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Remarks on Scattering on Scattering Manifolds

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Abstract

In this talk, we discuss scattering theory for a class of manifolds. We consider the asymptotic completeness and the microlocal properties of the scattering matrix. The space we consider is called *scattering manifolds* following R. Melrose, and we construct a time-dependent scattering theory for Schrödinger operators on such manifolds. In particular, we discuss an alternative approach to a theorem by R. Melrose and M. Zworski on the microlocal properties of the *absolute scattering matrix.* This work is partly in progress, and several theorems are preliminary.

Model: We consider an n-dimensional noncompact manifold (without boundary):

$$M = M_0 \cup M_\infty$$

where M_0 is relative compact, and M_∞ is diffeomorphic to $(1,\infty)\times\partial M$, where ∂M is a closed manifold without boundary. We consider ∂M as a boundary of M at infinity. We fix an identification map:

$$I : M_\infty \cong (1,\infty)\times\partial M \in (r,\theta), \quad r\in(1,\infty), \theta\in\partial M.$$

Let g^∂ be a Riemannian metric on ∂M, and we denote

$$g^\partial = \sum_{i,j} g^\partial_{ij}(\theta)d\theta^i d\theta^j, \quad \theta\in\partial M.$$

Definition: A Riemannian metric g^{cn} on M is called *conic* if it has the following form:

$$g^{cn} = dr^2 + r^2 g^\partial \quad \text{on } M_\infty,$$

where we identify M_∞ with $(1,\infty)\times\partial M$ as above.

Example 0: (Euclidean space) $M=\mathbb{R}^n$, $\partial M = S^{n-1}$, $g^\partial = d\theta^2$ is the surface metric on S^{n-1}. Then $g^{cn} = dr^2 + r^2 d\theta^2$ is the standard flat metric on $\mathbb{R}^n$ in the polar coordinate on $M_\infty = \{x \mid |x|>1\}$. The identification map is

$$I : r\theta \in M_\infty \mapsto (r,\theta) \in (1,\infty)\times S^{n-1}.$$

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This is a typical example and we should keep this in mind in the following argument.

Example 1: (Conic metric on $\mathbb{R}^n$) Let M and ∂M as in Example 0, but we introduce a different metric on S^{n-1}. Then we have a different conic metric structure on $\mathbb{R}^n$. For example, we can set $g^\partial = \alpha d\theta^2$ with $\alpha > 0$, and we have a different geometric structure.

Definition: A Riemannian metric g on M is called *scattering metric* if

$$g = g^{cn} + m,$$

where g^{cn} is the conic metric, and m is a symmetric 2-form such that

$$m = m^0(r,\theta)dr^2 + r\sum_{j=1}^{n-1} m_j^1(r,\theta)(drd\theta + d\theta dr) + r^2 \sum_{i,j=1}^{n-1} m_{ij}^2(r,\theta)d\theta^i d\theta^j$$

on M_∞, and the coefficients satisfy

$$\left|\partial_r^k \partial_\theta^\alpha m_*^\ell(r,\theta)\right| \le C_{k\alpha} r^{-\mu_\ell - k}, \quad (r,\theta) \in M_\infty$$

for any $k, \alpha, \ell = 0, 1, 2$ with $\mu_\ell > 0$.

Scattering metric was defined originally by R. Melrose [2], but here we use an equivalent, but different definition. (This formulation was introduced in [1]). We will assume the metric perturbation m is short-range type in the following sense:

Definition: A metric g on M is called *short-range type* if

$$\mu_0 > 1, \quad \mu_1 > 1/2, \quad \mu_2 > 0.$$

Let $\triangle_g$ be the Laplace-Beltrami operator on M corresponding to the Riemannian metric, i.e.,

$$\triangle_g = \frac{1}{\sqrt{G(x)}} \sum_{j,k=1}^n \partial_{x_j}\, g^{jk}(x) \sqrt{G(x)}\, \partial_{x_k}$$

where $G(x) = \det(g_{jk}(x))$ and $(g^{jk}) = (g_{jk})^{-1}$.

Definition: A potential function $V \in C^\infty(M;\mathbb{R})$ is called *short-range type* if there is $\mu_3 > 1$ such that for any α and k,

$$\left|\partial_r^k \partial_\theta^\alpha V(r,\theta)\right| \le C_{k\alpha} r^{-\mu_3 - k}, \quad (r,\theta) \in M_\infty.$$

In the following, we assume g and V are short-range type. We set

$$H = -\triangle_g + V(x) \quad \text{on } \mathcal{H} = L^2(M, \sqrt{G}\, dx).$$

Proposition 1. *H is essentially self-adjoint on $C_0^\infty(M)$. Moreover, $\sigma_{ess}(H) = [0,\infty)$; $\sigma_p(H)$ is discrete with possible accumulation points only at 0; $\sigma_{sc}(H) = \emptyset$; and $\sigma_{ac}(H) = [0,\infty)$.*

Idea of Proof. Let $j(r) \in C^\infty(\mathbb{R})$ be a smooth cut-off function such that

$$j(r) = \begin{cases} 1 & (r \geq 3/2), \\ 0 & (r \leq 1). \end{cases}$$

We use the Mourre theory with the conjugate operator:

$$A = \frac{1}{2i}\left(j(r) r \frac{\partial}{\partial r} + \frac{\partial}{\partial r} j(r) r + \frac{1}{2} j(r) r \partial_r (\log G(x)) \right)$$

on M^∞. Then the rest of the argument is similar to the Euclidean case. □

Scattering Theory: We first construct a *free system.* We might use $-\triangle_{g^{cn}}$ as the free system, but this operator itself is not very easy to handle. So, instead, we set

$$\begin{aligned} &H_{fr} = -\frac{\partial^2}{\partial r^2} \quad \text{on} \quad M_{fr} = \mathbb{R} \times \partial M, \\ &\mathcal{H}_{fr} = L^2(M_{fr}, dr \cdot \sqrt{g^\partial} d\theta) \\ &J : \mathcal{H}_{fr} \longrightarrow \mathcal{H}, \text{ where } J\varphi(r,\theta) = \begin{cases} 0 & \text{on } M_\infty^c \\ j(r)\left(\det g^\partial(\theta)/G(r,\theta)\right)^{1/4} \varphi(r,\theta) & \text{on } M_\infty. \end{cases} \end{aligned}$$

J is defined so that J is isometry on $L^2([3/2,\infty) \times \partial M)$. Note, asymptotically, $J\varphi \sim r^{-(n-1)/2}\varphi$ as $r \longrightarrow \infty$. In fact, for Examples 0 and 1, we have

$$J\varphi(r,\theta) = j(r) r^{-(n-1)/2} \varphi(r,\theta) \quad \text{for } \varphi \in L^2(\mathbb{R} \times S^{n-1}).$$

In this case, if we set $\varphi = e^{ikr}$, a generalized eigenfunction of H_{fr}, then

$$J\varphi = j(r) r^{-(n-1)/2} e^{ikr},$$

which is a spherical wave (generalized eigenfunction of $\triangle$ for large r).

We then set the wave operators:

$$W_\pm := \operatorname*{s-lim}_{t \to \pm\infty} e^{itH} J e^{-itH_{fr}} \; : \; \mathcal{H}_{fr} \longrightarrow \mathcal{H}.$$

The existence of $W_\pm$ is easy to show by the standard Cook-Kuroda method. We note g^{jk} has the form:

$$(g^{ij}) = (g_{ij})^{-1} = \begin{pmatrix} 1 + a_0 & r^{-1} a_1^t \\ r^{-1} a_1 & r^{-2} g_\partial + r^{-2} a_2 \end{pmatrix}$$

in the (r,θ) coordinate, where $\partial_r^k \partial_\theta^\alpha a_0 = O(r^{-1-\mu-k})$ and $\partial_r^k \partial_\theta^\alpha a_j = O(r^{-\mu-k})$ for $j = 1,2$ with some $\mu > 0$. Here we denote $g_\partial = (g^\partial)^{-1}$.

We then set

$$\mathcal{H}_{fr,\pm} = \left\{ \varphi \in \mathcal{H}_{fr} \;\middle|\; \operatorname{supp} \hat{\varphi} \subset \mathbb{R}_\pm \times \partial M \right\},$$

where $\hat{\varphi}$ is the Fourier transform of φ in r, i.e.,

$$\hat{\varphi}(\rho,\theta) = (\mathcal{F}\varphi)(\rho,\theta) = \int_{-\infty}^{\infty} e^{-i\rho r}\varphi(r,\theta)dr.$$

Then it is not difficult to see by the stationary phase method that

$$W_\pm(\mathcal{H}_{fr,\mp}) = 0,$$

and hence it is natural to consider $W_\pm : \mathcal{H}_{fr,\pm} \to \mathcal{H}$.

Theorem 2. *$W_\pm$ are isometry from $\mathcal{H}_{fr,\pm}$ to $\mathcal{H}$, and they are complete, i.e.,* Ran $W_\pm = \mathcal{H}_c(H)$. *Hence, in particular, the scattering operator defined by*

$$S = W_+^* W_- \ : \ \mathcal{H}_{fr,-} \to \mathcal{H}_{fr.+}$$

is unitary.

Idea of Proof. Let

$$\begin{aligned} H_\partial &= -\frac{1}{\sqrt{G(x)}}\sum_{j,k=1}^{n-1} \partial_{\theta_j} j(r) g_\partial^{jk}(\theta)\sqrt{G(x)}\,\partial_{\theta_k} \\ &= -\sum_{j,k=1}^{n-1} \partial_{\theta_j} j(r) g_\partial^{jk}(\theta)\,\partial_{\theta_k} + \text{(lower order terms)}. \end{aligned}$$

This operator is, roughly speaking, the pull-back of the Laplace-Beltrami operator on ∂M to M. By the Mourre theory, we can show

$$\langle j(r)r\rangle^{-\alpha}(H-\lambda\pm 0)^{-1}\langle j(r)r\rangle^{-\alpha} \in B(\mathcal{H}), \quad \lambda \in \mathbb{R}_+\setminus\sigma_p(H), \alpha > 1/2,$$

but these are not sufficient to show the completeness, since perturbation terms: $r^{-1}a_1\partial_r\partial_\theta$, $r^{-2}a_2\partial_\theta\partial_\theta$ are only of $O(r^{-\mu})$, $\mu > 0$, with respect to H. Instead, we show

$$\langle j(r)r\rangle^{-\alpha}(H_\partial+1)(H-\lambda\pm i0)^{-1}(H_\partial+1)^{-1}\langle j(r)r\rangle^{-\alpha} \in B(\mathcal{H}), \quad \lambda\in\mathbb{R}_+\setminus\sigma_p(H).$$

These estimates are proved by resolvent equations and commutator computations. These imply that

$$(H-\lambda\pm i0)^{-1} \ : \ (H_\partial+1)^{-1}\langle j(r)r\rangle^{-\alpha}\mathcal{H} \ \mapsto \ (H_\partial+1)^{-1}\langle j(r)r\rangle^{\alpha}\mathcal{H}$$

is bounded, and this is sufficient to show the completeness by using the abstract stationary scattering theory. □

Scattering Matrix: By the intertwining property, we have

$$H_{fr}S = SH_{fr} \ : \ \mathcal{H}_{fr,-} \to \mathcal{H}_{fr,+},$$

and hence

$$\rho^2(\mathcal{F}S\mathcal{F}^{-1}) = (\mathcal{F}S\mathcal{F}^{-1})\rho^2 \;:\; \hat{\mathcal{H}}_{fr,-} \to \hat{\mathcal{H}}_{fr,+},$$

where $\hat{\mathcal{H}}_{fr,\pm} = L^2(\mathbb{R}_\pm \times \partial M)$. Thus, $(\mathcal{F}S\mathcal{F}^{-1})$ commutes with multiplication by functions of ρ, and then we learn that $(\mathcal{F}S\mathcal{F}^{-1})$ is decomposed as

$$(\mathcal{F}S\mathcal{F}^{-1})\varphi(\rho,\theta) = (S(\rho)\varphi(-\rho))(\theta), \quad \rho > 0, \varphi \in \hat{\mathcal{H}}_{fr,+}$$

with $S(\rho) : L^2(\partial M) \to L^2(\partial M)$, unitary. $S(\rho)$ is called the *scattering matrix.*

Melrose-Zworski Theorem: Let

$$h(\theta,\omega) = \sum_{j,k} g_\partial^{jk}(\theta)\,\omega_j\omega_k \quad \text{for } (\theta,\omega) \in T^*\partial M$$

be the classical Hamiltonian on ∂M, and let $\exp tH_{\sqrt{h}}$ be the Hamilton flow generated by $\sqrt{h}$, which is in fact the geodesic flow. Then we can show

Theorem 3. *$S(\rho)$ is an FIO corresponding to the canonical transform* $\exp \pi H_{\sqrt{h}}$. *In particular,*

$$WF(S(\rho)\varphi) = \exp \pi H_{\sqrt{h}}(WF(\varphi)), \quad \varphi \in L^2(\partial M).$$

This result is a generalization of a result by R. Melrose and M. Zworksi [3], though they used different definition of the scattering matrix, which is called the *absolute scattering matrix.* The absolute scattering matrix is defined as follows: Let ψ be a generalized eigenfunction of H: $H\psi = \rho^2\psi$. Then ψ has an asymptotic form:

$$\psi(r,\theta) \sim r^{-(n-1)/2}(e^{ir\rho}\varphi_+(\theta) + e^{-ir\rho}\varphi_-(\theta)) \quad \text{as } r \to \infty$$

with some $\varphi_\pm \in L^2(\partial M)$. The map:

$$\tilde{S}(\rho) \;:\; \varphi_+ \mapsto \varphi_-$$

is well-defined and $\tilde{S}(\rho)$ is called the absolute scattering matrix since it is defined without using the time-dependent scattering theory. However, we can show

$$\tilde{S}(\rho) = -S(\rho)^{-1}$$

in our notation. As well as the formulation, the proof of Theorem 3 is considerably different from the one by Melrose and Zworski.

Example 0: (revisited) For the Euclidean case, $\exp \pi H_{\sqrt{h}}(\theta,\omega) = (-\theta,-\omega)$. Hence, the singularity of φ is mapped by the scattering matrix to the anti-podal points, which is well-known.

Example 1: (revisited) Let $n = 2$ and we set $g_\partial = \alpha g_0$ with $\alpha > 0$ and $g_0 = d\theta^2$, the standard length on S^1. Then $S(\rho)$ has a different microlocal propagation properties. Namely,

$$WF(S(\rho)\varphi) \subset \big\{\theta \pm \alpha\pi \;\big|\; \theta \in WF(\varphi)\big\}.$$

Classical Scattering Theory for Conic Metric: In order to understand the meaning of the Melrose-Zworski theorem, let us consider the classical scattering for the conic metric. Let $p(r,\theta,\rho,\omega)$ be the classical Hamiltonian for the conic metric:

$$p(r,\theta,\rho,\omega)=\rho^2+\frac{1}{\rho^2}\sum_{j,k}g_{\partial}^{jk}(\theta)\,\omega_j\omega_k,\quad r>0,\rho\in\mathbb{R},(\theta,\omega)\in T^*\partial M.$$

Let $(r(t),\theta(t),\rho(t),\omega(t))$ be the solution to the Hamiltonian equation:

$$\dot r=\frac{\partial p}{\partial\rho},\quad \dot\theta=\frac{\partial p}{\partial\omega},\quad \dot\rho=-\frac{\partial p}{\partial r},\quad \dot\omega=-\frac{\partial p}{\partial\theta},$$

with $r(0)=r_0$, $\theta(0)=\theta_0$, etc. It is easy to see $h(\theta(t),\omega(t))$ is invariant, i.e., $h(\theta(t),\omega(t))=h(\theta_0,\omega_0)=h_0$. Then we can solve equation for (r,ρ) easily to obtain

$$r(t)=\sqrt{4E_0t^2+4r_0\rho_0t+r_0^2},\quad E_0=p(r_0,\theta_0,\rho_0,\omega_0),$$

and (θ,ω) satisfies the equation:

$$\dot\theta=\frac{1}{r^2}\frac{\partial h}{\partial\omega},\quad \dot\omega=-\frac{1}{r^2}\frac{\partial h}{\partial\theta}.$$

So, by changing the time variable $t\mapsto\tau(t)=\int_0^t ds/r(s)^2$, we have

$$(\theta(t),\omega(t))=\exp(\tau(t)H_h)(\theta_0,\omega_0),$$

where $\exp(tH_h)$ is the Hamilton flow generated by h on $T^*\partial M$. As $t\to\pm\infty$, $\tau(t)$ converges to finite values:

$$\lim_{t\to\pm\infty}\tau(t)=\tau_\pm=\frac{1}{2\sqrt{h_0}}\left(\pm\frac{\pi}{2}-\tan\frac{r_0\rho_0}{\sqrt{h_0}}\right).$$

Hence we have

$$\lim_{t\to\pm\infty}(\theta(t),\omega(t))=(\theta_\pm,\omega_\pm)=\exp(\tau_\pm H_h)(\theta_0,\omega_0).$$

Similarly, we can show by straightforward computations,

$$\lim_{t\to\pm\infty}\rho(t)=\rho_\pm=\pm\sqrt{E_0},\quad \lim_{t\to\pm\infty}(r(t)-2t\rho(t))=r_\pm=\pm\frac{r_0\rho_0}{\sqrt{E_0}}$$

This gives us the explicit formula for the (inverse) classical scattering operator:

$$(W_\pm^{cl})^{-1}\ :\ (r_0,\rho_0,\theta_0,\omega_0)\mapsto(t_\pm,\rho_\pm,\theta_\pm,\omega_\pm)=\lim_{t\to\pm\infty}(r(t)-2t\rho(t),\rho(t),\theta(t),\omega(t)).$$

We note that the corresponding free Hamiltonian is simply given by ρ^2, which generates the free motion : $(r,\rho,\theta,\omega)\mapsto(r+2\rho t,\rho,\theta,\omega)$. By the formula, it is easy to show

$$(W_\pm^{cl})^{-1}:(\mathbb{R}_+\times\mathbb{R})\times(T^*\partial M)\ \to\ (\mathbb{R}\times\mathbb{R}_\pm)\times(T^*\partial M)$$

is diffeomorphic, and hence

$$S^{cl} = (W_+^{cl})^{-1} \circ W_-^{cl} \ : \ (\mathbb{R}_- \times \mathbb{R}) \times (T^*\partial M) \ \longrightarrow (\mathbb{R}_+ \times \mathbb{R}) \times (T^*\partial M)$$

is also diffeomorphic. In fact, we can easily show

$$S^{cl} \ : \ (r, \rho, \theta, \omega) \mapsto (-r, -\rho, \exp((\tau_+ - \tau_-)H_h)(\theta, \omega)),$$

and $\tau_+ - \tau_- = \pi/(2\sqrt{h_0})$. In general, we have $\exp(tH_q) = \exp((2t\sqrt{q})H_{\sqrt{q}})$ for $q \geq 0$, and hence we learn

$$\exp((\tau_+ - \tau_-)H_h)(\theta, \omega) = \exp(\pi H_{\sqrt{h}}).$$

Thus we have

$$S^{cl} = (-I) \otimes \exp(\pi H_{\sqrt{h}}),$$

and we realize that the Melrose-Zworski theorem is a quantization of this observation.

Scattering Calculus: In the proof of Theorem 3, we use the *scattering caluculus* following Melrose [2], but again in a quite different formulation. For $a \in C_0^\infty(T^*(\mathbb{R}_+ \times \partial M))$ (or $\in C_0^\infty(T^*(\mathbb{R} \times \partial M))$, we denote the *scattering quantization* by

$$A = a(\hbar r, \theta, D_r, \hbar D_\theta), \quad \hbar > 0.$$

Note the difference of the location of the semiclassical parameter $\hbar > 0$ from the usual semiclassical quantization $a(r, \theta, \hbar D_r, \hbar D_\theta)$. We identify $\mathbb{R}_+ \times \partial M$ with M_∞, and we consider A as an operator on $L^2(M, \sqrt{G}\, dx)$. For such an operator A, we consider

$$A(t) = e^{itH_{fr}} J^* e^{-itH} A\, e^{itH} J e^{-itH_{fr}}, \quad t \in \mathbb{R}.$$

$A(t)$ satisfies the Heisenberg equation:

$$\frac{d}{dt} A(t) = i[T(t), A(t)] + (\text{lower order error terms})$$

where

$$T(t) = e^{itH_{fr}}(HJ - JH_{fr})e^{-itH_{fr}} \sim j(r - 2tD_r) \cdot \frac{h(\theta, D_\theta)}{(r - 2tD_r)^2}$$

as $r \longrightarrow \infty$. We can construct the asymptotic solution to the Heisenberg equation:

$$A(t) = b_\hbar^t(\hbar r, \theta, D_r, \hbar D_\theta) \quad \text{where} \quad b_\hbar^t \sim b_0(\hbar^{-1}t; r, \theta, \rho.\omega) + O(\hbar),$$

and b_0 can be computed explicitly using the classical flow. We let $t \longrightarrow \pm\infty$ and we learn

$$\lim_{t \to \pm\infty} A(t) = W_\pm^* A W_\pm \sim b_\hbar^\pm(\hbar r, \theta, D_r, \hbar D_\theta),$$

where $b_\hbar^\pm \sim (a \circ W_\pm^{cl})(r, \theta, \rho, \omega) + O(\hbar)$. Using this procedure again, we learn

$$SAS^{-1} = c_\hbar(\hbar r, \theta, D_r, \hbar D_\theta), \quad \text{where} \quad c_\hbar \sim a \circ (S^{cl})^{-1} + O(\hbar).$$

If $A = a_1(D_r)\, a_2(\theta, \hbar D_t h)$, then we learn

$$SAS^{-1} \sim a_1(-D_r)(a_2 \circ \exp(\pi H_{\sqrt{\hbar}}))(\theta, D_\theta),$$

and hence

$$S(\rho) a_x(\theta, \hbar D_\theta) S(\rho)^{-1} \sim (a_2 \circ \exp(\pi H_{\sqrt{\hbar}}))(\theta, D_\theta).$$

Then Theorem 3 follows from an *inverse Egorov theorem.*

Finally we remark that this calculus can also be used to show the propagation properties of the *scattering wave front set* of Melrose, but we omit the detail here.

References

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