\}.$$

Above argument gives an upper bound of $r_J(I)$. But this is very big in general. In fact, in the above argument, we only see $I^{2^4} = JI^{2^4-1}$, that is, $r_J(I) \leq 2^4 - 1 = 15$. But $r_J(I) = 3$, i.e., $I^4 = JI^3$ holds.

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