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The dimension of a toric variety obtained from a numerical semigroup [1]

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Abstract

For a numerical semigroup H we define the toric dimension of H, which is denoted by $\mathrm{Tdim}\ H$. In the case where H is generated by three elements or H is a symmetric semigroup generated by four elements we calculate the range of $\mathrm{Tdim}\ H$. Moreover, we determine the range of $\mathrm{Tdim}\ H$ when H is a non-symmetric n-semigroup with $n = 4$ or 5 generated by four elements.

1 The toric dimension of a numerical semigroup

Let $\mathbb{N}_0$ be the additive semigroup of non-negative integers. A *numerical semigroup* H means a subsemigroup of $\mathbb{N}_0$ whose complement $\mathbb{N}_0 \backslash H$ is a finite set. We call the cardinality $\sharp(\mathbb{N}_0 \backslash H)$ the *genus* of H, which we denote by $g(H)$. Let $a_1, \ldots, a_m \in \mathbb{N}_0$. The semigroup generatd by $a_1, \ldots, a_m$ is denoted by $\langle a_1, \ldots, a_m \rangle$. We denote by $M(H) = \{a_1, a_2, \ldots, a_n\}$ the minimum set of generators for H. We set $c(H) = \min\{c \in \mathbb{N}_0 | c + \mathbb{N}_0 \subseteqq H\}$, which is called the *conductor* of H. We know that $c(H) \leqq 2g(H)$. A numerical semigroup H said to be *symmetric* if $c(H) = 2g(H)$.

Example 1.1 Let H be a numerical semigroup with $M(H) = \{a_1, a_2\}$. Then

$$g(H) = \frac{(a_1 - 1)(a_2 - 1)}{2} \text{ and } c(H) = (a_1 - 1)(a_2 - 1) = 2g(H).$$

Hence, every numerical semigroup generated by two elements is symmetric.

[1] This paper is an extended abstract and the details will appear elsewhere.

Example 1.2 Let $H = \langle 3,4,5 \rangle = \{0, 3 \longrightarrow\}$ where "$\rightarrow$" denotes the consecutive integers. Then we have $g(H) = 2$ and $c(H) = 3$. Hence $H = \langle 3,4,5 \rangle$ is non-symmetric.

In this paper k denotes an algebraically closed field of characteristic 0. Let $M(H) = \{a_1, a_2, \ldots, a_n\}$. We define a k-algebra homomorphism

$$\varphi_H : k[X_1, \ldots, X_n] \longrightarrow k[H] = k[\ t^h\ ]_{h \in H}$$

sending X_i to t^{a_i}. It is important to study about the ideal Ker φ_H for investigating a relation between H and an affine toric variety.

Example 1.3 Let H be a numerical semigroup with $M(H) = \{a_1, a_2\}$. Then Ker $\varphi_H = (X_1^{a_2} - X_2^{a_1})$.

Example 1.4 Let H be a numerical semigroup with $M(H) = \{a_1 = 3, a_2 = 4, a_3 = 5\}$. Then Ker $\varphi_H = (X_1^3 - X_2X_3, X_2^2 - X_1X_3, X_3^2 - X_1^2X_2)$.

Example 1.5 Let H be a numerical semigroup with $M(H) = \{a_1 = 4, a_2 = 5, a_3 = 6\}$. Then $H = \{0, 4, 5, 6, 8 \longrightarrow\}$. Hence, $g(H) = 4$ and $c(H) = 8$, which implies that H is symmetric. Then Ker $\varphi_H = (X_1^3 - X_3^2, X_2^2 - X_1X_2)$.

Let H be a numerical semigroup with $\sharp M(H) = n$. It is said to be *l-dimensionally toric* if there exists an affine toric variety X of dimension l such that we have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \overset{{}^a\varphi_H}{\hookrightarrow} & \mathbb{A}^n = \text{Spec } k[X_1, \ldots, X_n] \\ \downarrow & \square & \downarrow^{a}\eta \\ X & \overset{\iota}{\hookrightarrow} & \mathbb{A}^{n+l-1} = \text{Spec } k[Y_1, \ldots, Y_{n+l-1}] \end{array}$$

where ι is a closed immersion and $g_j = \eta(Y_j)$'s are non-constant monomials.

Here, we review the notion of an affine toric variety. Let $\mathbb{G}_m = k^\times$ be the multiplicative group. We set $T = \mathbb{G}_m^l$. An affine variety X is called an *l-dimensionally affine toric variety* if it contains T as a dense open subset and the multiplication map on T extends to $T \times X$ as follows:

$$\begin{array}{ccc} T \times T & \hookrightarrow & T \times X \\ \downarrow \text{multip} & & \downarrow^{\exists} \\ T & \hookrightarrow & X \end{array}$$

Example 1.6 Let H be a numerical semigroup with $M(H) = \{a_1, a_2\}$. Then H is 1-dimensionally toric. In fact, we have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \overset{{}^a\varphi_H}{\hookrightarrow} & \mathbb{A}^2 = \text{Spec } k[X_1, X_2] \\ \downarrow & \square & \downarrow^{a}\eta \\ \mathbb{A}^1 = \text{Spec } k[T] & \overset{{}^a\psi}{\hookrightarrow} & \mathbb{A}^2 = \text{Spec } k[Y_1, Y_2] \end{array}$$

where $\eta(Y_1) = X_1^{a_2}$, $\eta(Y_2) = X_2^{a_1}$ and $\psi(Y_i) = T$ for $i = 1, 2$.

Example 1.7 Let H be a numerical semigroup with $M(H) = \{3, 4, 5\}$. Then H is 4-dimensionally toric. In fact, we have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \overset{{}^a\varphi_H}{\hookrightarrow} & \mathbb{A}^3 = \text{Spec } k[X_1, X_2, X_3] \\ \downarrow & & \downarrow^{a}\eta \\ \text{Spec } k[T^s]_{s \in S} & \overset{{}^a\psi_S}{\hookrightarrow} & \mathbb{A}^6 = \text{Spec } k[Y_1, \dots, Y_6] \end{array}$$

where η sends $Y_1, \dots, Y_6$ to $X_1, X_1^2, X_2, X_2, X_3$, X_3 respectively and S is the *saturated* subsemigroup of $\mathbb{Z}^4$ (i.e., $r \in \mathbb{Z}^4$ with $nr \in S$ for some non-zero $n \in \mathbb{N}_0$ implies that $r \in S$) generated by e_i's, $(1, 1, -1, 0)$ and $(-1, 0, 1, 1)$ where e_i is the vector whose i-th component is 1 and all the other components are 0 for $i = 1, 2, 3, 4$.

For a numerical semigroup H we want to know the minimum l where H is l-dimensionally toric. For this reason we introduce the notion of the toric dimension of a numerical semigroup as follows: We set

$$\text{Tdim } H = \min\{l \mid H \text{ is } l - \text{dimensionally toric}\},$$

which is called the *toric dimension* of H. If H is not l-dimensionally toric for any l , then we set Tdim $H = \infty$.

Example 1.8 Let H be a numerical semigroup with $M(H) = \{a_1, a_2\}$. Then Tdim $H = 1$.

Example 1.9 Tdim $\langle 3, 4, 5 \rangle \leqq 4$.

Remark 1.1 *If* Tdim $H < \infty$, *then* H *is Weierstrass, i.e., there exists a pointed non-singular curve* (C, P) *such that*

$$H = \{n \in \mathbb{N}_0 | \exists f \in k(C) \ s.t. \ (f)_\infty = nP\}$$

where $k(C)$ *is the field of rational functions on* C. (See [4])

Example 1.10 We have Tdim $\langle 13,14,15,16,17,18,20,22,23\rangle = \infty$.

Proof. Buchweitz [2] showed that the above semigroup is not Weierstrass. $\square$

For any $n \geqq 2$ we give numerical semigroups H with $\sharp M(H) = n$ and Tdim $H = 1$.

Theorem 1.2 *Let $n \geqq 2$ and $2 \leqq r_n < r_{n-1} < \cdots < r_2 < r_1$ with $(r_i, r_j) = 1$ if $i \neq j$. We set $a_i = r_1 \cdots r_{i-1} r_{i+1} \cdots r_n$ for $i = 1, \ldots, n$. Let $H = \langle a_1, a_2, \ldots, a_n\rangle$. Then we have the following:*
i) $(a_1, a_2, \ldots, a_n) = 1$, *hence H is a numerical semigroup.*
ii) $M(H) = \{a_1, a_2, \ldots, a_n\}$.
iii) *H is symmetric.*
iv) Ker φ_H *is generated by* $X_i^{r_i} - X_{i+1}^{r_{i+1}}$*'s,* $i = 1, 2, \ldots, n-1$.
v) Tdim $H = 1$.

Proof. It is not difficult to prove i) $\sim$ iv).
v) We have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \overset{{}^a\varphi_H}{\hookrightarrow} & \mathbb{A}^n = \text{Spec } k[X_1, \ldots, X_n] \\ \downarrow & \square & \downarrow{}^a\eta \\ \mathbb{A}^1 = \text{Spec } k[T] & \overset{{}^a\psi}{\hookrightarrow} & \mathbb{A}^n = \text{Spec } k[Y_1, \ldots, Y_n] \end{array}$$

where $\eta(Y_i) = X_i^{r_i}$ and $\psi(Y_i) = T$ for $i = 1, \ldots, n$. Hence we get Tdim $H = 1$. $\square$

2 The toric dimension of a numerical semigroup generated by three elements

Example 2.1 Let $H = \langle 4,6,5\rangle$. We set $a_1 = 4, a_2 = 6, a_3 = 5$. This semigroup is symmetric, but not the type as in Theorem 1.2. We have Tdim $H \leqq 2$.

Proof. We have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \overset{{}^a\varphi_H}{\hookrightarrow} & \mathbb{A}^3 = \text{Spec } k[X_1, X_2, X_3] \\ \downarrow & \square & \downarrow{}^a\eta \\ \mathbb{A}^2 = \text{Spec } k[T_1, T_2] & \overset{{}^a\psi}{\hookrightarrow} & \mathbb{A}^4 = \text{Spec } k[Y_1, \ldots, Y_4] \end{array}$$

where $\eta(Y_1) = X_1$, $\eta(Y_2) = X_2^2$, $\eta(Y_3) = X_2$, $\eta(Y_4) = X_3^2$ and $\psi(Y_1) = T_1$, $\psi(Y_2) = T_1^3$,$\psi(Y_3) = T_2$, $\psi(Y_4) = T_1T_2$. Hence, we get Tdim $H \leqq 2$.

For a numerical semigroup H with $M(H) = \{a_1, \dots, a_n\}$ we set

$$\alpha_i = \min\{\beta \in \mathbb{N}_0 > 0 \mid \beta a_i \in \langle a_1, \cdots, \check{a}_i, \cdots, a_n\rangle\}$$

for $i = 1, \dots, n$.

Proposition 2.1 *Let H be a numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$. If H is symmetric, then* Tdim $H \leqq 2$.

Proof. By the result of Herzog [3] if we renumber a_1, a_2, a_3, we may assume

$$\alpha_1 a_1 = \alpha_2 a_2,\ \alpha_3 a_3 = \beta_1 a_1 + \beta_2 a_2.$$

If $(\beta_1, \beta_2) = (\alpha_1, 0)$ or $(0, \alpha_2)$, the same way as the proof in Theorem 1.2 works well. Hence, Tdim $H = 1$. Otherwise, the similar way to that of $H = \langle 4, 5, 6\rangle$ works well. So, we get Tdim $H \leqq 2$. □

Proposition 2.2 *Let H be a numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$. If H is non-symmetric, then* $2 \leqq$ Tdim $H \leqq 4$.

Proof. Using the result of Herzog [3] the similar way to that of $H = \langle 3, 4, 5\rangle$ works well. So, we get Tdim $H \leqq 4$. If Tdim $H = 1$, then the ideal Kerφ_H is a complete intersection. Hence, we get $2 \leqq$ Tdim H. □

Remark 2.3 *H with $M(H) = \{3, 4, 5\}$. We can get* Tdim $H \leqq 2$, *hence* Tdim $H = 2$.

Proof. We have a fiber product

$$\begin{array}{ccc} \operatorname{Spec} k[H] & \overset{{}^a\varphi_H}{\hookrightarrow} & \mathbb{A}^3 = \operatorname{Spec} k[X_1, X_2, X_3] \\ \downarrow & \square & \downarrow {}^a\eta \\ \operatorname{Spec} k[T^s]_{s \in S} & \overset{{}^a\psi_S}{\hookrightarrow} & \mathbb{A}^4 = \operatorname{Spec} k[Y_1, \dots, Y_4] \end{array}$$

where η sends $Y_1, \dots, Y_4$ to X_2, X_1, X_1^2, X_3 respectively and $S = \check{\sigma} \cap M$ is the saturated subsemigroup of $\mathbb{Z}^2$ generated by e_i's, $3e_1 - 2e_2$ and $2e_1 - e_2$. Here, $\sigma = \sigma_{2,3} = \mathbb{R}_0(1, 0) + \mathbb{R}_0(2, 3)$ where $\mathbb{R}_0$ denotes the set of non-negative real numbers. □

Theorem 2.4 *We give the toric dimension of H with $\sharp M(H) = 3$ in the following table:*

Tdim H	Symmetric	Non-symmetric
1	$\exists$	$\times$
2	$\bigcirc$	$\exists$
3	$\times$	$\bigcirc$
4	$\times$	$\bigcirc$
$\geqq 5$	$\times$	$\times$

"$\bigcirc$" means that such a semigroup probably exists.

3 On the toric dimension of a numerical semigroup generated by four elements

In this section we will consider the toric dimension of H with $\sharp M(H) = 4$. First we study about the symmetric case. By the result of Bresinsky [1] the symmetric semigroups are divided into three types. We explain these three kinds of symmetric semigroups by examples.

Example 3. 1 Let $H = \langle 10, 25, 14, 21 \rangle$. We set $a_1 = 10, a_2 = 25, a_3 = 14, a_4 = 21$. We have (minimal) relations

$$5a_1 = 2a_2,\ 3a_3 = 2a_4,\ a_1 + a_2 = a_3 + a_4.$$

In this case, $g(H) = 29$ and $c(H) = 58$.

Example 3. 2 The semigroup $\langle 30, 42, 70, 105 \rangle$ as in Theorem 1.2 is a special case of the above type. In this case, let $r_1 = 2$, $r_2 = 3$, $r_3 = 5$ and $r_4 = 7$ in Theorem 1.2.

Example 3. 3 Let $H = \langle 8, 12, 10, 19 \rangle$. We set $a_1 = 8, a_2 = 12, a_3 = 10, a_4 = 19$. We have (minimal) relations

$$3a_1 = 2a_2,\ 2a_3 = a_1 + a_2,\ 2a_4 = 2a_1 + a_2 + a_3,$$

which are similar to $H = \langle 4, 5, 6 \rangle$ in Example 1.5. In this case, $g(H) = 17$ and $c(H) = 34$.

Example 3.4 Let $H = \langle 5,7,8,9\rangle$. We set $a_1 = 5, a_2 = 7, a_3 = 8, a_4 = 9$. We have (minimal) relations

$$3a_1 = a_2+a_3,\ 2a_2 = a_1+a_4,\ 2a_3 = a_2+a_4,\ 2a_4 = 2a_1+a_3,\ 2a_1+a_2 = a_3+a_4,$$

which are similar to $H = \langle 3,4,5\rangle$ in Example 1.4. In this case, $g(H) = 6$ and $c(H) = 12$.

For the above three types of symmetric numerical semigroup H generated by four elements we will construct an affine toric variety of which fiber product is Spec $k[H]$.

Example 3.5 Let $H = \langle 8,12,10,19\rangle$. We have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \stackrel{^a\varphi_H}{\hookrightarrow} & \mathbb{A}^4 = \text{Spec } k[X_1,\ldots,X_4] \\ \downarrow & \square & \downarrow^{a}\eta \\ \mathbb{A}^3 = \text{Spec } k[T_1,T_2,T_3] & \stackrel{^a\psi}{\hookrightarrow} & \mathbb{A}^6 = \text{Spec } k[Y_1,\ldots,Y_6] \end{array}$$

where

$$\eta(Y_1) = X_1,\ \eta(Y_2) = X_2,\ \eta(Y_3) = X_3,\ \eta(Y_4) = X_2^2,\ \eta(Y_5) = X_3^2,\ \eta(Y_6) = X_4^2,$$
$$\psi(Y_i) = T_i,\ i = 1,2,3,\ \psi(Y_4) = T_1^3,\ \psi(Y_5) = T_1T_2,\ \psi(Y_6) = T_1^2T_2T_3.$$

Hence, we get Tdim $H \leqq 3$.

Example 3.6 Let $H = \langle 5,7,8,9\rangle$. We have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \stackrel{^a\varphi_H}{\hookrightarrow} & \mathbb{A}^4 = \text{Spec } k[X_1,\ldots,X_4] \\ \downarrow & \square & \downarrow^{a}\eta \\ \text{Spec } k[T^s]_{s\in S} & \stackrel{^a\psi_S}{\hookrightarrow} & \mathbb{A}^8 = \text{Spec } k[Y_1,\ldots,Y_8] \end{array}$$

Let S be the saturated subsemigroup of $\mathbb{Z}^5$ generated by $e_1,\ldots,e_5, e_1+e_2-e_3, e_3+e_4-e_1, e_1+e_2-e_3-e_4+e_5$. The ring homomorphism η sends $Y_1,\ldots,Y_8$ to X_1, X_1^2, X_2, X_2, X_3, X_3, X_4, X_4 respectively. ψ_S sends $Y_1,\ldots,Y_8$ to $T_1,\ldots,T_5, T_1T_2T_3^{-1}, T_1^{-1}T_3T_4, T_1T_2T_3^{-1}T_4^{-1}T_5$ respectively. Hence we get Tdim $H \leqq 5$.

Example 3.7 Let $H = \langle 10,25,14,21\rangle$. We have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \stackrel{^a\varphi_H}{\hookrightarrow} & \mathbb{A}^4 = \text{Spec } k[X_1,\ldots,X_4] \\ \downarrow & \square & \downarrow^{a}\eta \\ \text{Spec } k[T^s]_{s\in S} & \stackrel{^a\psi_S}{\hookrightarrow} & \mathbb{A}^7 = \text{Spec } k[Y_1,\ldots,Y_7] \end{array}$$

where S is the saturated subsemigroup of $\mathbb{Z}^4$ generated by $e_1, \ldots, e_4, e_1 + e_3 - e_4$. The morphism η sends $Y_1, \ldots, Y_7$ to X_1, X_3^3, X_2, X_3, X_2^2, X_4^2, X_4 respectively and ψ_S sends $Y_1, \ldots, Y_7$ to $T_1, \ldots, T_4, T_1^5, T_2, T_1T_3T_4^{-1}$ respectively. Hence, we get Tdim $H \leqq 4$.

Theorem 3.1 *We give the toric dimension of a symmetric numerical semigroup H with $\sharp M(H) = 4$ in the following table:*

Tdim H	Symmetric
1	$\nexists$
2	○
3	○Ex.3.3(3.5)
4	○Ex.3.1(3.7)
5	○Ex.3.4(3.6)
$\geqq 6$	×

A numerical semigroup H is called an *n-semigroup* if n is the minimum positive integer in H. In the case of a non-symmetric numerical semigroup H with $\sharp M(H) = 4$ we will investigate the toric dimensions of 4-semigroups and 5-semigroups. By the result of [4] the 4-semigroups are divided into two types. We explain these two kinds of 4-semigroups by examples.

Example 3.8 Let $H = \langle 4, 9, 10, 15 \rangle$. We set $a_1 = 4, a_2 = 9, a_3 = 10, a_4 = 15$. We have (minimal) relations

$$5a_1 = 2{a_3}^{t\mathrm{iv})},\ 2a_2 = 2a_1 + {a_3}^{\mathrm{v})},\ 2a_4 = 5a_1 + {a_3}^{\mathrm{vi})},$$

$$a_1 + a_4 = a_2 + {a_3}^{t\mathrm{i})},\ a_1 + 2a_3 = a_2 + {a_4}^{t\mathrm{iii})},\ 4a_1 + a_2 = a_3 + {a_4}^{t\mathrm{ii})}.$$

In this case, $g(H) = 7$ and $c(H) = 12$. We have a fiber product

$$\begin{array}{ccc}
\mathrm{Spec}\ k[H] & \overset{^a\varphi_H}{\hookrightarrow} & \mathbb{A}^4 = \mathrm{Spec}\ k[X_1, \ldots, X_4] \\
\downarrow & \square & \downarrow^{a}\eta \\
\mathrm{Spec}\ k[T^s]_{s \in S} & \overset{^a\psi_S}{\hookrightarrow} & \mathbb{A}^7 = \mathrm{Spec}\ k[Y_1, \ldots, Y_7]
\end{array}$$

where S is the saturated subsemigroup of $\mathbb{Z}^4$ generated by $e_1, \ldots, e_4, e_1 + e_2 - e_3$, $e_2 - e_3 + e_4$, $2e_1 - e_3 - e_4$. The morphism η sends $Y_1, \ldots, Y_7$ to X_2, X_3, X_1, X_3, X_4, X_1^4, X_1 respectively and ψ_S is determined by the above minimum set of generators of S. Hence, we get Tdim $H \leqq 4$.

Example 3.9 Let $H = \langle 4, 9, 11, 14\rangle$. We set $a_1 = 4, a_2 = 9, a_3 = 11, a_4 = 14$. We have (minimal) relations

$$5a_1 = a_2 + a_3,\ 2a_2 = a_1 + a_4,\ 2a_3 = 2a_1 + a_4,$$

$$2a_4 = 2a_1 + a_2 + a_3,\ 4a_1 + a_2 = a_3 + a_4,\ 3a_1 + a_3 = a_2 + a_4.$$

In this case, $g(H) = 7$ and $c(H) = 11$. We have a fiber product

$$\begin{array}{ccc} \operatorname{Spec} k[H] & \overset{{}^a\varphi_H}{\hookrightarrow} & \mathbb{A}^4 = \operatorname{Spec} k[X_1, \ldots, X_4] \\ \downarrow & \square & \downarrow^{a}\eta \\ \operatorname{Spec} k[T^s]_{s\in S} & \overset{{}^a\psi_S}{\hookrightarrow} & \mathbb{A}^9 = \operatorname{Spec} k[Y_1, \ldots, Y_9] \end{array}$$

where S is the saturated subsemigroup of $\mathbb{Z}^6$ generated by $e_1, \ldots, e_6, e_1 + e_2 + e_3 - e_4, -e_1 + e_4 + e_5, e_1 + e_3 - e_4 + e_6$. The morphism η sends $Y_1, \ldots, Y_9$ to X_1, X_1^2, X_1^2, X_2, X_2, X_3, X_3, X_4, X_4 respectively and ψ_S is determined by the above minimum set of generators of S. Hence, we get Tdim $H \leqq 6$.

Theorem 3.2 *We give the toric dimension of a non-symmetric* 4*-semigroup* H *with* $\sharp M(H) = 4$ *in the following table:*

Tdim H	Non-symmetric 4-semigroup
1	×
2	○
3	○
4	○Ex.3.8
5	○
6	○Ex.3.9
$\geqq 7$	×

By the result of [5] the 5-semigroups are divided into two types. We explain these two kinds of 5-semigroups by examples.

Example 3.10 Let $H = \langle 5, 7, 13, 16\rangle$. We set $a_1 = 5, a_2 = 7, a_3 = 13, a_4 = 16$. We have (minimal) relations

$$4a_1 = a_2 + a_3,\ 3a_2 = a_1 + a_4,\ 2a_3 = 2a_1 + a_4,$$

$$2a_4 = a_1 + 2a_2 + a_3,\ 3a_1 + 2a_2 = a_3 + a_4,\ 2a_1 + a_3 = a_2 + a_4.$$

In this case, $g(H) = 8$ and $c(H) = 12$. We have a fiber product

$$\begin{array}{ccc} \mathrm{Spec}\ k[H] & \overset{{}^a\varphi_H}{\hookrightarrow} & \mathbb{A}^4 = \mathrm{Spec}\ k[X_1, \ldots, X_4] \\ \downarrow & \square & \downarrow^{a}\eta \\ \mathrm{Spec}\ k[T^s]_{s \in S} & \overset{{}^a\psi_S}{\hookrightarrow} & \mathbb{A}^9 = \mathrm{Spec}\ k[Y_1, \ldots, Y_9] \end{array}$$

where S is the saturated subsemigroup of $\mathbb{Z}^6$ generated by $e_1, \ldots, e_6, e_1 + e_2 + e_3 - e_4$, $e_2 + e_3 - e_4 + e_5$, $-e_2 + e_4 + e_6$. The morphism η sends $Y_1, \ldots, Y_9$ to X_1, X_1^2, X_1, X_3, X_2^2, X_3, X_2, X_4, X_4 respectively and ψ_S is determined by the above minimum set of generators of S. Hence, we get Tdim $H \leqq 6$.

Example 3.11 Let $H = \langle 5, 13, 21, 22 \rangle$. We set $a_1 = 5, a_2 = 13, a_3 = 21, a_4 = 22$. We have (minimal) relations

$$7a_1 = a_2 + a_4,\ 2a_2 = a_1 + a_3,\ 2a_3 = 4a_1 + a_4,$$

$$2a_4 = 2a_1 + a_2 + a_3,\ 6a_1 + a_2 = a_3 + a_4.$$

In this case, $g(H) = 16$ and $c(H) = 30$. We have a fiber product

$$\begin{array}{ccc} \mathrm{Spec}\ k[H] & \overset{{}^a\varphi_H}{\hookrightarrow} & \mathbb{A}^4 = \mathrm{Spec}\ k[X_1, \ldots, X_4] \\ \downarrow & \square & \downarrow^{a}\eta \\ \mathrm{Spec}\ k[T^s]_{s \in S} & \overset{{}^a\psi_S}{\hookrightarrow} & \mathbb{A}^9 = \mathrm{Spec}\ k[Y_1, \ldots, Y_9] \end{array}$$

where S is the saturated subsemigroup of $\mathbb{Z}^6$ generated by $e_1, \ldots, e_6, e_1 + e_2 + e_3 - e_4$, $-e_1 + e_4 + e_5$, $-e_1 - e_2 + e_4 + e_5 + e_6$. The morphism η sends $Y_1, \ldots, Y_9$ to X_1, X_1^4, X_1^2, X_2, X_2, X_3, X_4, X_3, X_4 respectively and ψ_S is determined by the above minimum set of generators of S. Hence, we get Tdim $H \leqq 6$.

Theorem 3.3 *We give the toric dimension of a non-symmetric* 5*-semigroup* H *with* $\sharp M(H) = 4$ *in the following table:*

Tdim H	Non-symmetric 5-semigroup
1	$\times$
2	○
3	○
4	○
5	○
6	○ Ex.3.10,3.11
$\geqq 7$	$\times$

Problem 1 What is the minimum number n such that Tdim $H \leqq n$ for any non-symmetric 6-semigroup H with $\sharp M(H) = 4$?

Problem 2 Have we Tdim $H < \infty$ for any non-symmetric numerical semigroup H with $\sharp M(H) = 4$?

References

[1] H. Bresinsky, *Symmetric semigroups of integers generated by* 4 *elements.* Manuscripta Math. **17** (1975) 205-219.

[2] R.O. Buchweitz, *On Zariski's criterion for equisingularity and non-smoothable monomial curves.* preprint 113, University of Hannover, 1980.

[3] J. Herzog, *Generators and relations of abelian semigroups and semigroup rings*, Manuscripta Math. **3** (1970) 175-193.

[4] J. Komeda, *On Weierstrass points whose first non-gaps are four.* J. reine angew. Math. **341** (1983) 68–86.

[5] J. Komeda, *On the existence of Weierstrass points whose first non-gaps are five.* Manuscripta Math. **76** (1992) 193–211.