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EINSTEIN HOMOGENEOUS MANIFOLDS AND GEOMETRIC INVARIANT THEORY

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ABSTRACT. The only known examples until now of noncompact homogeneous Einstein manifolds are standard solvmanifolds: solvable Lie groups endowed with a left invariant metric such that if $\mathfrak{s}$ is the Lie algebra, $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$ and $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is the orthogonal decomposition then $[\mathfrak{a}, \mathfrak{a}] = 0$. This is a very natural algebraic condition which has played an important role in many aspects of homogeneous Riemannian geometry. The aim of this note is to give an idea of the proof, and mainly of the tools used in it, of the fact that any Einstein solvmanifold must be standard. The proof of the theorem involves a somewhat extensive study of the natural $\mathrm{GL}_n(\mathbb{R})$-action on the vector space $V = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, from a geometric invariant theory point of view. We had to adapt a stratification for reductive groups actions on projective algebraic varieties introduced by F. Kirwan, to get a $\mathrm{GL}_n(\mathbb{R})$-invariant stratification of V satisfying many nice properties which are relevant to our problem.

1. INTRODUCTION

The construction of Einstein metrics on manifolds is a classical problem in differential geometry and general relativity. A Riemannian manifold is called *Einstein* if its Ricci tensor is a scalar multiple of the metric. We refer to [Besse 87] for a detailed exposition on Einstein manifolds (see also the surveys in [Lebrun-Wang 99]). In the homogeneous case, the following main general question is still open, in both compact and noncompact cases:

> **Problem 1**. Which homogeneous spaces G/K admit a G-invariant Einstein Riemannian metric?

We refer to [Böhm-Wang-Ziller 04] and the references therein for an update in the compact case. In the noncompact case, the only known examples until now are all of a very particular kind; namely, simply connected solvable Lie groups endowed with a left invariant metric (so called *solvmanifolds*). According to the following long standing conjecture, these might exhaust all the possibilities for noncompact homogeneous Einstein manifolds.

> **Alekseevskii's conjecture** [Besse 87, 7.57]. If G/K is a homogeneous Einstein manifold of negative scalar curvature then K is a maximal compact subgroup of G (or equivalently, G/K is a solvmanifold).

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The conjecture is wide open, and it is known to be true only for dim ≤ 5, a result which follows from the complete classification in these dimensions given in [Nikonorov 05]. One of the most intriguing questions related to this conjecture is the following:

Problem 2. Are there Einstein left invariant metrics on $\mathrm{SL}_n(\mathbb{R})$, $n \geq 3$?

Indeed, a reason to consider Alekseevskii's conjecture as too optimistic is the fact proved in [Dotti-Leite 82] that the above Lie groups do admit left invariant metrics of negative Ricci curvature (and also does any complex simple Lie group, see [Dotti-Leite-Miatello 84]). However, an inspection of the eigenvalues of the Ricci tensors in [Dotti-Leite 82] shows that they are far from being close to each other, giving back some hope.

Anyway, even if one is very optimistic and believe that the conjecture is true, a classification of Einstein metrics in the noncompact homogeneous case would still be just a dream, as the following problem is also open:

Problem 3. Which solvable Lie groups admit an Einstein left invariant metric?

Examples are irreducible symmetric spaces of noncompact type and some deformations, Damek-Ricci spaces, the radical of any parabolic subgroup of a semisimple Lie group (see [Tamaru 07]), and several more, including continuous families depending on various parameters (see [L. 04], [L.-Will 06] and [Nikolayevsky 08] for further information). Every known example of an Einstein solvmanifold S satisfies the following additional condition: if $\mathfrak{s}$ is the Lie algebra of S, $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$ and $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is the orthogonal decomposition relative to the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{s}$ which determines the metric, then

$$[\mathfrak{a}, \mathfrak{a}] = 0.$$

A solvmanifold with such a property is called *standard*. This is a very simple algebraic condition which has nevertheless played an important role in many aspects of homogeneous Riemannian geometry:

- [Azencott-Wilson 76] Any homogeneous manifold of nonpositive sectional curvature is a standard solvmanifold.
- [Heber 06] All harmonic noncompact homogeneous manifold are standard solvmanifolds (with $\dim \mathfrak{a} = 1$).
- [Gindikin-Piatetskii Shapiro-Vinberg 67] Kähler-Einstein noncompact homogeneous manifolds are all standard solvmanifolds.
- [Alekseevskii 75, Cortés 96] Every quaternionic Kähler solvmanifold (completely real) is standard.

Standard Einstein solvmanifolds were extensively investigated in [Heber 98], where many remarkable structural and uniqueness results are derived, by assuming only the standard condition. A natural question arises:

Problem 4. Is every Einstein solvmanifold standard?

Partial results on this question were obtained in [Heber 98] and [Schueth 04], who gave several sufficient conditions. The answer was known to be yes in dimension ≤ 6 and follows from a complete classification obtained in [Nikitenko-Nikonorov 06]. On the other hand, it is proved in [Nikolayevsky 06b] that many classes of nilpotent Lie algebras can not be the nilradical of a non-standard Einstein solvmanifold.

The aim of this note is to give an idea of the proof, and mainly of the tools used in it, of the following result.

Theorem. [L. 07] Any Einstein solvmanifold is standard.

The proof of the theorem involves a somewhat extensive study of the natural $\mathrm{GL}_n(\mathbb{R})$-action on the vector space $V = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, from a geometric invariant theory point of view. We recall that V can be viewed as a vector space containing the space of all n-dimensional Lie algebras as an algebraic subset. We had to adapt a stratification for reductive groups actions on projective algebraic varieties given in [Kirwan 84, Section 12] (algebraically closed case), to get a $\mathrm{GL}_n(\mathbb{R})$-invariant stratification of V satisfying many nice properties which are relevant to our problem (see Theorem 2.2). Kirwan's construction, in turn, is based on instability results proved in [Kempf 78] and [Hesselink 78]. We note that any $\mu \in V$ is *unstable* (i.e. $0 \in \overline{\mathrm{GL}_n(\mathbb{R}).\mu}$). The strata are parameterized by a finite set $\mathcal{B}$ of diagonal $n \times n$ matrices, and each $\beta \in \mathcal{B}$ is (up to conjugation) the 'most responsible' direction for the instability of each μ in the stratum $\mathcal{S}_\beta$, in the sense that $e^{-t\beta}.\mu \to 0$, as $t \to \infty$ faster that any other one-parameter subgroup having a tangent vector of the same norm. Such a stratification is intimately related to the moment map $m : V \longrightarrow \mathfrak{gl}_n(\mathbb{R})$ for the action above, specially to the functional square norm of m and its critical points.

We finally mention that the geometric invariant theory point of view considered in this paper has also proved to be very useful in the study standard Einstein solvmanifolds (see for instance [Heber 98], [Payne 05], [L.-Will 06], [Nikolayevsky 07], [Will 08] and [Nikolayevsky 08]). The algebraic subset $\mathcal{N} \subset V$ of all nilpotent Lie algebras parameterizes a set of $(n+1)$-dimensional rank-one (i.e. $\dim \mathfrak{a} = 1$) solvmanifolds $\{S_\mu : \mu \in \mathcal{N}\}$, containing the set of all those which are Einstein in that dimension. The stratum of μ determines the eigenvalue type of a potential Einstein solvmanifold $S_{g.\mu}$, $g \in \mathrm{GL}_n(\mathbb{R})$ (if any), and so the stratification provides a convenient tool to produce existence results as well as obstructions for nilpotent Lie algebras to be the nilradical of an Einstein solvmanifold. Furthermore, S_μ is Einstein if and only if μ is a critical point of the square norm of the moment map.

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2. The variety of nilpotent Lie algebras

Let us consider the vector space

$$V = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n = \{\mu : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n : \mu \text{ bilinear and skew-symmetric}\},$$

on which there is a natural linear action of $\mathrm{GL}_n(\mathbb{R})$ on the left given by

$$(1)\quad g.\mu(X, Y) = g\mu(g^{-1}X, g^{-1}Y), \qquad X, Y \in \mathbb{R}^n, \quad g \in \mathrm{GL}_n(\mathbb{R}), \quad \mu \in V.$$

The space of all n-dimensional nilpotent Lie algebras can be parameterized by the set

$$\mathcal{N} = \{\mu \in V : \mu \text{ satisfies the Jacobi identity and is nilpotent}\},$$

and it is an algebraic subset of V as the Jacobi identity and the nilpotency condition can both be expressed as zeroes of polynomial functions. Note that $\mathcal{N}$ is $\mathrm{GL}_n(\mathbb{R})$-invariant and Lie algebra isomorphism classes are precisely $\mathrm{GL}_n(\mathbb{R})$-orbits.

The canonical inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ defines an $\mathrm{O}(n)$-invariant inner product on V by

$$(2)\qquad \langle \mu, \lambda \rangle = \sum_{ij} \langle \mu(e_i, e_j), \lambda(e_i, e_j) \rangle = \sum_{ijk} \langle \mu(e_i, e_j), e_k \rangle \langle \lambda(e_i, e_j), e_k \rangle,$$

where $\{e_1, ..., e_n\}$ is the canonical basis of $\mathbb{R}^n$. A Cartan decomposition for the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$ is given by $\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{so}(n) \oplus \mathrm{sym}(n)$, that is, in skew-symmetric and symmetric matrices respectively. We consider the following $\mathrm{Ad}(\mathrm{O}(n))$-invariant inner product on $\mathfrak{gl}_n(\mathbb{R})$,

(3)

$$\langle \alpha, \beta \rangle = \mathrm{tr}\, \alpha\beta^{\mathrm{t}} = \sum_i \langle \alpha e_i, \beta e_i \rangle = \sum_{ij} \langle \alpha e_i, e_j \rangle \langle \beta e_i, e_j \rangle, \qquad \alpha, \beta \in \mathfrak{gl}_n(\mathbb{R}).$$

Remark 2.1. There have been several abuses of notation concerning inner products. Recall that $\langle \cdot, \cdot \rangle$ has been used to denote an inner product on $\mathbb{R}^n$, V and $\mathfrak{gl}_n(\mathbb{R})$.

The action of $\mathfrak{gl}_n(\mathbb{R})$ on V obtained by differentiation of (1) is given by

$$(4)\qquad \pi(\alpha)\mu = \alpha\mu(\cdot, \cdot) - \mu(\alpha \cdot, \cdot) - \mu(\cdot, \alpha \cdot), \qquad \alpha \in \mathfrak{gl}_n(\mathbb{R}), \quad \mu \in V.$$

We note that $\pi(\alpha)\mu = 0$ if and only if $\alpha \in \mathrm{Der}(\mu)$, the Lie algebra of derivations of the algebra μ, and also that $\pi(\alpha)^t = \pi(\alpha^t)$ for any $\alpha \in \mathfrak{gl}_n(\mathbb{R})$, due to the choice of canonical inner products everywhere. Let $\mathfrak{t}$ denote the set of all diagonal $n \times n$ matrices. If $\{e'_1, ..., e'_n\}$ is the basis of $(\mathbb{R}^n)^*$ dual to the canonical basis then

$$\{v_{ijk} = (e'_i \wedge e'_j) \otimes e_k : 1 \leq i < j \leq n, \ 1 \leq k \leq n\}$$

is a basis of weight vectors of V for the action (1), where v_{ijk} is actually the bilinear form on $\mathbb{R}^n$ defined by $v_{ijk}(e_i, e_j) = -v_{ijk}(e_j, e_i) = e_k$ and zero otherwise. The corresponding weights $\alpha_{ij}^k \in \mathfrak{t}$, $i < j$, are given by

$$\pi(\alpha)v_{ijk} = (a_k - a_i - a_j)v_{ijk} = \langle \alpha, \alpha_{ij}^k \rangle v_{ijk}, \quad \forall \alpha = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \in \mathfrak{t},$$

where $\alpha_{ij}^k = E_{kk} - E_{ii} - E_{jj}$ and $\langle \cdot, \cdot \rangle$ is the inner product defined in (3). As usual, E_{rs} denotes the matrix whose only nonzero coefficient is 1 in the entry rs. From now on, we will always denote by μ_{ij}^k the structure coefficients of a vector $\mu \in V$ with respect to this basis:

$$\mu = \sum \mu_{ij}^k v_{ijk}, \qquad \mu_{ij}^k \in \mathbb{R}, \qquad \text{i.e.} \quad \mu(e_i, e_j) = \sum \mu_{ij}^k e_k.$$

Let $\mathfrak{t}^+$ denote the Weyl chamber of $\mathfrak{gl}_n(\mathbb{R})$ given by

$$\mathfrak{t}^+ = \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \in \mathfrak{t} : a_1 \leq \ldots \leq a_n \right\}.$$

We summarize in the following theorem some properties of the $\mathrm{GL}_n(\mathbb{R})$-invariant stratification of the vector space V defined in [L. 07]. Such a stratification is an adaptation of the one given by F. Kirwan in [Kirwan 84, Section 12] for complex reductive Lie group representations.

Theorem 2.2. [L. 07] *There exists a finite subset $\mathcal{B} \subset \mathfrak{t}^+$, and for each $\beta \in \mathcal{B}$ a GL_n-invariant subset $\mathcal{S}_\beta \subset V$ (a stratum) such that*

$$V \setminus \{0\} = \bigcup_{\beta \in \mathcal{B}} \mathcal{S}_\beta \qquad \textit{(disjoint union)}.$$

If $\mu \in \mathcal{S}_\beta$ then

(5) $\langle [\beta, D], D \rangle \geq 0 \qquad \forall\, D \in \mathrm{Der}(\mu) \qquad$ (*equality holds* $\Leftrightarrow [\beta, D] = 0$)

and

(6) $\qquad \beta + ||\beta||^2 I$ *is positive definite* $\forall\, \beta \in \mathcal{B}$.

If in addition

(7) $$\min\left\{ \langle \beta, \alpha_{ij}^k \rangle : \mu_{ij}^k \neq 0 \right\} = ||\beta||^2,$$

then

(8) $$\mathrm{tr}\, \beta D = 0 \quad \forall\, D \in \mathrm{Der}(\mu),$$

and

(9) $\langle \pi\left(\beta + ||\beta||^2 I\right) \mu, \mu \rangle \geq 0 \qquad$ (*equality holds* $\Leftrightarrow \beta + ||\beta||^2 I \in \mathrm{Der}(\mu)$).

Moreover, condition (7) *is always satisfied by some $g.\mu$ with $g \in \mathrm{O}(n)$.*

Given a finite subset X of $\mathfrak{t}$, denote by $\mathrm{CH}(X)$ the convex hull of X and by $\mathrm{mcc}(X)$ the *minimal convex combination of* X, that is, the (unique) vector of minimal norm in $\mathrm{CH}(X)$. Each nonzero $\mu \in V$ uniquely determines an element $\beta_\mu \in \mathfrak{t}$ given by

$$\beta_\mu = \mathrm{mcc}\left\{ \alpha_{ij}^k : \mu_{ij}^k \neq 0 \right\}, \qquad \mu = \sum \mu_{ij}^k v_{ijk}.$$

We note that β_μ is always nonzero since $\operatorname{tr}\alpha_{ij}^k = -1$ for all $i < j$ and consequently $\operatorname{tr}\beta_\mu = -1$. If $\mu \in \mathcal{S}_\beta$ satisfies condition (7) then $\beta = \beta_\mu$ (see [L. 07, Theorem 2.10, (iv)]), and hence for an arbitrary μ we still have that $\beta = \beta_{g.\mu}$ for some $g \in \mathrm{O}(n)$. This implies that $\operatorname{tr}\beta = -1$ for any $\beta \in \mathcal{B}$.

3. Proof of the theorem

We now apply the results described in Section 2 to prove that Einstein solvmanifolds are all standard.

Let S be a solvmanifold, that is, a simply connected solvable Lie group endowed with a left invariant Riemannian metric. Let $\mathfrak{s}$ be the Lie algebra of S and let $\langle\cdot,\cdot\rangle$ denote the inner product on $\mathfrak{s}$ determined by the metric. We consider the orthogonal decomposition $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n} = [\mathfrak{s},\mathfrak{s}]$. A solvmanifold S is called *standard* if $[\mathfrak{a},\mathfrak{a}] = 0$. The mean curvature vector of S is the only element $H \in \mathfrak{a}$ which satisfies $\langle H, A\rangle = \operatorname{tr}\operatorname{ad} A$ for any $A \in \mathfrak{a}$. If B denotes the symmetric map defined by the Killing form of $\mathfrak{s}$ relative to $\langle\cdot,\cdot\rangle$ then $B(\mathfrak{a}) \subset \mathfrak{a}$ and $B|_{\mathfrak{n}} = 0$ as $\mathfrak{n}$ is contained in the nilradical of $\mathfrak{s}$. The Ricci operator Ric of S is given by (see for instance [Besse 87, 7.38]):

$$\operatorname{Ric} = R - \tfrac{1}{2}B - S(\operatorname{ad} H), \tag{10}$$

where $S(\operatorname{ad} H) = \frac{1}{2}(\operatorname{ad} H + \operatorname{ad} H^t)$ is the symmetric part of $\operatorname{ad} H$ and R is the symmetric operator defined by

$$\langle Rx, y\rangle = -\tfrac{1}{2}\sum_{ij}\langle[x,x_i],x_j\rangle\langle[y,x_i],x_j\rangle + \tfrac{1}{4}\sum_{ij}\langle[x_i,x_j],x\rangle\langle[x_i,x_j],y\rangle, \tag{11}$$

for all $x, y \in \mathfrak{s}$, where $\{x_i\}$ is any orthonormal basis of $(\mathfrak{s},\langle\cdot,\cdot\rangle)$.

It is proved in [L. 06, Propositions 3.5, 4.2] that R is the only symmetric operator on $\mathfrak{s}$ such that

$$\operatorname{tr} RE = \tfrac{1}{4}\langle\pi(E)[\cdot,\cdot],[\cdot,\cdot]\rangle, \qquad \forall E \in \operatorname{End}(\mathfrak{s}), \tag{12}$$

where we are considering $[\cdot,\cdot]$ as a vector in $\Lambda^2\mathfrak{s}^*\otimes\mathfrak{s}$, $\langle\cdot,\cdot\rangle$ is the inner product defined in (2) and π is the representation given in (4) (see the notation in Section 2 and replace $\mathbb{R}^n$ with $\mathfrak{s}$). This is equivalent to say that

$$m([\cdot,\cdot]) = \tfrac{4}{\|[\cdot,\cdot]\|^2}R,$$

where $m : \Lambda^2\mathfrak{s}^*\otimes\mathfrak{s} \longrightarrow \operatorname{sym}(\mathfrak{s})$ is the moment map for the action of $\mathrm{GL}(\mathfrak{s})$ on $\Lambda^2\mathfrak{s}^*\otimes\mathfrak{s}$ (see [Kirwan 84], [Ness 84], [Mumford-Fogarty-Kirwan 94], [L.-Will 06]). Thus the anonymous tensor R in formula (10) for the Ricci operator is precisely the value of the moment map at the Lie bracket $[\cdot,\cdot]$ of $\mathfrak{s}$ (up to scaling).

Remark 3.1. Recall that actually each point of the variety of Lie algebras

$$\mathcal{L} = \{[\cdot,\cdot]' \in \Lambda^2\mathfrak{s}^*\otimes\mathfrak{s} : [\cdot,\cdot]' \text{ satisfies Jacobi}\}$$

can be identified with a Riemannian manifold; namely, the simply connected Lie group with Lie algebra $(\mathfrak{s},[\cdot,\cdot]')$ endowed with the left invariant metric determined by a fixed inner product $\langle\cdot,\cdot\rangle$ in $\mathfrak{s}$. Moreover, any left invariant metric in that dimension is isometric to a point in $\mathcal{L}$. The fact that $m([\cdot,\cdot]') = R$ up to scaling has been used in [L. 06] and [L.-Will 06] to get geometric results on left invariant metrics from the well known nice convexity properties of the functional square norm of m.

We therefore obtain from (10) and (12) that S is an Einstein solvmanifold with $\operatorname{Ric} = cI$, if and only if, for any $E \in \operatorname{End}(\mathfrak{s})$,

$$\operatorname{tr}\left(cI + \tfrac{1}{2}B + S(\operatorname{ad}H)\right)E = \tfrac{1}{4}\langle\pi(E)[\cdot,\cdot],[\cdot,\cdot]\rangle. \tag{13}$$

Let S be an Einstein solvmanifold with $\operatorname{Ric} = cI$. We can assume that S is not unimodular by using [Dotti 82], thus $H \neq 0$ and $\operatorname{tr}\operatorname{ad}H = ||H||^2 > 0$. By letting $E = \operatorname{ad}H$ in (13) we get

$$c = -\tfrac{\operatorname{tr}S(\operatorname{ad}H)^2}{\operatorname{tr}S(\operatorname{ad}H)} < 0. \tag{14}$$

In order to apply the results in Section 2, we identify $\mathfrak{n}$ with $\mathbb{R}^n$ via an orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathfrak{n}$ and we set $\mu := [\cdot,\cdot]|_{\mathfrak{n}\times\mathfrak{n}}$. In this way, μ can be viewed as an element of $\mathcal{N} \subset V$. If $\mu \neq 0$ then μ lies in a unique stratum $\mathcal{S}_\beta$, $\beta \in \mathcal{B}$, by Theorem 2.2, and it is easy to see that we can assume (up to isometry) that μ satisfies (7), so that one can use all the additional properties stated in the theorem. In particular, the following crucial technical result follows. Consider $E_\beta \in \operatorname{End}(\mathfrak{s})$ defined by

$$E_\beta = \begin{bmatrix} 0 & 0 \\ 0 & \beta + ||\beta||^2 I \end{bmatrix},$$

that is, $E|_{\mathfrak{a}} = 0$ and $E|_{\mathfrak{n}} = \beta + ||\beta||^2 I$.

Lemma 3.2. *If $\mu \in \mathcal{S}_\beta$ satisfies (7) then $\langle\pi(E_\beta)[\cdot,\cdot],[\cdot,\cdot]\rangle \geq 0$.*

We then apply (13) to $E_\beta \in \operatorname{End}(\mathfrak{s})$ and obtain from Lemma 3.2 and (14) that

$$-\tfrac{\operatorname{tr}S(\operatorname{ad}H)^2}{\operatorname{tr}S(\operatorname{ad}H)}\operatorname{tr}E_\beta + \operatorname{tr}S(\operatorname{ad}H)E_\beta \geq 0. \tag{15}$$

By using that $\operatorname{tr}\beta = -1$ we get

$$\begin{aligned} \operatorname{tr}E_\beta^2 &= \operatorname{tr}(\beta^2 + ||\beta||^4 I + 2||\beta||^2\beta) = ||\beta||^2(1 + n||\beta||^2 - 2) \\ &= ||\beta||^2(-1 + n||\beta||^2) = ||\beta||^2 \operatorname{tr}E_\beta. \end{aligned} \tag{16}$$

On the other hand, we have that

$$\operatorname{tr}S(\operatorname{ad}H)E_\beta = \operatorname{tr}\operatorname{ad}H|_{\mathfrak{n}}(\beta + ||\beta||^2) = ||\beta||^2 \operatorname{tr}S(\operatorname{ad}H) \tag{17}$$

by (8). We now use (15), (16) and (17) and obtain

$$\operatorname{tr}S(\operatorname{ad}H)^2 \operatorname{tr}E_\beta^2 \leq (\operatorname{tr}S(\operatorname{ad}H)E_\beta)^2,$$

a 'backwards' Cauchy-Schwartz inequality. This turns all inequalities which appeared in the proof of Lemma 3.2 into equalities, in particular:

$$\tfrac{1}{4}\sum_{rs}\langle(\beta + ||\beta||^2 I)[A_r, A_s], [A_r, A_s]\rangle = 0,$$

where $\{A_i\}$ is an orthonormal basis of $\mathfrak{a}$. We therefore get that $\mathfrak{a}$ is abelian since $\beta + ||\beta||^2 I$ is positive definite by (6), concluding the proof of the theorem.

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