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On the spectrum of Dirac operators

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1 Introduction

Let α_j, $j = 1,2,3$, $\beta = \alpha_0$ be Hermite (symmetric) 4×4 matrices which satisfy the following anti-commuting relations.

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}, \quad 0 \leq \forall j,\ \forall k \leq 3, \tag{1.1}$$

where δ_{jk} denotes Kronecker's delta. The Dirac operator with which we are concerned is defined by

$$H_D u = c\sum_{j=1}^{3} \alpha_j D_{x_j} u + mc^2\beta u + V(x)u,\ x \in \mathbf{R}^3, \tag{1.2}$$

where c is the speed of light, m is a non-negative number and V is a real-valued function defined on $\mathbf{R}^3$. It holds that the free Dirac operator

$$H_0 = c\alpha \cdot D_x + mc^2\beta,$$

is essentially self-adjoint on $C_0^\infty(\mathbf{R}^3;\ \mathbf{C}^4)$ in $\mathcal{H} = L^2(\mathbf{R}^3;\ \mathbf{C}^4)$ and

$$\sigma(H_0) = (-\infty, -mc^2] \cup [mc^2, +\infty).$$

in view of the identity $H_0^2 = (-c^2\Delta + m^2c^4)I_4$.

When the potential decays at infinity, we may expect that the spectrum is almost equal to that of the free operator. In fact when the potentials V are short range type it holds that
(1) $\sigma_{ess}(H_D) = (-\infty, -mc^2] \cup [mc^2, \infty)$,
(2) $\sigma_{sc}(H_D) = \emptyset$,
(3) $\sigma_p(H_D) \subset [-mc^2, mc^2]$ is an at most countable set whose elements can only accumulate to the points $\pm mc^2$ (O.Yamada [12]). This result has been extended to a class of long range type potentials (V. Vogelsang [11]).

On the other hand if we allow the potential to be diverge at infinity, the situation is dramatically changed. In fact it turns out (H. Kalf, T. Okaji and O. Yamada [8]) that the spectrum of H_D coincides with $\mathbf{R}$ and there exist no eigenvalues if potentials fulfill the following conditions. There exists a positive number δ such that as $|x| \to \infty$,

$$V(x) - q(|x|)I = \mathcal{O}(|x|^{-1/2-\delta} q^{1/2}(|x|)),$$

$$\partial_r (V(x) - q(|x|)I) = \mathcal{O}(|x|^{-1-\delta} q(|x|)),$$

where $q(r)$ is a real valued $C^2([0,\infty)$ function diverging at infinity, which satisfies the following conditions.

i) $\inf_{r \in I_a} q(r) > 0$, ii) $[q'(r)]_- = \mathcal{O}(r^{-1-\delta} q)$,

iii) $q'(r) = \mathcal{O}(r^{-1/2-\delta} q^{3/2})$, iv) $q''(r) = \mathcal{O}(r^{-1-\delta} q^2)$

Our purpose is to investigate the spectrum of Dirac operators with potentials neither decaying nor diverging. In this paper we consider potentials homogeneous of degree zero.

2 Main results

We always assume that
(V1) $V \in C^\infty(\mathbf{R}^3 \backslash \{0\})$ is homogeneous of degree zero.

Let $V_+ = \max_{\omega \in S^2} V(\omega)$, $V_- = \min_{\omega \in S^2} V(\omega)$, $\Sigma_V = \{\omega \in S^2;\ \nabla V(\omega) = 0\}$ and define the threshold set

$$\tau(H_D) = (V(\Sigma_V) + mc^2) \cup (V(\Sigma_V) - mc^2).$$

Then our first result for the Dirac operator is as follows.

Theorem 2.1 *Suppose that (V-1) holds, m is nonnegative and Σ_V is at most countable. Then*
1) $\sigma(H_D) = (-\infty, V_+ - mc^2] \cup [V_- + mc^2, +\infty)$
2) $\sigma_p(H_D)$ *is an at most countable set whose elements can only accumulate to* $\tau(H_D)$,
3) $\sigma_{sc}(H_D) = \emptyset$.

We can improve the above result when the light speed c is large enough.

Theorem 2.2 *Suppose that (V1) holds and m is positive. Then there exists a positive constant c_0 such that the spectrum of H_D is purely absolutely continuous if $c \geq c_0$.*

For Schrödinger operators

$$H_S = -\frac{1}{2}\triangle + V(x) \quad \text{in } L^2(\mathbf{R}^d), \quad d \geq 3$$

it is well known that the same conclusion is true for more general potentials. In fact Lavine [7] proved that if $(x \cdot D)V \leq 0$ (repulsive), then the spectrum of H_D is purely absolutely continuous. In connection to the study of asymptotic behavior of solutions Herbst [3] has proved a uniform limiting absorption principle for homogeneous potentials of degree zero by use of complex dilation method. Later Agmon, Cruz and Herbst [1] applied Mourre theory to generalize it and Hassel, Melrose and Vasy [2] investigated more general operator from the view point of propagation of singularities.

3 Idea of proof of Theorem 2.1

Define a unitary operator $(Uf)(x) = (h/c)^{-3/2} f(hx/c)$. Then

$$U^{-1}cD_xU = hD_x := p \text{ and } U^{-1}VU = V.$$

Let $H = U^{-1}H_DU = \alpha \cdot p + mc^2\beta + V$ and define a selfadjoint operator A_1, called conjugate operator

$$A_1 = \frac{1}{2h}\left(H_0^{-1}px + xpH_0^{-1}\right) - \frac{\gamma}{2h}(G + G^*)$$

where $H_0 = \alpha \cdot p + mc^2\beta$, $\gamma > 0$ is a small parameter and $G = E_h^{-2} p \cdot \nabla(\tilde{V}(x))$. Here

$$E_h = \begin{cases} \sqrt{|hD|^2 + m^2c^4}, & \text{if } m > 0 \\ \sqrt{|hD|^2 + 1}, & \text{if } m = 0. \end{cases}$$

and $\tilde{V}(x) = (1 - \chi_\varepsilon(x))|x|^2 V(x)$ with $\chi_\varepsilon \in C_0^\infty(\mathbf{R}^3)$ satisfying

$$\chi_\varepsilon(x) = \begin{cases} 0 & |x| \geq 2\varepsilon \\ 1 & |x| \leq \varepsilon \end{cases}, \quad \varepsilon > 0.$$

If $m > 0$,then $\dfrac{hD}{H_0} = hDH_0/E_h^2$ is called the classical velocity operator ([10]).

Theorem 2.1 is a simple consequence of Mourre estimates [9] outside the threshold $\tau(H)$.

Theorem 3.1 *Let $I \subset \mathbf{R}\backslash\tau(H)$. Then there exist positive constant $\delta > 0$ and a compact operator K in $L^2(\mathbf{R}^3;\ \mathbf{C}^4)$ such that*

$$E_H(I)[iH, A_1]E_H(I) \geq \delta E_H(I) + K,$$

where E_H is the spectral projection of H.

The most essential step to prove Mourre estimate is the following result.

Lemma 3.2
(i) $\quad [iH_0, A_1] = -h^2\Delta E_h^{-2}$,
(ii) if ε, γ and h are small enough, then there exists a positive constant C such that

$$[iH, A_1] \geq \frac{1}{2}E_h^{-1}\left\{|hD|^2 + W(x, D)\right\}E_h^{-1},$$
$$W(x, D) = \gamma|x|^2|\nabla V(x)|^2 - C(\gamma + h)|hD|,$$

(iii) $\quad [[H, iA_1], iA_1] \leq C$

Once the Mourre estimate is verified, it holds that the (local) limiting absorption principle.

Theorem 3.3 *If $J \subset\subset \mathbf{R}\backslash\{\tau(H) \cup \sigma_p(H)\}$, then for any $s > 1/2$,*

$$\sup_{\lambda\in J,\ \varepsilon>0} \|\langle x\rangle^{-s}(H - \lambda \mp i\varepsilon)^{-1}\langle x\rangle^{-s}\|_{\mathbf{B}(L^2)} < \infty.$$

4 Uniform limiting absorption principle

To prove Theorem 2.2, we shall establish a uniform limiting absorption principle which is derived from the one for relativistic Schrödinger operators H_R.

Define a unitary operator $(Uf)(x) = (mc)^{-3/2}f(mcx)$. Then

$$U^{-1}cD_xU = mc^2D_x \text{ and } U^{-1}VU = V.$$

Then

$$U^{-1}H_DU = mc^2\left\{\alpha\cdot D_x + \beta + \frac{1}{mc^2}V\right\}$$

Let $p = D_x$ and

$$H = \alpha\cdot p + \beta + \frac{1}{mc^2}V, \quad H_R = \sqrt{|p|^2+1} + \frac{1}{mc^2}V.$$

It is known that there exists a unitary operator T, called Foldy-Wouthuysen-Tani transform defined later explicitly such that

$$T(\alpha\cdot p + \beta)T^{-1} = \begin{pmatrix} EI_2 & 0 \\ 0 & -EI_2 \end{pmatrix},$$

where $E = \sqrt{|p|^2+1}$ is called the relativistic Schrödinger operators. Consider two unitary operator $T_\pm$ on $L^2(\mathbf{R}^3;\ \mathbf{C}^4)$

$$T_\pm = \sqrt{\frac{E+1}{2E}}I_4 \pm \sqrt{\frac{E-1}{2E}}\beta\frac{\alpha\cdot D}{|D|} = \frac{1}{\sqrt{2E}}\left(\sqrt{E+1}I_4 \pm \beta\frac{\alpha\cdot D}{\sqrt{E+1}}\right).$$

Then it holds that

$$T_+^* = T_-, \quad T_+T_- = I_4.$$

$T = T_+$ is called Foldy-Wouthuysen-Tani transform T ([10]).

The result in Theorem 2.1 can be improved if we consider H as a perturbation of a pair of relativistic Schrödinger operators because

$$H_R = \sqrt{|D|^2+1} + \frac{1}{mc^2}V(x)$$

has a nice property as follows.

Theorem 4.1 *Let $m > 0$ and $H_R = \sqrt{|D|^2+1} + \frac{1}{mc^2}V$. Suppose that (V1). Then there exist positive constants L and c_0 such that*

$$\sup_{\lambda\in\mathbf{R},\varepsilon>0} \|\langle x\rangle^{-1}(H_R - \lambda \mp i\varepsilon)^{-1}\langle x\rangle^{-1}\|_{\mathbf{B}(L^2)} < L, \ \ for\ all\ c > c_0$$

Corollary 4.2 *If c is large enough, then the spectrum of H_R is purely absolutely continuous.*

We can show that the conclusion of Corollary 4.2 is true without limitation on c by using the Mourre theory and absence of eigenvalues of H_R. In applying a perturbation argument, however, we need a uniform estimate as in Theorem 4.1

Theorem 4.3 *Suppose that (V1) . Then there exists a positive constant* c_0 *such that if* $c > c_0$*, then it holds that*

$$\sup_{\lambda \in \mathbf{R}, \varepsilon > 0} \|\langle x \rangle^{-1}(THT^{-1} - \lambda \mp i\varepsilon)^{-1}\langle x \rangle^{-1}\|_{\mathbf{B}(L^2)} < \infty.$$

Corollary 4.4 *If* c *is large enough, then the spectrum of* H_D *is purely absolutely continuous.*

5 Proof of Theorem 4.3

Approximate H by a pair of $\pm H_R$ via FWT transform.

$$\begin{aligned} & T_+(\alpha \cdot D + \beta + V)T_- \\ &= \begin{pmatrix} \sqrt{|D|^2+1} + \frac{1}{mc^2}V & 0 \\ 0 & -(\sqrt{|D|^2+1} - \frac{1}{mc^2}V) \end{pmatrix} + W, \end{aligned} \tag{5.1}$$

where

$$W = \frac{1}{mc^2}\left\{T_+VT_- - V\right\}.$$

Let $\tilde{H} = T_+(\alpha \cdot D + \beta)T_- + (mc^2)^{-1}V$. Then $T_+HT_- = \tilde{H} + W$. We shall use the following results to handle the remainder term W.

Lemma 5.1 *Suppose*

$$\sup_{\lambda \in J,\ \varepsilon > 0} \|\langle x \rangle^{-s}(\tilde{H} - \lambda \mp i\varepsilon)^{-1}\langle x \rangle^{-s}\| = M < \infty.$$

If $\|\langle x \rangle^{s} W(x) \langle x \rangle^{s}\| < \frac{1}{M}$*, then the same conclusion is valid for* H *replaced by* $H + W$.

Lemma 5.2 (**Remainder estimate**): *Let* $\tilde{V}(x) = (1 - \chi_\varepsilon(x))V(x)$. *Then*

$$T_+\tilde{V}T_- - \tilde{V} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}.$$

where

$$\langle x\rangle W_{jj}\langle x\rangle \in \mathcal{L}(\mathcal{H}), \quad \langle x\rangle^{1/2}W_{jk}\langle x\rangle^{1/2} \in \mathcal{L}(\mathcal{H}), \ j \neq k.$$

Proof: Let

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \text{with } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then it holds that

$$T_\pm = a + b, \quad a = \begin{pmatrix} A_+ & 0 \\ 0 & A_+ \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \pm A_- \\ \mp A_- & 0 \end{pmatrix}.$$

Here

$$A_+ = \frac{1}{\sqrt{2E}}\sqrt{E+1}, \quad A_- = \frac{\sigma\cdot D}{\sqrt{2E}\sqrt{E+1}}.$$

Note that $a^* = a, \ b^* = -b$ and

$$\begin{aligned} T_+VT_- - V &= \frac{1}{2}\left[[a,V],a\right] - \frac{1}{2}\left[[b,V],b\right] \\ &\quad + \frac{1}{2}\left([a,V]b + b[a,V]\right) - \frac{1}{2}\left([b,V]a + a[b,V]\right). \quad (5.2) \end{aligned}$$

To derive the conclusion we use a calculus of ΨDO. Let $g = (|x|^2+1)^{-1}dx^2 + (|\xi|^2+1)^{-1}d\xi^2$ be a metric on $\mathbf{R}^{2d}$. A smooth function $a(x,\xi)$ defined on $\mathbf{R}^d \times \mathbf{R}^d$ belongs to a class of symbols $S^m_\ell(g)$ if

$$\forall\alpha, \ \beta, \ |\partial_x^\beta\partial_\xi^\alpha a(x,\xi)| \le C\langle x\rangle^{\ell-|\beta|}\langle\xi\rangle^{m-|\alpha|}, \quad (x,\xi) \in \mathbf{R}^d \times \mathbf{R}^d$$

Define the pseudo-differential operator $\mathrm{OP}(a)$ with symbol a by

$$\mathrm{OP}(a)u(x) = (2\pi)^{-d}\int e^{i(x-y)\cdot\xi}a\left(\frac{x+y}{2},\xi\right)u(y)dyd\xi, \ u \in C_0^\infty(\mathbf{R}^d)$$

It is easily verified that W_{jk} is a 2×2 matrix-valued pseudo-differential operator with symbol

$$W_{jj}(x,\xi) \in S^{-2}_{-2}(g), \quad W_{jk}(x,\xi) \in S^{-1}_{-1}(g), \ j \neq k.$$

Q.E.D.

Let $\hat{H} = \begin{bmatrix} H_+ & W_{12} \\ W_{21} & H_- \end{bmatrix}$ where

$$H_+ = E + \frac{1}{mc^2}V, \quad H_- = -(E - \frac{1}{mc^2}V)$$

Then $\left(U^{-1}H_D U - z\right) u = f$ is equivalent to

$$(\hat{H} - \frac{1}{mc^2}z) \begin{bmatrix} u_+ \\ u_- \end{bmatrix} = \frac{1}{mc^2} \begin{bmatrix} f_+ \\ f_- \end{bmatrix},$$

which means that if $\zeta = z(mc^2)^{-1}$,

$$(H_+ - \zeta)u_+ - W_{12}(H_- - \zeta)^{-1}W_{21}u_+ = \frac{1}{mc^2}\left[f_+ - W_{12}(H_- - \zeta)^{-1}f_-\right],$$

$$(H_- - \zeta)u_- - W_{21}(H_+ - \zeta)^{-1}W_{12}u_- = \frac{1}{mc^2}\left[f_- - W_{21}(H_+ - \zeta)^{-1}f_+\right]$$

In virtue of

$$\sigma(H_+) \cap \sigma(H_-) = \emptyset, \quad if \ \frac{1}{mc^2}(V_+ - V_-) < 2$$

it follows from Lemma 5.1 with $s = 0$ and $s = 1$ that

$$\|\langle x\rangle^{-1}u_\pm\| \leq C\left(\|\langle x\rangle f_+\| + \|\langle x\rangle f_-\|\right).$$

6 Proof of Theorem 4.1

We shall apply weakly conjugate operator method to H_R (a weak version of Mourre estimates).

This method is applied for many cases. One of them treats the free Dirac operator with positive mass $\alpha \cdot D + m\beta$ (Iftimocvici and Măntoiu [6].)

In our case we consider relativistic Schrödinger operators with homogeneous potential.

$$A_2 = \frac{1}{2}\left(E^{-1}D_x \cdot x + x \cdot D_x E^{-1}\right), \quad E = (|D|^2 + 1)^{1/2}$$

Lemma 6.1 *There exist positive numbers c_0 and δ such that*

$$\langle [iH_R, A_2]u, u\rangle \geq \delta \|B_0^{1/2}u\|^2$$
$$B_0^{-1/2}[[H_R, iA_2], iA_2]B_0^{-1/2} \in \mathbf{B}(L^2(\mathbf{R}^3)),$$

where $B_0 = |D|^2(|D|^2+1)^{-1}$.

Proof: A simple computation gives

$$[iE, A_2] = |D|^2(|D|^2+1)^{-1} = B_0 > 0.$$

Moreover

$$\begin{aligned} 2[iV, A_2] &= E^{-1}D\cdot xiV + D\cdot xE^{-1}iV - iVx\cdot DE^{-1} - iVE^{-1}x\cdot D \\ &\quad + 3E^{-1}V - 3VE^{-1} \\ &= 2E^{-1}D\cdot ixV - 2iVx\cdot DE^{-1} + [D\cdot x, E^{-1}]iV - iV[E^{-1}, x\cdot D] \\ &\quad + 3E^{-1}V - 3VE^{-1} \end{aligned}$$

where $[E^{-1}, ix\cdot D] = [E^{-1}, D\cdot ix] = E^{-1}[ix\cdot D, E]E^{-1} = B_0E^{-1}$. Let $V_1(x) = \chi(x)V(x)$ and $V_2(x) = (1-\chi(x))V(x)$ with $\chi \in C_0^\infty(\mathbf{R}^3)$. Note that

$$\langle E^{-1}D_j u, x_jV_1u\rangle = \langle \frac{D_j}{|D|}B_0^{1/2}u, x_j\langle x\rangle V_1\langle x\rangle^{-1}B_0^{-1/2}B_0^{1/2}u\rangle.$$

By virtue of

$$B_0^{-1/2} = \sqrt{1+|D|^{-2}} \leq 1 + |D|^{-1},$$

and Hardy's inequality

$$\||x|^{-1}u\|_{L^2(\mathbf{R}^d)} \leq C_d\||D|u\|_{L^2(\mathbf{R}^d)} \text{ with } C_d = \frac{2}{d-2},$$

it holds that

$$\|\langle x\rangle^{-1}B_0^{-1/2}v\| \leq (1+C_3)\|v\|,$$

Since

$$\sup|x_j\langle x\rangle V_1(x)| \leq \max_{\mathrm{supp}\chi}\{|x|(1+|x|)\}\,\|V\|_\infty,$$

if we take $\mathrm{supp}\chi \subset \{|x| \le 1/2\}$, we obtain

$$\left|\langle E^{-1}D \cdot xiV_1 u, u\rangle\right| \le \frac{3}{4}(1+C_0)\|V\|_\infty \|B_0^{1/2}u\|^2.$$

Similarly

$$\left|\langle [D\cdot x, E^{-1}]iV_1 u, u\rangle\right| \le \frac{3}{2}(1+C_0)\|V\|_\infty \|B_0^{1/2}u\|^2,$$

$$\left|\langle E^{-1}V_1 u, u\rangle\right| \le \frac{9}{4}(1+C_0)^2\|V\|_\infty \|B_0^{1/2}u\|^2.$$

To deal with V_2 we now use the identities

$$\begin{aligned}
-[iV_2, A_2] =& \frac{1}{2}\left([E^{-1}, iV_2]D\cdot x + x\cdot D[E^{-1}, iV_2]\right) \\
&+ \frac{1}{2}\left(E^{-1}[D\cdot x, V_2] - [iV_2, x\cdot D]\right), \\
[E^{-1}, iV_2] =& E^{-1}[iV_2, E]E^{-1}, \\
[E, iV_2] =& E^{-1}D\cdot\nabla V + K(x,D),
\end{aligned}$$

where K is a ΨDO with symbol satisfying

$$K(x,\xi) = (\xi^2 + m^2c^2)^{-3/2}\xi^2\Delta V + (\xi^2+m^2c^2)^{-1/2}\Delta V + \cdots,$$

$$\forall\alpha,\ \beta,\ |\partial_x^\beta\partial_\xi^\alpha K(x,\xi)| \le C\langle x\rangle^{-2-|\beta|}\langle\xi\rangle^{-|\alpha|}, \quad (x,\xi)\in\mathbf{R}^d\times\mathbf{R}^d.$$

Thus, it holds that

$$|\langle E^{-1}K(x,D)E^{-1}u, u\rangle| \le C\|B_0^{1/2}u\|^2.$$

Therefore if we take c to be large, then

$$\langle [iH_R, A_2]u, u\rangle \ge (1 - C_1(mc^2)^{-1}\|V\|_\infty)\|B_0^{1/2}u\|^2 \ge \delta\|B_0^{1/2}u\|^2.$$

Q.E.D.

Let

$$F_\varepsilon = \langle u, (H_R - \lambda \mp i\varepsilon B)^{-1}u\rangle$$

where $B = [H, iA_2]$. Then it holds that

$$\left|\frac{dF_\varepsilon}{d\varepsilon}\right| \le C\varepsilon^{-1/2}|F_\varepsilon|^{1/2}\|\langle x\rangle u\|_{L^2}.$$

Integrating it on $[\varepsilon, \varepsilon_0] \subset (0, 1)$ with aid of a Gronwall-type lemma and taking the limit

$$\lim_{\varepsilon \to +0} F_\varepsilon = F_0,$$

we can conclude that

$$|F_0| \leq C \|\langle x \rangle u\|_{L^2}^2.$$

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