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Infinite dimensional stochastic processes associated with the Exotic Laplacians

UN CIG JI*
DEPARTMENT OF MATHEMATICS
RESEARCH INSTITUTE OF MATHEMATICAL FINANCE
CHUNGBUK NATIONAL UNIVERSITY
CHEONGJU, 361-763 KOREA
E-MAIL: uncigji@cbucc.chungbuk.ac.kr

AND

KIMIAKI SAITÔ†
DEPARTMENT OF MATHEMATICS
MEIJO UNIVERSITY
NAGOYA 468, JAPAN
E-MAIL: ksaito@ccmfs.meijo-u.ac.jp

Abstract

In this paper we present a construction of an infinite dimensional separable Hilbert space $H_{\mathrm{E,a}}$ on which the Exotic trace plays as the usual trace and give an infinite dimensional stochastic process associated with the Exotic Laplacian by extending recent results in [13]. This implies that the Exotic Laplacian plays as the Gross Laplacian in the Boson Fock space $\Gamma(H_{\mathrm{E,a}})$ over the Hilbert space $H_{\mathrm{E,a}}$. The Hilbert space $H_{\mathrm{E,a}}$ is directly constructed by using the Exotic trace. Motivated by the constructions in [2] and extending the results in [13], we introduce an example of $H_{\mathrm{E,a}}$ as a direct sum of some Hilbert spaces with the norm induced from the Exotic trace. Then the stochastic process associated with the Exotic Laplacian is given by an infinite dimensional Brownian motion based on an orthonormal basis for $H_{\mathrm{E,a}}$.

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1 Introduction

The Exotic Laplacian was introduced by L. Accardi [4]. It also has been discussed in [1] and others. The Lévy Laplacian introduced in [19] is one of Exotic Laplacians. The Lévy Laplacian has been studied by many authors from several different aspects in [3, 5, 20, 21, 25] and references cited therein. In particular, the Lévy Laplacian was distinguishably studied in white noise theory [7, 8, 16, 17, 18, 26, 27, 28]. Recently, in [2], the Cesàro Hilbert space associated with the Lévy Laplacian was constructed by completing a pre-Hilbert bundle. We also obtained a similarity between the Gross Laplacian and the Lévy Laplacian in [13].

Main purpose of this paper is to construct an infinite dimensional separable Hilbert space $H_{\mathrm{E,a}}$ on which the Exotic trace plays as the usual trace, generalizing recent results in our previous paper [13]. This implies that the Exotic Laplacian plays as the Gross Laplacian in the Boson Fock space $\Gamma(H_{\mathrm{E,a}})$ over the Hilbert space $H_{\mathrm{E,a}}$. The Hilbert space $H_{\mathrm{E,a}}$ is directly constructed by using the Exotic trace. Extending the construction in [13], we introduce an example of H_{L} as a direct sum of some Hilbert spaces with the norm induced from the Exotic trace. However this space is slightly different from the Cesàro Hilbert space, since the norm of the Cesàro Hilbert space is not exactly equal to the Exotic norm (Remark 2.2). Then the stochastic process associated with the Exotic Laplacian is given by an infinite dimensional Brownian motion based on an orthonormal basis for $H_{\mathrm{E,a}}$. Thus the Exotic Laplacian strongly depends on the space in which the associated infinite dimensional Brownian motion moves.

The paper is organized as follows. In Section 2, we construct an infinite dimensional Hilbert space $H_{\mathrm{E,a}}$ on which the Exotic trace plays as the usual trace. In addition, motivated by the Cesàro Hilbert space in [2], we introduce an example of the space $H_{\mathrm{E,a}}$. In Section 3, we give a nuclear rigging of Fock spaces based on the space $H_{\mathrm{E,a}}$. In Section 4, we give a similarity between the Gross Laplacian and the Exotic Laplacian. Precisely, we prove that the Exotic Laplacian coincides with the Gross Laplacian acting on some domain in $\Gamma(H_{\mathrm{E,a}})$. Based on this result, we give a stochastic process associated with the Exotic Laplacian in Section 5.

2 Basic Gelfand Triples

Let H be a complex Hilbert space and let E be a countably Hilbert nuclear space such that

$$E \subset H \subset E^* \tag{2.1}$$

is a complex Gelfand triple, where H is identified with its dual space. More precisely, we construct the complex triplet as follows: Let $\{\lambda_k\}_{k=1}^{\infty}$ be an increasing sequence of positive real numbers such that

$$1 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda_j^{-2} < \infty. \tag{2.2}$$

For each $p \in \mathbf{R}$, define

$$|\,\xi\,|_{H,p}^2 = \sum_{j=1}^{\infty} \lambda_j^{2p} |\alpha_j|^2, \qquad \xi = \sum_{j=1}^{\infty} \alpha_j e_j \in H$$

where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H. Let $p \geq 0$. Put $E_p = \{\xi \in H\,;\, |\,\xi\,|_{H,p} < \infty\}$ and let E_{-p} be the completion of H with respect to $|\cdot|_{H,-p}$. Then we have a Gelfand triple

$$E = \operatorname*{proj\,lim}_{p\to\infty} E_p \subset H \subset E^* = \operatorname*{ind\,lim}_{p\to\infty} E_{-p}. \tag{2.3}$$

Let J be the conjugate operator on H defined by

$$J\xi = \sum_{i=1}^{\infty} \bar{\alpha}_i e_i, \qquad \xi = \sum_{i=i}^{\infty} \alpha_i e_i \in H.$$

Then the real parts of E, H and E^* are subspaces invariant under the action of J and are denoted by $E_{\mathbf{R}}$, $H_{\mathbf{R}}$ and $E^*_{\mathbf{R}}$, respectively. Then we obtain a real Gelfand triple:

$$E_{\mathbf{R}} \subset H_{\mathbf{R}} \subset E^*_{\mathbf{R}}. \tag{2.4}$$

The inner product on $H_{\mathbf{R}}$ and the canonical $\mathbf{C}$-bilinear form on $E^*_{\mathbf{R}} \times E_{\mathbf{R}}$ is denoted by the notation $\langle \cdot, \cdot \rangle$.

Let $\ell^2 = \left\{(\alpha_j);\ \alpha_j \in \mathbf{C}, \sum_{j=1}^{\infty} |\alpha_j|^2 < \infty\right\}$. The space ℓ^2 is the Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\ell^2}$ given by $\langle \alpha, \beta \rangle_{\ell^2} = \sum_{j=1}^{\infty} \alpha_j \overline{\beta_j}$ for $\alpha = (\alpha_j), \beta = (\beta_j) \in \ell^2$. Take $a \in \mathbf{N}$ arbitrarily. Let $\mathfrak{C}$ be a countable family of double sequences $f : \mathbf{N} \times \mathbf{N} \to \mathbf{C}$ satisfying the following conditions:

(C1) for each $j, k \in \mathbf{N}$, the limit $\lim_{N\to\infty} \frac{1}{N^a} \sum_{n=1}^{N} f(n,j)\overline{f(n,k)}$ exists and for any $\alpha = (\alpha_j) \in \ell^2$ with $\alpha \neq 0$,

$$\sum_{j,k=1}^{\infty} \alpha_j \overline{\alpha_k} \lim_{N\to\infty} \frac{1}{N^a} \sum_{n=1}^{N} f(n,j)\overline{f(n,k)} \gneq 0;$$

(C2) for any $f, g \in \mathfrak{C}$ with $f \neq g$, and all $j, k \in \mathbf{N}$

$$\lim_{N\to\infty} \frac{1}{N^a} \sum_{n=1}^{N} f(n,j)\overline{g(n,k)} = 0;$$

(C3) for all $j \in \mathbf{N}$,

$$\sum_{n=1}^{\infty} f(n,j) e_n \in E^*.$$

We can take $\mathfrak{C}$ to be a class which contains the following sequences as examples:

$f_\nu(n,j) = c_j n^{\frac{a-1}{2}} e^{in\lambda_{\nu,j}}$, $\nu = 1, 2, \ldots$, where all $\lambda_{\nu,j}$, $\nu, j \in \mathbf{N}$, are different numbers with $\lambda_{\nu,1} < \lambda_{\nu,2} < \cdots$, $\nu = 1, 2, \ldots$, and $(c_j)_{j=1}^{\infty} \in \ell^2$ with $c_j \neq 0$ for all j.

Motivated by the Cesàro Hilbert space introduced in [2], we define the spaces H_f for $f \in \mathfrak{C}$ by

$$H_f = LS\Big\{s_j(f);\ j \in \mathbf{N}\Big\},$$

where LS means a linear span and $s_j(f) = \sum_{n=1}^{\infty} f(n,j)e_n$ for each $j \in \mathbf{N}$. We introduce a norm $|\cdot|_0$ by

$$|\xi|_0^2 = \lim_{N\to\infty} \frac{1}{N^a} \sum_{n=1}^{N} |\langle \xi, e_n\rangle|^2, \quad \xi \in H_f.$$

Let $\overline{H_f}$ be the completion of H_f with respect to $|\cdot|_0$. Then the space $\overline{H_f}$ becomes a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathrm{E,a}}$ given by

$$\langle \xi, \eta\rangle_{\mathrm{E,a}} = \lim_{N\to\infty} \frac{1}{N^a} \sum_{n=1}^{N} \langle \xi, e_n\rangle \overline{\langle \eta, e_n\rangle}, \quad \xi, \eta \in \overline{H_f}.$$

Proposition 2.1 *For any $f, g \in \mathfrak{C}$ with $f \neq g$, the Hilbert spaces H_f and H_g are orthogonal.*

PROOF. The proof is immediate from the condition (C2). ∎

Let $H_{\mathrm{E,a}} = \bigoplus_{f\in\mathfrak{C}} \overline{H_f}$. Then the space $H_{\mathrm{E,a}}$ is an infinite dimensional separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathrm{E,a}}$ which is extended to the inner product on $H_{\mathrm{E,a}}$. From now on we take $H_{\mathrm{E,a}}$ to be this space.

Remark 2.2 In [2], the Cesàro Hilbert space is introduced by

$$H_c = \Big\{\int_{(0,\pi)} \alpha(\lambda) s_\lambda d\lambda;\ \alpha \in L^2((0,\pi), d\lambda)\Big\},$$

where $s_\lambda = \sum_{n=1}^{\infty} e^{in\lambda} e_n$ for any $\lambda \in (0,\pi)$. The space H_c is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{H_c}$ given by

$$\begin{aligned}\left\langle \int_{(0,\pi)} \alpha(\lambda) s_\lambda d\lambda, \int_{(0,\pi)} \beta(\lambda) s_\lambda d\lambda \right\rangle_{H_c} &= \int_{(0,\pi)} \alpha(\lambda)\overline{\beta(\lambda)} \langle s_\lambda, s_\lambda\rangle_{\mathrm{a}} d\lambda \\ &= \int_{(0,\pi)} \alpha(\lambda)\overline{\beta(\lambda)} d\lambda, \quad \alpha, \beta \in L^2((0,\pi), d\lambda).\end{aligned}$$

The inner product $\langle\cdot,\cdot\rangle_{H_c}$ is different from the inner product $\langle\cdot,\cdot\rangle_{\mathrm{E,1}}$. In fact, we have

$$\left\langle \int_{(0,\pi)} \alpha(\lambda) s_\lambda d\lambda, \int_{(0,\pi)} \beta(\lambda) s_\lambda d\lambda \right\rangle_{\mathrm{E,1}} = 0.$$

Let A be a selfadjoint operator (densely defined) in $H_{\mathrm{E,a}}$ satisfying the condition:

- $\inf \mathrm{Spec}\,(A) > 1$ and A^{-1} is of Hilbert-Schmidt type.

Then there exist a sequence

$$1 < \ell_1 \le \ell_2 \le \ell_3 \le \dots, \qquad \| A^{-1} \|_{\mathrm{HS}}^2 = \sum_{k=1}^{\infty} \ell_k^{-2} < \infty,$$

and an orthonormal basis $\{e_{a,k}\}_{k=1}^{\infty}$ of $H_{\mathrm{E,a}}$ such that $Ae_{a,k} = \ell_k e_{a,k}$, $k = 1, 2, \dots$. By Condition (C1) we see that $s_j(f)$, $j = 1, 2, \dots$, are linearly independent for all $f \in \mathfrak{C}$. Therefore by the Gram-Schmidt orthogonalization we have an orthonormal system $\{\widetilde{s}_j(f)\}_{j=1}^{\infty}$. Collecting $\{\widetilde{s}_j(f)\}_{j=1}^{\infty}$ for all $f \in \mathfrak{C}$ we can take an example of $\{e_{a.k}\}_{k=1}^{\infty}$ satisfying the above properties. From now on, we take this orthonormal basis. For $p \in \mathbf{R}$ we define

$$| \xi |_p^2 = | A^p \xi |_0^2 = \sum_{k=1}^{\infty} \ell_k^{2p} | \langle \xi, e_{a,k} \rangle_{\mathrm{E,a}} |^2, \qquad \xi \in H_{\mathrm{E,a}}.$$

Now let $p \ge 0$. We put $\mathcal{N}_p = \{\xi \in H_{\mathrm{E,a}} \, ; \, | \xi |_p < \infty\}$ and define $\mathcal{N}_{-p}$ to be the completion of $H_{\mathrm{E,a}}$ with respect to $| \cdot |_{-p}$. Thus we obtain a chain of Hilbert spaces $\{\mathcal{N}_p \, ; \, p \in \mathbf{R}\}$ and consider their limit spaces:

$$\mathcal{N} = \operatorname*{proj\,lim}_{p \to \infty} \mathcal{N}_p, \qquad \mathcal{N}^* = \operatorname*{ind\,lim}_{p \to \infty} \mathcal{N}_{-p}$$

which are mutually dual spaces. Note also that $\mathcal{N}$ becomes a countably Hilbert nuclear space. Identifying $H_{\mathrm{E,a}}$ with its dual space, we obtain a complex Gelfand triple:

$$\mathcal{N} \subset H_{\mathrm{E,a}} \subset \mathcal{N}^*. \tag{2.5}$$

Remark 2.3 The bilinear form τ_{a} defined by

$$\langle \tau_{\mathrm{a}}, z \otimes w \rangle = \lim_{N \to \infty} \frac{1}{N^a} \sum_{i=1}^{N} \langle z, e_i \rangle \, \overline{\langle w, e_i \rangle}, \quad z, w \in H_{\mathrm{E,a}},$$

is called the *Exotic trace*. Then by definition we obtain that for any $z, w \in H_{\mathrm{E,a}}$,

$$\langle \tau_{\mathrm{a}}, z \otimes w \rangle = \langle z, w \rangle_{\mathrm{E,a}}.$$

Therefore, the Exotic trace coincides with the usual trace on $H_{\mathrm{E,a}}$ and hence τ_{a} can be represented by

$$\tau_{\mathrm{a}} = \sum_{k=1}^{\infty} e_{a,k} \otimes e_{a,k}.$$

Theorem 2.4 *The Exotic trace belongs to* $\mathcal{N}_{-1/2} \otimes \mathcal{N}_{-1/2}$.

PROOF. By definition we have

$$| \tau_{\mathrm{a}} |_{-1/2}^2 = \sum_{k=1}^{\infty} \ell_k^{-2} | e_{a,k} \otimes e_{a,k} |_0^2 = \|A^{-1}\|_{\mathrm{HS}}^2 < \infty$$

which follows the proof. ∎

3 Nuclear Riggings of Fock Spaces

Now, we construct a rigging of (Boson) Fock space based on the basic Gelfand triple (2.3). For each $p \in \mathbf{R}$ let $\Gamma(E_p)$ be the Fock space over the Hilbert space E_p, i.e.,

$$\Gamma(E_p) = \left\{ \phi = (f_n)_{n=0}^{\infty} \, ; \, f_n \in E_p^{\hat{\otimes} n}, \, \| \phi \|_{H,p}^2 = \sum_{n=0}^{\infty} n! \, | f_n |_{H,p}^2 < \infty \right\}.$$

Then by identifying $\Gamma(H)$ with its dual space, we have a chain of Fock spaces:

$$\cdots \subset \Gamma(E_p) \subset \Gamma(E_0) = \Gamma(H) \subset \Gamma(E_{-p}) \subset \cdots$$

and a Gelfand triple

$$(E) = \operatorname*{proj\,lim}_{p \to \infty} \Gamma(E_p) \subset \Gamma(H) \subset (E)^* = \operatorname*{ind\,lim}_{p \to \infty} \Gamma(E_{-p}). \tag{3.1}$$

An *exponential vector* (or also called a *coherent vector*) associated with $\xi \in E$ is defined by

$$\phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \ldots, \frac{\xi^{\otimes n}}{n!}, \ldots \right). \tag{3.2}$$

Since $\phi_\xi \in (E)$, the *S-transform* of an element $\Phi \in (E)^*$ is defined by

$$S\Phi(\xi) = \langle\!\langle \Phi, \, \phi_\xi \rangle\!\rangle \, , \qquad \xi \in E,$$

where $\langle\!\langle \cdot, \, \cdot \rangle\!\rangle$ is the canonical $\mathbf{C}$-bilinear form on $(E)^* \times (E)$ which takes the form:

$$\langle\!\langle \Phi, \, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \, \langle F_n, \, f_n \rangle \, , \qquad \Phi = (F_n) \in (E)^*, \quad \phi = (f_n) \in (E). \tag{3.3}$$

Every element $\Phi \in (E)^*$ is uniquely specified by its S-transform $S\Phi$ since $\{\phi_\xi \, ; \, \xi \in E\}$ spans a dense subspace of (E).

A complex-valued function F on E is called a *U-functional* if F is Gâteaux entire and there exist constants $C, K \geq 0$ and $p \geq 0$ such that

$$|F(\xi)| \leq C \exp\left(K \, | \xi |_{H,p}^2 \right), \qquad \xi \in E.$$

Theorem 3.1 [24] *A $\mathbf{C}$-valued function F on E is the S-transform of an element in $(E)^*$ if and only if F is a U-functional.*

Remark 3.2 The Bochner-Minlos Theorem admits the existence of a probability measure μ on $E_{\mathbf{R}}^*$ such that

$$\int_{E_{\mathbf{R}}^*} e^{i\langle x, \xi \rangle} d\mu(x) = e^{-\frac{1}{2}\langle \xi, \xi \rangle}, \qquad \xi \in E_{\mathbf{R}}.$$

Then the famous Wiener-Itô-Segal isomorphism between $\Gamma(H)$ and $L^2(E^*, \mu)$ is a unitary isomorphism uniquely determined by the correspondence:

$$\phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \ldots, \frac{\xi^{\otimes n}}{n!}, \ldots \right) \quad \longleftrightarrow \quad e^{\langle x, \xi \rangle - \langle \xi, \xi \rangle / 2} = \phi_\xi(x), \;\; \xi \in E.$$

The Gelfand triple obtained from (3.1) through the Wiener-Itô-Segal isomorphism is denoted also by

$$(E) \subset L^2(E^*, \mu) \subset (E)^*$$

which is referred to as the *Hida–Kubo–Takenaka space.* An element of (E) (resp. $(E)^*$) is called a test (resp. generalized) white noise function.

By same argument with the Gelfand triple (2.5), we construct a rigging of Fock spaces:

$$(\mathcal{N}) = \operatorname{proj}\lim_{p\to\infty} \Gamma(\mathcal{N}_p) \subset \cdots \subset \Gamma(\mathcal{N}_p) \subset \Gamma(\mathcal{N}_0)$$
$$= \Gamma(H_{\mathrm{E,a}}) \subset \Gamma(\mathcal{N}_{-p}) \subset \cdots \subset (\mathcal{N})^* = \operatorname{ind}\lim_{p\to\infty} \Gamma(\mathcal{N}_{-p}), \quad (3.4)$$

where $\Gamma(\mathcal{N}_p)$ is the Fock space over the Hilbert space $\mathcal{N}_p$, i.e.,

$$\Gamma(\mathcal{N}_p) = \left\{ \phi = (f_n)_{n=0}^{\infty}\,;\, f_n \in \mathcal{N}_p^{\hat{\otimes} n},\ \| \phi \|_p^2 = \sum_{n=0}^{\infty} n!\, | f_n |_p^2 < \infty \right\}, \qquad p \in \mathbf{R}.$$

The $S_{\mathrm{E,a}}$-*transform* of an element $\Phi \in (\mathcal{N})^*$ is defined by

$$S_{\mathrm{E,a}}\Phi(\xi) = \langle\!\langle \Phi,\, \phi_\xi \rangle\!\rangle_{\mathrm{E,a}}, \qquad \xi \in \mathcal{N},$$

where ϕ_ξ is the exponential vector defined as in (3.2) and $\langle\!\langle \cdot,\, \cdot \rangle\!\rangle_{\mathrm{E,a}}$ is the canonical $\mathbf{C}$-bilinear form on $(\mathcal{N})^* \times (\mathcal{N})$ which takes the form:

$$\langle\!\langle \Phi,\, \phi \rangle\!\rangle_{\mathrm{E,a}} = \sum_{n=0}^{\infty} n!\, \langle F_n,\, f_n \rangle_{\mathrm{E,a}}, \qquad \Phi = (F_n) \in (\mathcal{N})^*, \quad \phi = (f_n) \in (\mathcal{N}). \qquad (3.5)$$

Then as a similar result to Theorem 3.1 we prove that a $\mathbf{C}$-valued function F on $\mathcal{N}$ is the $S_{\mathrm{E,a}}$-transform of an element in $(\mathcal{N})^*$ if and only if F is Gâteaux entire and there exist constants $C, K \ge 0$ and $p \ge 0$ such that

$$|F(\xi)| \le C \exp\left(K\, |\, \xi\, |_p^2\right), \qquad \xi \in \mathcal{N}.$$

4 Infinite Dimensional Laplacians

Let $\mathfrak{X}$ be a locally convex nuclear space. A function $F : \mathfrak{X} \to \mathbf{C}$ is said to be an element of class $C^2(\mathfrak{X})$ if F is twice (continuously) Fréchet differentiable in each variable, i.e., there exist two continuous maps

$$\xi \longmapsto F'(\xi) \in \mathfrak{X}^*, \quad \xi \longmapsto F''(\xi) \in \mathcal{L}(\mathfrak{X}, \mathfrak{X}^*), \qquad \xi \in \mathfrak{X}$$

such that

$$F(\xi + \eta) = F(\xi) + \langle F'(\xi),\, \eta \rangle + \frac{1}{2} \langle F''(\xi)\eta,\, \eta \rangle + \varepsilon(\eta)$$

for any $\eta \in \mathfrak{X}$, where the error terms satisfy

$$\lim_{t\to 0} \frac{\varepsilon(t\eta)}{t} \to 0, \qquad \eta \in \mathfrak{X}.$$

From the kernel theorem, we use the common symbol $F''(\xi)$, i.e.,

$$\langle F''(\xi)\eta, \eta\rangle = \langle F''(\xi), \eta\otimes\eta\rangle = F''(\xi)(\eta,\eta) = \widetilde{D}_\eta^2 F(\xi),$$

where $\widetilde{D}_\eta$ is the Fréchet differentiation in the direction η, i.e.,

$$\widetilde{D}_\eta F(\xi) = \lim_{t\to 0}\frac{1}{t}\left[F(\xi+t\eta) - F(\xi)\right] = \frac{d}{dt}F(\xi+t\eta)\Big|_{t=0}.$$

4.1 Gross Laplacian

The Gross Laplacian Δ_{G} acting on $(\mathcal{N})$ is defined by

$$\Delta_{\mathrm{G}}\phi = S_{\mathrm{E,a}}^{-1}\left(\sum_{k=1}^{\infty}\widetilde{D}_{e_{a,k}}^2\right)S_{\mathrm{E,a}}\phi, \qquad \phi\in(\mathcal{N}),$$

where $\{e_{a,k}\}_{k=1}^{\infty}$is an orthonormal basis of $H_{\mathrm{E,a}}$. Then for any $\phi = (f_n)_{n=0}^{\infty}\in(\mathcal{N})$ we have

$$\Delta_{\mathrm{G}}\phi = \left((n+2)(n+1)\tau_{\mathrm{a}}\widehat{\otimes}^2 f_{n+2}\right), \tag{4.1}$$

see [6, 16, 23]. Moreover, we have the following

Theorem 4.1 *The Gross Laplacian is a continuous linear operator acting on $(\mathcal{N})$.*

PROOF. Let $\phi = (f_n)_{n=0}^{\infty}\in(\mathcal{N})$. Then for any $p>1/2$ and $q>0$ we obtain from (4.1) that

$$\begin{aligned}\|\Delta_{\mathrm{G}}\phi\|_p^2 &\le \sum_{n=0}^{\infty}(n+2)!(n+2)(n+1)\ell_1^{-2q(n+2)}\,|\tau_{\mathrm{a}}|_{-p}^2\,|f_{n+2}|_{p+q}^2\\ &\le C_{\mathrm{E,a}}(q)\,|\tau_{\mathrm{a}}|_{-p}^2\,\|\phi\|_{p+q}^2,\end{aligned}$$

where

$$C_{\mathrm{E,a}}(q) = \sup_{n\ge 0}(n+2)(n+1)\ell_1^{-2q(n+2)} < \infty.$$

Therefore, for any $p>1/2$ and $q>0$ we have

$$\|\Delta_{\mathrm{G}}\phi\|_p \le \sqrt{C_{\mathrm{E,a}}(q)}\,|\tau_{\mathrm{a}}|_{-p}\,\|\phi\|_{p+q}, \qquad \phi\in(\mathcal{N})$$

which follows the proof. ∎

4.2 The Exotic Laplacians

Let $\mathrm{Dom}(\Delta_{\mathrm{E,a}})$ denote the set of all $\Phi\in(E)^*$ such that the limit

$$\widetilde{\Delta}_{\mathrm{E,a}}S\Phi(\xi) = \lim_{N\to\infty}\frac{1}{N^{\mathrm{a}}}\sum_{k=1}^{N}\langle (S\Phi)''(\xi),\, e_k\otimes e_k\rangle, \quad \xi\in E$$

exists for each $\xi\in E$ and a functional $\widetilde{\Delta}_{\mathrm{E,a}}(S\Phi)$ is the S-transform. Then the *Exotic Laplacian* $\Delta_{\mathrm{E,a}}\Phi$ defined on $Dom(\Delta_{\mathrm{E,a}})$ is defined by

$$\Delta_{\mathrm{E,a}}\Phi = S^{-1}(\widetilde{\Delta}_{\mathrm{E,a}}S\Phi), \quad \Phi\in\mathrm{Dom}(\Delta_{\mathrm{E,a}}).$$

Theorem 4.2 *Any element $\phi \in (E)^* \cap (\mathcal{N})$ is in* $\mathrm{Dom}(\Delta_{\mathrm{E,a}})$. *Moreover, if $\phi = (f_n)_{n=0}^{\infty}$, then we have*

$$\Delta_{\mathrm{E,a}}\phi = \left((n+2)(n+1)\tau_{\mathrm{a}}\widehat{\otimes}^2 f_{n+2}\right). \tag{4.2}$$

PROOF. Let $\phi = (f_n) \in (\mathcal{N})$. Then we can easily show that

$$(S\phi)''(\xi)(e_k, e_k) = \sum_{n=0}^{\infty}(n+2)(n+1)\left\langle (e_k \otimes e_k)\widehat{\otimes}^2 f_{n+2}, \xi^{\otimes n}\right\rangle$$

which implies from the definition of Levy trace that

$$\begin{aligned}\widetilde{\Delta}_{\mathrm{E,a}}S\phi(\xi) &= \lim_{N\to\infty}\frac{1}{N^{\mathrm{a}}}\sum_{k=1}^{N}\sum_{n=0}^{\infty}(n+2)(n+1)\left\langle (e_k \otimes e_k)\widehat{\otimes}^2 f_{n+2}, \xi^{\otimes n}\right\rangle \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)\left\langle \tau_{\mathrm{a}}\widehat{\otimes}^2 f_{n+2}, \xi^{\otimes n}\right\rangle.\end{aligned}$$

Therefore, by applying (3.3) we prove (4.2). ∎

By Theorem 4.2 and (4.1), the Exotic Laplacian coincides with the Gross Laplacian on $(E)^*\cap(\mathcal{N})$, which is dense in $(\mathcal{N})$. Therefore, we can consider $(\mathcal{N})$ as the domain of the Exotic Laplacian. On the other hand, every harmonic function in $(E)^*$ associated with the Exotic Laplacian belongs to $(\mathcal{N})$ as the zero element. Hence, the vector space $(\mathcal{N})\oplus(E)^*_{\mathrm{a,h}}$ can be considered as the reasonable domain of the Exotic Laplacian, where $(E)^*_{\mathrm{a,h}}$ is the linear space of all harmonic functions in $(E)^*$ associated with $\Delta_{\mathrm{E,a}}$. Note that for the study of heat equation and (infinite dimensional) stochastic process associated with the Exotic Laplacian, the harmonic functions associated with the Laplacian are not necessary. Therefore, from now on we consider $(\mathcal{N})$ as the domain of the Exotic Laplacian which is defined by (4.2) for each $\phi = (f_n) \in (\mathcal{N})$ and then, by Theorem 4.1, the Exotic Laplacian is a continuous linear operator acting on $(\mathcal{N})$.

5 One-parameter group and Stochastic process associated with the Exotic Laplacian

5.1 One-Parameter Group

The *symbol* of a continuous linear operator $\Xi \in \mathcal{L}((\mathcal{N}), (\mathcal{N})^*)$ is defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta\rangle\rangle_{\mathrm{E,a}}, \qquad \xi, \eta \in \mathcal{N}.$$

An operator $\Xi \in \mathcal{L}((\mathcal{N}), (\mathcal{N})^*)$ is uniquely specified by the symbol since $\{\phi_\xi\,;\, \xi \in \mathcal{N}\}$ spans a dense subspace of $(\mathcal{N})$. Moreover, we have an analytic characterization of symbols.

Theorem 5.1 [22] *A $\mathbf{C}$-valued function Θ on $\mathcal{N} \times \mathcal{N}$ is the symbol of an operator $\Xi \in \mathcal{L}((\mathcal{N}), (\mathcal{N})^*)$ if and only if*

(i) *Θ is Gâteaux entire;*

(ii) *there exist* $C \geq 0$, $K \geq 0$ *and* $p \geq 0$ *such that*

$$|\Theta(\xi, \eta)| \leq C \exp K\left(|\xi|_p^2 + |\eta|_p^2 \right), \qquad \xi, \eta \in \mathcal{N}.$$

Moreover, Θ *is the symbol of an operator* $\Xi \in \mathcal{L}((\mathcal{N}), (\mathcal{N}))$ *if and only if* Θ *satisfies (i) and*

(iii) *for any* $p \geq 0$ *and* $\epsilon > 0$, *there exist constants* $C \geq 0$ *and* $q \geq 0$ *such that*

$$|\Theta(\xi, \eta)| \leq C \exp \epsilon \left(|\xi|_{p+q}^2 + |\eta|_{-p}^2 \right), \qquad \xi, \eta \in \mathcal{N}.$$

By applying Theorem 5.1 we can easily see that for each $t \in \mathbf{R}$ there exists a unique operator $P_t \in \mathcal{L}((\mathcal{N}), (\mathcal{N}))$ such that

$$\widehat{P_t}(\xi, \eta) = \langle\!\langle P_t \phi_\xi, \phi_\eta \rangle\!\rangle_{\mathrm{E,a}} = e^{t\langle \tau_{\mathrm{a}}, \xi \otimes \xi \rangle + \langle \xi, \eta \rangle_{\mathrm{E,a}}}, \qquad \xi, \eta \in \mathcal{N}.$$

In fact, if we put $\Theta = \widehat{P_t}$ for fixed $t \in \mathbf{R}$, then Θ satisfies the conditions (i) and (iii) in Theorem 5.1 which proves the existence of the operator P_t. Moreover, for any $\phi = (f_n) \in (\mathcal{N})$, $P_t\phi$ is given by

$$P_t \phi = \left(\sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} t^m (\tau_{\mathrm{a}}^{\otimes m} \widehat{\otimes}_{2m} f_{n+2m}) \right). \tag{5.1}$$

Theorem 5.2 $\{P_t; t \in \mathbf{R}\}$ *becomes a regular one-parameter group of operators acting on* $(\mathcal{N})$ *with infinitesimal generator* $\Delta_{\mathrm{E,a}}$.

The proof is a simple modification of the proof of Theorem 4.3 in [6].

5.2 An infinite dimensional stochastic process generated by the Laplacians

Let $\{\mathbf{X}_t; t \geq 0\}$ be a $(\mathcal{N})$-valued stochastic process. Then we can write the process in the form $\mathbf{X}_t = (\mathbf{X}_{t,n})$. The expectation $E[\mathbf{X}_t]$ of $\mathbf{X}_t$ is given by

$$E[\mathbf{X}_t] = (E[\mathbf{X}_{t,n}])$$

if $(E[\mathbf{X}_{t,n}])$ exists in $(\mathcal{N})$.

For $\eta \in \mathcal{N}$ let $\mathbf{T}_\eta$ be a translation operator defined on $(\mathcal{N})^*$ by

$$\mathbf{T}_\eta \Phi = \sum_{k=0}^{\infty} \frac{1}{k!} D_\eta^k \Phi, \qquad \Phi \in (\mathcal{N})^*,$$

where D_η is defined by $D_\eta \Phi = S_{\mathrm{E,a}}^{-1}(\widetilde{D}_\eta S_{\mathrm{E,a}} \Phi)$. Then we have

$$D_\eta \Phi = ((n+1) \langle F_{n+1}, z \rangle_{\mathrm{E,a}}), \qquad \Phi = (F_n).$$

and

$$\mathbf{T}_\eta \Phi = \left(\sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} \left\langle F_{n+k}, \eta^{\otimes k} \right\rangle_{\mathrm{E,a}} \right). \tag{5.2}$$

Therefore, for given $z \in \mathcal{N}^*$ and $\Phi \in (\mathcal{N})^*$, it is natural to define $\mathbf{T}_z \Phi$ by (5.2) whenever the right hand side of (5.2) is well defined as an element in $(\mathcal{N})^*$.

Proposition 5.3 *For all $z \in \mathcal{N}^*$, the operator $\mathbf{T}_z$ is in $\mathcal{L}((\mathcal{N}), (\mathcal{N}))$. Furthermore, for any $p \geq 0$, $q > 0$ with $|z|_{-(p+q)} < \infty$, it holds that*

$$\|\mathbf{T}_z\phi\|_p \leq \frac{\|\phi\|_{p+q}}{(1-\ell_1^{-2q})} \exp\left(\frac{|z|^2_{-(p+q)}}{2(1-\ell_1^{-2q})}\right), \qquad \phi \in (\mathcal{N}).$$

The proof is given by a similar method of the proof of Theorem 4.2.3 in [23].

Let $\{B_k(t); t \geq 0\}$, $k = 1, 2, \ldots$, be an infinite sequence of independent one dimensional Brownian motions and $\{\mathbf{B}_t; t \geq 0\}$ an infinite dimensional stochastic process defined by

$$\mathbf{B}_t = \sum_{k=1}^{\infty} B_k(t) e_{a,k}, \qquad t \geq 0. \tag{5.3}$$

Lemma 5.4 *For all $t \geq 0$ we have $\mathbf{B}_t \in \mathcal{N}^*$* (a.e.).

PROOF. By definition, we can check that

$$\begin{aligned} E[|\mathbf{B}_t|^2_{-p}] &= E\left[\sum_{k=1}^{\infty} \ell_k^{-2p} |\langle \mathbf{B}_t, e_{a,k}\rangle_{\mathrm{E,a}}|^2\right] \\ &= \sum_{k=1}^{\infty} \ell_k^{-2p} E[|B_k(t)|^2] \\ &= t\sum_{k=1}^{\infty} \ell_k^{-2p} < \infty, \quad t \geq 0 \end{aligned}$$

for any $p \geq 1$ which implies the assertion. ∎

Theorem 5.5 *Let $\phi \in (\mathcal{N})$. Then the equality*

$$P_t\phi = E[\mathbf{T}_{\mathbf{B}_{2t}}\phi]$$

holds for $t \geq 0$.

PROOF. Let $\phi = (f_n) \in (\mathcal{N})$. Then by (5.2) we have

$$\mathbf{T}_{\mathbf{B}_{2t}}\phi = \left(\sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} \left\langle f_{n+k}, \mathbf{B}_{2t}{}^{\otimes k}\right\rangle_{\mathrm{E,a}}\right) \tag{5.4}$$

and by (5.3) we have

$$\left\langle f_{n+k}, \mathbf{B}_{2t}{}^{\otimes k}\right\rangle_{\mathrm{E,a}} = \sum_{\ell_1,\ldots,\ell_k=1}^{\infty} \left[\prod_{j=1}^{k} B_{\ell_j}(2t)\right] \left\langle f_{n+k}, \bigotimes_{j=1}^{k} e_{L,\ell_j}\right\rangle_{\mathrm{E,a}}.$$

Therefore, we have

$$
\begin{aligned}
&S_{\mathrm{E,a}}[E[\mathbf{T}_{\mathbf{B}_{2t}}\phi]](\xi)\\
&= \sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{n=0}^{\infty}\sum_{j=1}^{m}\frac{(n+2m)!}{j!}\left\langle f_{n+2m}, \sum_{\substack{\ell_1,\ldots,\ell_j\\ \text{all different}}}\left(\sum_{\nu=1}^{j}e_{a,\ell_\nu}^{\otimes 2}\right)^{\otimes m}\widehat{\otimes}\xi^{\otimes n}\right\rangle_{\mathrm{E,a}}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(n+2m)!}{m!}t^m\left\langle f_{n+2m}, \left(\sum_{\ell=1}^{\infty}e_{a,\ell}^{\otimes 2}\right)^{\otimes m}\widehat{\otimes}\xi^{\otimes n}\right\rangle_{\mathrm{E,a}}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(n+2m)!}{m!}t^m\langle \tau_{\mathrm{a}}^{\otimes m}\widehat{\otimes}_{2m}f_{n+2m}, \xi^{\otimes n}\rangle_{\mathrm{E,a}}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
E[\mathbf{T}_{\mathbf{B}_{2t}}\phi] &= \left(\sum_{m=0}^{\infty}\frac{(n+2m)!}{n!m!}t^m(\tau_{\mathrm{a}}^{\otimes m}\widehat{\otimes}_{2m}f_{n+2m})\right)\\
&= P_t\phi.
\end{aligned}
$$

∎

By Theorem 5.5, we can consider $\{\mathbf{B}_{2t}\}_{t\geq 0}$ as a stochastic process generated by the Lévy Laplacian $\Delta_{\mathrm{E,a}}$. This is one of reasons why the Exotic Laplacian is called 'Laplacian'.

References

[1] L. Accardi: *Yang-Mills ewquations and Lévy-Laplacians*, Dirichlet Forms and Stochastic Processes, Walter de Gruyter (1995) 1-24.

[2] L. Accardi, A. Barhoumi and H. Ouerdiane: *A quantum approach to Laplace operators*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, Vol. **9**, (2006) 215-248.

[3] L. Accardi, P. Gibilisco and I. V. Volovich: *The Lévy Laplacian and the Yang-Mills equations*, Russ. J. Math. Phys. **2** (1994) 235-250.

[4] L. Accardi and O. G. Smoliyanov: *On Laplacians and Traces*, Conferenze del Seminario di Matematica dell'Universit'a di Bari, N.250, 1993.

[5] L. Accardi and O. G. Smoliyanov: *The trace formulae for Lévy-Gaussian measures and their applications*, in "Mathematical Approach to Fluctuations Vol. II (T. Hida, ed.)" pp.31-47, World Scientific 1995.

[6] D. M. Chung and U. C. Ji: *Transformation groups on white noise functionals and their applications*, Appl. Math. Optim. **37** (1998), 205–223.

[7] D. M. Chung, U. C. Ji and K. Saitô: *Cauchy problems associated with the Lévy Laplacian in white noise analysis*, Infinite Dimensional Analysis, Quantum Probability and Related Topics **2** (1999), 131–153.

[8] D. M. Chung, U. C. Ji and K. Saitô: *Notes on a C_0-group generated by the Lévy Laplacian*, Proc. Amer. Math. Soc **130** (2001), 1197–1206.

[9] T. Hida: "Analysis of Brownian Functionals," Carleton Math. Lect. Notes no. 13, Carleton University, Ottawa, 1975.

[10] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: "White noise: An Infinite Dimensional Calculus", Kluwer Academic Publishers, 1993.

[11] T. Hida, N. Obata and K. Saitô: *Infinite dimensional rotations and Laplacians in terms of white noise calculus*, Nagoya Math. J. **128** (1992), 65-93.

[12] U. C. Ji: *Integral kernel operators on regular generalized white noise functions*, Bull. Korean Math. Soc. **37** (2000), 601–618.

[13] U. C. Ji and K. Saitô: *A similarity between the Gross Laplacian and the Lévy Laplacian*, Infinite Dimensional Analysis, Quantum Probability and Related Topics **10** (2007), 261–276.

[14] H.-H. Kuo: *On Laplacian operators of generalized Brownian functionals*, in "Stochastic Processes and Applications (K. Itô and T. Hida, eds.)," pp. 119–128, Lect. Notes in Math. Vol. 1203, Springer–Verlag, 1986.

[15] H.-H. Kuo: *Lectures on white noise analysis*, Soochow J. Math. **18** (1992), 229-300.

[16] H.-H. Kuo: "White noise distribution theory," CRC Press, 1996.

[17] H.-H. Kuo, N. Obata and K. Saitô: *Lévy Laplacian of generalized functions on a nuclear space*, J. Funct. Anal., **94** (1990), 74-92.

[18] H.-H. Kuo, N. Obata and K. Saitô: *Diagonalization of the Lévy Laplacian and related stable processes*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, **5** (2002), 317-331.

[19] P. Lévy: Lecons d'analyse fonctionnelle, Gauthier-Villars, Paris, 1922.

[20] P. Lévy: Problèms concrets d'analyse fonctionnelle, Gauthier-Villars, Paris, 1951.

[21] N. Obata: *A characterization of the Lévy Laplacian in terms of infinite dimensional rotation groups*, Nagoya Math. J., **118** (1990), 111-132.

[22] N. Obata: *An analytic characterization of symbols of operators on white noise functionals*, J. Math. Soc. Japan, **45** (1993), 421-445.

[23] N. Obata: "White Noise Calculus and Fock Space," Lect. Notes in Math. Vol. 1577, Springer–Verlag, 1994.

[24] J. Potthoff and L. Streit: *A characterization of Hida distributions*, J. Funct. Anal. **101** (1991), 212-229.

[25] K. Saitô: *Itô's formula and Lévy Laplacian I and II*, Nagoya Math. J. **108** (1987) 67-76, **123** (1991) 153-169.

[26] K. Saitô: *A group generated by the Lévy Laplacian and the Fourier-Mehler transform*, Pitman Research Notes in Math. Ser. 310 (1994) 274-288.

[27] K. Saitô: *A (C_0)-group generated by the Lévy Laplacian*, Stoch. Anal. and Appl. **16** (1998), 567–584.

[28] K. Saitô: *A (C_0)-group generated by the Lévy Laplacian II*, Infinite Dimensional Analysis, Quantum Probability and Related Topics **1** (1998), 425–437.

[29] Saitô, K.: A stochastic process generated by the Lévy Laplacian, *Acta Appl. Math.* **63** (2000), 363–373.