数理解析研究所講究録
第 1610 巻 2008 年 18-28

Orientability of real resultant singularities.

A. I. Esterov[1]

1 Real resultant singularities

Let $\mathbb{K}$ be the field of real or complex numbers. For a finite set $A \in \mathbb{Z}^n$, denote the space of Laurent polynomials $\{\sum_{(a_1,\ldots,a_n)\in A} c_{a_1,\ldots,a_n} t_1^{a_1}\ldots t_n^{a_n} \mid c_{a_1,\ldots,a_n} \in \mathbb{K}\}$ by $\mathbb{K}[A]$.

DEFINITION 1. For finite sets $\Sigma_i \subset \mathbb{Z}^N$, $i = 1,\ldots,I$, *the resultant variety* $R_{\mathbb{K}}(\Sigma_1,\ldots,\Sigma_I) \subset \mathbb{K}[\Sigma_1]\oplus\ldots\oplus\mathbb{K}[\Sigma_I]$ is defined as the closure of the set

$$\Bigl\{(g_1,\ldots,g_I) \mid g_i \in \mathbb{K}[\Sigma_i],\ \exists(t_1,\ldots,t_N) \in (\mathbb{K}\setminus\{0\})^N :$$
$$g_1(t_1,\ldots,t_N) = \ldots = g_I(t_1,\ldots,t_N) = 0\Bigr\}.$$

One simple example of a resultant variety is the set of all degenerate matrices in the space of all $N \times I$ matrices with entries in $\mathbb{K}$. The papers [5], [6] study the intersection number of the variety $R_{\mathbb{C}}(\Sigma_1,\ldots,\Sigma_I)$ with the image of a complex analytic map $f : \mathbb{C}^m \to \mathbb{C}[\Sigma_1]\oplus\ldots\oplus\mathbb{C}[\Sigma_I]$ in terms of the Newton polyhedra of the components of f. To discuss the real version of this computation, we need to know if the intersection number is well defined for the variety $R_{\mathbb{R}}(\Sigma_1,\ldots,\Sigma_I)$: in particular, if this variety is equidimensional, orientable, if it has no boundary etc. The main result of this paper (Theorem 4 below) answers this question, if the sets $\Sigma_1,\ldots,\Sigma_I$ are in general position in a sense.

For a finite set $\Sigma \subset \mathbb{Z}^N$, we denote the real vector space and the lattice, spanning all vectors of the form $a-b$, $a \in \Sigma$, $b \in \Sigma$, by $\mathrm{Lin}_{\mathbb{R}}(\Sigma)$ and $\mathrm{Lin}_{\mathbb{Z}}(\Sigma)$ respectively.

DEFINITION 2. A collection of sets $\Sigma_i \subset \mathbb{Z}^N$, $i = 1,\ldots,I$, is said to be *consistent* if $J - \dim \mathrm{Lin}_{\mathbb{R}}(\Sigma_{i_1}+\ldots+\Sigma_{i_J}) \leq I - \dim \mathrm{Lin}_{\mathbb{R}}(\Sigma_1+\ldots+\Sigma_I)$ for every subset $\{i_1,\ldots,i_J\} \subset \{1,\ldots,I\}$. It is said to be *essential* if, in addition, $J - \dim \mathrm{Lin}_{\mathbb{R}}(\Sigma_{i_1}+\ldots+\Sigma_{i_J}) < I - \dim \mathrm{Lin}_{\mathbb{R}}(\Sigma_1+\ldots+\Sigma_I)$ for every subset $\{i_1,\ldots,i_J\} \subsetneq \{1,\ldots,I\}$, and $\mathrm{Lin}_{\mathbb{R}}(\Sigma_1+\ldots+\Sigma_I)\cap\mathbb{Z}^N = \mathrm{Lin}_{\mathbb{Z}}(\Sigma_1+\ldots+\Sigma_I)$.

[1]Supported in part by RFBR-JSPS-06-01-91063, RFBR-07-01-00593 and INTAS-05-7805.

DEFINITION 3. A collection of subsets $\widetilde{\Sigma}_1 \subset \Sigma_1, \ldots, \widetilde{\Sigma}_I \subset \Sigma_I$ is called a *codimension 1 face* of the collection $\Sigma_1, \ldots, \Sigma_I$, if the sum $\widetilde{\Sigma}_1 + \ldots + \widetilde{\Sigma}_I$ is contained in a codimension 1 face of the convex hull of the sum $\Sigma_1 + \ldots + \Sigma_I$, and the collection $\widetilde{\Sigma}_1, \ldots, \widetilde{\Sigma}_I$ is a maximal collection of subsets of $\Sigma_1, \ldots, \Sigma_I$ with this property.

THEOREM 4. *1) The variety $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ is equidimensional, and its codimension equals* $\max\limits_{\{i_1,\ldots,i_J\}\subset\{1,\ldots,I\}} J - \dim \mathrm{Lin}_{\mathbb{R}}(\Sigma_{i_1} + \ldots + \Sigma_{i_J})$.
2) If $I > \dim \mathrm{Lin}_{\mathbb{R}}(\Sigma_1 + \ldots + \Sigma_I) + 1$, the collection $\Sigma_1, \ldots, \Sigma_I$ is essential, and every its consistent codimension 1 face is essential, then the codimension of the singular locus of $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ in $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ is greater then 1.
3) If, under the assumptions of Part 2, there exists $a \in \mathbb{Z}^N$, such that the shifted lattice $a + 2\mathbb{Z}^N$ does not intersect the shifted lattice $\widetilde{\Sigma}_1 + \ldots + \widetilde{\Sigma}_I + \mathrm{Lin}_{\mathbb{Z}}(\widetilde{\Sigma}_1 + \ldots + \widetilde{\Sigma}_I)$ for every consistent codimension 1 face $(\widetilde{\Sigma}_1, \ldots, \widetilde{\Sigma}_I)$ of the collection $(\Sigma_1, \ldots, \Sigma_I)$, then the smooth locus of $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ is orientable.

We study only essential collections $\Sigma_1, \ldots, \Sigma_I$, because the general case can be reduced to this one in the same way as for complex resultant varieties (see [7], section 2). Other assumptions of this theorem can not be omitted. In particular, if $I = \dim \mathrm{Lin}_{\mathbb{R}}(\Sigma_1 + \ldots + \Sigma_I) + 1$, then the hypersurface $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ may have self-intersections of codimension 1; if $I < \dim \mathrm{Lin}_{\mathbb{R}}(\Sigma_1 + \ldots + \Sigma_I) + 1$, then $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ may have a boundary; if the affine hulls of essential codimension 1 faces of the collection $(\Sigma_1, \ldots, \Sigma_I)$ intersect every shifted lattice of the form $a + 2\mathbb{Z}^n$, then the smooth locus of $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ is not orientable.

However, we can get rid of the assumptions of Part 2 as follows. A point x of a piecewise-smooth set $V \subset \mathbb{R}^n$ is said to be *weakly smooth*, if an open neighborhood of x is homeomorphic to the product of an open disc D and the bouquet of open segments I_i. Under this assumption, an orientation of the smooth locus of V near x is called an orientation of V at the point x, if it is induced by some orientations of D and I_i. The *weak smooth locus* of V is the set of all weakly smooth points, and the *weak singular locus* is its complement. The intersection number of piecewise-smooth sets of complementary dimension is well defined, if their weakly smooth loci are orientable, and the codimensions of their weakly singular loci are greater than 1.

This leads to the following version of Theorem 4:

1) If the collection $\Sigma_1, \ldots, \Sigma_I$ is essential, then the codimension of the weak singular locus of $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ in $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ is greater then 1.

2) If, in addition, there exists $a \in \mathbb{Z}^n$, such that the shifted lattice $a + 2\mathbb{Z}^n$ does not intersect the shifted lattice $\widetilde{\Sigma}_1 + \ldots + \widetilde{\Sigma}_I + \mathrm{Lin}_{\mathbb{Z}}(\widetilde{\Sigma}_1 + \ldots + \widetilde{\Sigma}_I)$ for every consistent codimension 1 face $(\widetilde{\Sigma}_1, \ldots, \widetilde{\Sigma}_I)$ of the collection $(\Sigma_1, \ldots, \Sigma_I)$, then the weak smooth locus of $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ is orientable.

Lemma 8 below explicitely describes orientations of $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$.

2 Proof of Theorem 4

Parts 1 and 2 easily follow from the corresponding facts about complex resultant varieties, and Part 3 of Theorem 4 is the main result of this paper.

Part 1. We denote *the real locus* $V \cap \mathbb{R}^q$ of a complex analytic set $V \subset \mathbb{C}^q$ by $\mathbb{R}V$.

LEMMA 5. *If* $p : \mathbb{C}^q \to \mathbb{C}^r$ *is a complex analytic map such that* $p(\mathbb{R}^q) \subset \mathbb{R}^r$*, and* $V \subset \mathbb{C}^q$ *is an irreducible complex analytic set such that* $\dim_{\mathbb{R}} \mathbb{R}V = \dim_{\mathbb{C}} V$ *at every point of* $\mathbb{R}V$*, then* $\dim_{\mathbb{R}} p(\mathbb{R}V) = \dim_{\mathbb{C}} p(V)$ *at every point of* $p(\mathbb{R}V)$.

Proof. We denote the set of all points $y \in V$ such that $\dim_{\mathbb{C}} p(V) + \dim_{\mathbb{C}} p^{(-1)}\big(p(y)\big) \cap V > \dim_{\mathbb{C}} V$ by V_0. Obviously, $\dim_{\mathbb{C}} V_0 < \dim_{\mathbb{C}} V$, hence $\dim_{\mathbb{R}} \mathbb{R}V_0 \leq \dim_{\mathbb{C}} V_0 < \dim_{\mathbb{C}} V = \dim_{\mathbb{R}} \mathbb{R}V$, hence $\mathbb{R}V \setminus \mathbb{R}V_0$ is everywhere dense in $\mathbb{R}V$, hence the subanalytic set $P = p(\mathbb{R}V \setminus \mathbb{R}V_0)$ is everywhere dense in $p(\mathbb{R}V)$. We note that

$$\dim_{\mathbb{R}} p^{(-1)}(x) \cap \mathbb{R}V \leq \dim_{\mathbb{C}} V - \dim_{\mathbb{C}} p(V) \tag{1}$$

for every point $x \in P$, because $p^{(-1)}(x) \cap \mathbb{R}V \subset p^{(-1)}(x) \cap V$ and

$$\dim_{\mathbb{C}} p^{(-1)}(x) \cap V = \dim_{\mathbb{C}} V - \dim_{\mathbb{C}} p(V).$$

If we assume that the dimension of the set P is smaller than $\dim_{\mathbb{C}} p(V)$ at every point of its open non-empty subset P_0, then, by inequality (1), the dimension of $\mathbb{R}V$ at every point of its open subset $p^{(-1)}(P_0) \cap \mathbb{R}V$ is smaller than $\dim_{\mathbb{C}} p(V) + \big(\dim_{\mathbb{C}} V - \dim_{\mathbb{C}} p(V)\big) = \dim_{\mathbb{C}} V$, which contradicts the condition $\dim_{\mathbb{R}} \mathbb{R}V = \dim_{\mathbb{C}} V$. Thus, the set of all points of P, where $\dim_{\mathbb{R}} P = \dim_{\mathbb{C}} p(V)$, is everywhere dense in P. $\square$

Let V be the set of all points $(t, \varphi_1, \ldots, \varphi_I) \in (\mathbb{C} \setminus \{0\})^N \times (\mathbb{C}[\Sigma_1] \oplus \ldots \oplus \mathbb{C}[\Sigma_I])$ such that $\varphi_1(t) = \ldots = \varphi_I(t) = 0$, and let p be the projection of the product $(\mathbb{C} \setminus \{0\})^N \times (\mathbb{C}[\Sigma_1] \oplus \ldots \oplus \mathbb{C}[\Sigma_I])$ to the second multiplier. The variety V is irreducible and $\dim_{\mathbb{C}} V = \dim_{\mathbb{R}} \mathbb{R}V$, because V and $\mathbb{R}V$ are $\mathbb{K}$-vector bundles of the same rank over $(\mathbb{K} \setminus \{0\})^N$, where $\mathbb{K}$ stands for $\mathbb{C}$ and $\mathbb{R}$ respectively. Since the closures of $p(V)$ and $p(\mathbb{R}V)$ equal $R_{\mathbb{C}}(\Sigma_1, \ldots, \Sigma_I)$ and $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ respectively, the lemma above gives the equality $\operatorname{codim}_{\mathbb{R}} R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I) = \operatorname{codim}_{\mathbb{C}} R_{\mathbb{C}}(\Sigma_1, \ldots, \Sigma_I)$. The latter codimension is equal to $\max\limits_{\{i_1, \ldots, i_J\} \subset \{1, \ldots, I\}} J - \dim \operatorname{Lin}_{\mathbb{R}}(\Sigma_{i_1} + \ldots + \Sigma_{i_J})$ by Theorem 2.12 from [7].

Part 2. For a fan Γ in $\mathbb{R}^n$, we denote the corresponding toric variety by $\mathbb{CT}^{\Gamma}$, and denote its real locust by $\mathbb{RT}^{\Gamma}$ (see details in [1] or, for the smooth case, in [2]). The inclusion $(\mathbb{K} \setminus \{0\}) \subset \mathbb{KT}^{\Gamma}$ induces the inclusion $(\mathbb{K} \setminus \{0\})^N \times (\mathbb{K}[\Sigma_1] \oplus \ldots \oplus \mathbb{K}[\Sigma_I]) \subset \mathbb{KT}^{\Gamma} \times (\mathbb{K}[\Sigma_1] \oplus \ldots \oplus \mathbb{K}[\Sigma_I])$ for $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$, we denote the compactification of V and $\mathbb{R}V$ in the latter space by $\mathbb{C}U$ and $\mathbb{R}U$ respectively, and denote the extension of the projection $p : V \to R_{\mathbb{C}}(\Sigma_1, \ldots, \Sigma_I)$ to $\mathbb{C}U$ by the same letter p. Since $R_{\mathbb{K}}(\Sigma_1, \ldots, \Sigma_I) = p(\mathbb{K}U)$, the singular locus of $R_{\mathbb{K}}(\Sigma_1, \ldots, \Sigma_I)$ is contained in the union of the following three sets:

$p(S_{1,\mathbb{K}})$, where $S_{1,\mathbb{K}} \subset \mathbb{K}U$ is the singular locus of $\mathbb{K}U$,

$p(S_{2,\mathbb{K}})$, where $S_{2,\mathbb{K}} \subset \mathbb{K}U$ is the set of all smooth points x such that $dp|_{\mathbb{K}U}(x)$ is degenerate,

$p(S_{3,\mathbb{K}})$, where $S_{3,\mathbb{K}} \subset \mathbb{K}U$ is the set of all points x such that x is not the only point in its fiber $p^{(-1)}(p(x)) \cap \mathbb{K}U$.

Obviously, $S_{i,\mathbb{R}} \subset S_{i,\mathbb{C}}$. We assume (without loss of generality) that the dimension of the convex hull of $\Sigma_1 + \ldots + \Sigma_I$ equals N, and that the fan Γ is dual to the convex hull of $\Sigma_1 + \ldots + \Sigma_I$. If, under these assumptions, the collection $\Sigma_1, \ldots, \Sigma_I$ is essential, then $\dim_{\mathbb{C}} S_{i,\mathbb{C}} \leq \dim_{\mathbb{C}} V - 2$ by Lemma 4.3 from [7].

Part 3. Instead of orientations of $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$, we discuss its coorientations.

Definition 6. A *coorientation* of a piecewise-smooth set $M \subset \mathbb{R}^n$ is a choise of orientations of the smooth locus of M and $\mathbb{R}^n$, up to the simultaneous reverting of these orientations.

The function s_i on the set $(\mathbb{R} \setminus \{0\})^N \times (\mathbb{R}[\Sigma_1] \oplus \ldots \oplus \mathbb{R}[\Sigma_I])$ is defined by the equation $s_i(t, \varphi_1, \ldots, \varphi_I) = \varphi_i(t)$. The smooth variety $\mathbb{R}V \subset (\mathbb{R} \setminus$

$\{0\})^N \times (\mathbb{R}[\Sigma_1] \oplus \ldots \oplus \mathbb{R}[\Sigma_I])$ is the complete intersection defined by the equations $s_1 = \ldots = s_I = 0$. The coordinate system $ds_1, \ldots, ds_I$ on a fiber of the normal bundle $\mathrm{N}(\mathbb{R}V)$ defines its orientation, which we refer to as *the tautological orientation.*

Again, without loss of generality, we assume that the dimension of the convex hull of $\Sigma_1 + \ldots + \Sigma_I$ equals N, and the proof of Part 2 of Theorem 4 gives the following fact as a byproduct. There exists an open dense subset D in the smooth locus of $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$, such that the projection $p : \mathbb{R}V \to R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ induces a diffeomorphism of D and its preimage $p^{(-1)}(D) \cap \mathbb{R}V$. If $y \in \mathbb{R}V$ is the preimage of a point $x \in D$ under this diffeomorphism, then, identifying the tangent spaces $\mathrm{T}_x D$ and $\mathrm{T}_y(\mathbb{R}V)$, we have the identity

$$\mathrm{N}_y(\mathbb{R}V) + \mathrm{T}_x D = \mathrm{T}_y(\mathbb{R} \setminus \{0\})^N + \mathrm{T}_x(\mathbb{R}[\Sigma_1] \oplus \ldots \oplus \mathbb{R}[\Sigma_I]).$$

Suppose that the tautological orientation of the first term of this identity and orientations O_1, O_2 and O_3 of the other three terms induce the same orientation of the two sides of this identity. Then the coorientation (O_1, O_3) of D is said to be *induced by the orientation O_2 of* $(\mathbb{R} \setminus \{0\})^N$.

Below we describe all orientations of $(\mathbb{R}\setminus\{0\})^N$, such that the corresponding induced coorientation of D can be extended to the coorientation of the whole $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$.

DEFINITION 7. If an orientation of the group $(\mathbb{R} \setminus \{0\})^n$ is invariant with respect to the action of $(\mathbb{R} \setminus \{0\})^n$ on itself, and coincides with the standard orientation at the point $(1, \ldots, 1)$, then it is called *the invariant orientation of* $(\mathbb{R} \setminus \{0\})^n$. For an arbitrary (not necessary linear) function $r : \mathbb{Z}_2^n \to \mathbb{Z}_2$, the r-orientation of $(\mathbb{R} \setminus \{0\})^n$ is the orientation that differs from the invariant one by $(-1)^{r(a_1,\ldots,a_n)}$ at the point $\big((-1)^{a_1}, \ldots, (-1)^{a_n}\big)$ for every $(a_1, \ldots, a_n) \in \mathbb{Z}_2^n$.

For a codimension 1 face $(\widetilde{\Sigma}_1, \ldots, \widetilde{\Sigma}_I)$ of the collection $(\Sigma_1, \ldots, \Sigma_I)$, consider a primitive normal covector of the convex hull of the sum $\widetilde{\Sigma}_1 + \ldots + \widetilde{\Sigma}_I$. This covector is unique modulo 2, we denote it by $\gamma_{(\widetilde{\Sigma}_1,\ldots,\widetilde{\Sigma}_I)} \in \mathbb{Z}_2^N$, and denote its value at an arbitrary point of $\widetilde{\Sigma}_1 + \ldots + \widetilde{\Sigma}_I$ by $g_{(\widetilde{\Sigma}_1,\ldots,\widetilde{\Sigma}_I)} \in \mathbb{Z}_2$.

Let $O_{\Sigma_1,\ldots,\Sigma_I}$ be the set of all functions $r : \mathbb{Z}_2^N \to \mathbb{Z}_2$, such that $r(a+\gamma_F) = r(a) + g_F + 1$ for every consistent codimension 1 face F of the collection $\Sigma_1, \ldots, \Sigma_I$. Let $q(\Sigma_1, \ldots, \Sigma_I)$ be the codimension of the linear span of the covectors γ_F. We note that, if there exists a point $a \in \mathbb{Z}^N$ mentioned in the formulation of Part 3 of Theorem 4, then the set $O_{\Sigma_1,\dots,\Sigma_I}$ consists of $2^{2^{q(\Sigma_1,\dots,\Sigma_I)}}$ elements, and $O_{\Sigma_1,\dots,\Sigma_I} = \varnothing$ otherwise. Hence, Part 3 of Theorem 4 is a corollary of the following statement.

LEMMA 8. *Under the assumptions of Part 3 of Theorem 4, the coorientation of the set D, induced by the r-orientation of $(\mathbb{R}\setminus\{0\})^N$, can be extended to the variety $R_{\mathbb{R}}(\Sigma_1,\dots,\Sigma_I)$ for every $r \in O_{\Sigma_1,\dots,\Sigma_I}$.*

The proof is given below and implies the following remarks.

1. The variety $R_{\mathbb{R}}(\Sigma_1,\dots,\Sigma_I)$ may admit more than two coorientations because its smooth locus may have more than one connected component; namely, it has $2^{q(\Sigma_1,\dots,\Sigma_I)}$ components.

2. Lemma 8 describes all possible orientations of the smooth locus of the variety $R_{\mathbb{R}}(\Sigma_1,\dots,\Sigma_I)$ under the assumptions of Part 3 of Theorem 4. However, it may not describe all orientations of the weak smooth locus (which is defined one paragraph after the statement of Theorem 4). For example, if $R_{\mathbb{C}}(\Sigma_1,\dots,\Sigma_I)$ is a hypersurface, then it is given by a certain square-free real polynomial, which is called the $(\Sigma_1,\dots,\Sigma_I)$-resultant (see [3], [4]). The differential of this polynomial defines a coorientation of the smooth locus of $R_{\mathbb{R}}(\Sigma_1,\dots,\Sigma_I)$ that can be extended to the weak smooth locus of $R_{\mathbb{R}}(\Sigma_1,\dots,\Sigma_I)$. However, if the codimension of the singular locus of $R_{\mathbb{R}}(\Sigma_1,\dots,\Sigma_I)$ equals 1, then this coorientation can not be induced by an orientation of $(\mathbb{R}\setminus\{0\})^N$.

3 Proof of Lemma 8

If $(\Sigma'_1,\dots,\Sigma'_I)$ is a codimension 1 face of $(\Sigma_1,\dots,\Sigma_I)$, then the convex hull of $\Sigma'_1+\dots+\Sigma'_I$ is a codimension 1 face of the convex hull of $\Sigma_1+\dots+\Sigma_I$, and the primitive external normal covector to this face is denoted by $\tilde{\gamma}_{(\Sigma'_1,\dots,\Sigma'_I)} \in (\mathbb{Z}^N)^*$. Let Γ be the dual fan of the convex hull of the sum $\Sigma_1+\dots+\Sigma_I$, and let $\mathbb{RT}^\Gamma$ be the corresponding real toric variety. If γ generates a 1-dimensional cone of the fan Γ, then the corresponding codimension 1 orbit of $\mathbb{RT}^\Gamma$ is denoted by W_γ. Let $\mathbb{RT}^\Gamma_1$ be the union of $(\mathbb{R}\setminus\{0\})^N$ and the orbits $W_{\tilde{\gamma}_F}$, where F runs over all consistent codimension 1 faces of $(\Sigma_1,\dots,\Sigma_I)$.

The inclusions $(\mathbb{R}\setminus\{0\})^N \subset \mathbb{RT}^\Gamma_1 \subset \mathbb{RT}^\Gamma$ induce the inclusions

$$(\mathbb{R}\setminus\{0\})^N \times (\mathbb{R}[\Sigma_1]\oplus\dots\oplus\mathbb{R}[\Sigma_I]) \subset$$

$$\mathbb{RT}^\Gamma_1 \times (\mathbb{R}[\Sigma_1]\oplus\dots\oplus\mathbb{R}[\Sigma_I]) \subset$$

$$\mathbb{R}\mathbb{T}^\Gamma \times (\mathbb{R}[\Sigma_1] \oplus \ldots \oplus \mathbb{R}[\Sigma_I]),$$

we denote the closure of the variety $\mathbb{R}V$ in the second and the third of these spaces by U_1 and U respectively, and denote the extension of the projection $p : \mathbb{R}V \to R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ to U by the same letter p.

LEMMA 9. *There exists an open subanalytic subset $\tilde{D}$ in the smooth locus of $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$, such that the codimension of $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I) \setminus \tilde{D}$ in $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ is at least 2, and the projection $p : U_1 \to R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ induces a diffeomorphism between $\tilde{D}$ and its preimage $p^{(-1)}(\tilde{D}) \cap U_1$.*

Proof. We define $\tilde{D}$ as the interior of the complement to the union of $p(S_{1,\mathbb{R}}), p(S_{2,\mathbb{R}}), p(S_{3,\mathbb{R}})$ and $p(U \setminus U_1)$. The codimension of $p(S_{1,\mathbb{R}}), p(S_{2,\mathbb{R}}), p(S_{3,\mathbb{R}})$ in $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ was estimated as a part of the proof of Theorem 4, Part 2. To estimate the codimension of $p(U \setminus U_1)$, we choose an arbitrary cone C in the fan Γ, such that the corresponding orbit W of $\mathbb{R}\mathbb{T}^\Gamma$ is not contained in U_1, and estimate the codimension of the set

$$p\Big(U \cap (W \times (\mathbb{R}[\Sigma_1] \oplus \ldots \oplus \mathbb{R}[\Sigma_I]))\Big)$$

as follows. We choose a covector γ in the relative interior of C, and denote by Σ_i^γ the subset of Σ_i, where γ attains its maximum as a function on Σ_i. Then the set

$$p\Big(U \cap (W \times (\mathbb{R}[\Sigma_1] \oplus \ldots \oplus \mathbb{R}[\Sigma_I]))\Big)$$

is the preimage of $R_{\mathbb{R}}(\Sigma_1^\gamma, \ldots, \Sigma_I^\gamma)$ under the natural projection $\mathbb{R}[\Sigma_1] \oplus \ldots \oplus \mathbb{R}[\Sigma_I] \to \mathbb{R}[\Sigma_1^\gamma] \oplus \ldots \oplus \mathbb{R}[\Sigma_I^\gamma]$, and we can estimate the codimension of $R_{\mathbb{R}}(\Sigma_1^\gamma, \ldots, \Sigma_I^\gamma)$ by Theorem 4, Part 1. $\square$

To extend the desired coorientation from D to $\tilde{D}$, which would prove Lemma 8, we need the following notation. The support function $\Sigma(\cdot) : (\mathbb{R}^N)^* \to \mathbb{R}$ of a finite set $\Sigma \subset \mathbb{Z}^N$ is defined as follows: its value $\Sigma(\gamma)$ at a covector $\gamma \in (\mathbb{R}^N)^*$ equals the maximal value of γ as a function on Σ. The set of all primitive generators of 1-dimensional cones of the fan Γ is denoted by Γ_1. For every $\gamma \in \Gamma_1$, the corresponding codimension 1 orbit of the complex toric variety $\mathbb{C}\mathbb{T}^\Gamma$ is denoted by W_γ. If the support function of Σ is linear on every cone of the fan Γ, then the variety $\mathbb{C}\mathbb{T}^\Gamma$ carries an ample line bundle $\mathcal{I}_\Sigma$ with a meromorphic section τ_Σ, such that the divisor of zeros and poles of τ_Σ equals $\sum_{\gamma \in \Gamma_1} \Sigma(\gamma) \cdot W_\gamma$. The pair $(\mathcal{I}_\Sigma, \tau_\Sigma)$ is uniquely defined by this condition. The section τ_Σ generates a real line bundle on $\mathbb{R}\mathbb{T}^\Gamma$, and defines an orientation of its fibers over $(\mathbb{R}\setminus\{0\})^N \subset \mathbb{RT}^\Gamma$; we denote this real line bundle by $\mathbb{R}\mathcal{I}_\Sigma$, and refer this orientation to as *the Σ-orientation.*

The Σ_i-orientation of the line bundles $\mathbb{R}\mathcal{I}_{\Sigma_i}$ for $i = 1, \dots, I$ and the r-orientation of $(\mathbb{R}\setminus\{0\})^N$ define an orientation on the total space of the bundle $\bigoplus_i \mathbb{R}\mathcal{I}_{\Sigma_i}$ over $(\mathbb{R}\setminus\{0\})^N$, and we refer this orientation to as *the r-orientation on the total space of the bundle $\bigoplus_i \mathbb{R}\mathcal{I}_{\Sigma_i}$ over $(\mathbb{R}\setminus\{0\})^N$.*

LEMMA 10. *The r-orientation on the total space of the vector bundle $\bigoplus_i \mathbb{R}\mathcal{I}_{\Sigma_i}$ over $(\mathbb{R}\setminus\{0\})^N$ can be extended to the total space of this bundle over $\mathbb{RT}_1^\Gamma$ if and only if $r \in O_{\Sigma_1,\dots,\Sigma_I}$.*

We denote the latter total space by M.

Proof. Every codimension 1 orbit W of $\mathbb{RT}_1^\Gamma$ corresponds to a 1-dimensional cone of Γ, generated by a primitive covector γ. If we choose a coordinate system so that $\gamma = (1, 0, \dots, 0)$, then the inclusion $(\mathbb{R}\setminus\{0\})^N \subset \mathbb{RT}_1^\Gamma$ extends to the inclusion $\mathbb{R}^1 \times (\mathbb{R}\setminus\{0\})^{N-1} \subset \mathbb{RT}_1^\Gamma$, such that the image of $\{0\} \times (\mathbb{R}\setminus\{0\})^{N-1}$ equals W. The line bundles $\mathbb{R}\mathcal{I}_{\Sigma_i}$ are trivial on $\mathbb{R}^1 \times (\mathbb{R}\setminus\{0\})^{N-1}$, and we identify them with the trivial line bundle $\mathbb{I}$ on $\mathbb{R}^1 \times (\mathbb{R}\setminus\{0\})^{N-1}$. Thus, we can consider the Σ_I-orientation of the line bundle $\mathbb{R}\mathcal{I}_{\Sigma_i}$ on $(\mathbb{R}\setminus\{0\})^N$ as an orientation of the trivial line bundle $\mathbb{I}$ on $(\mathbb{R}\setminus\{0\})^N$, consider the r-orientation of the total space of the vector bundle $\bigoplus_i \mathbb{R}\mathcal{I}_{\Sigma_i}$ over the set $(\mathbb{R}\setminus\{0\})^N$ as an orientation of the total space of $\underbrace{\mathbb{I}\oplus\dots\oplus\mathbb{I}}_{I}$ over $(\mathbb{R}\setminus\{0\})^N$, and our aim is to verify if the r-orientation can be extended to the total space of $\underbrace{\mathbb{I}\oplus\dots\oplus\mathbb{I}}_{I}$ over $\mathbb{R}^1 \times (\mathbb{R}\setminus\{0\})^{N-1}$.

For arbitrary $a_2, \dots, a_N$ in $\mathbb{Z}_2$, we denote by $W^{a_2,\dots,a_N}$ the connected component of $W = \{0\}\times(\mathbb{R}\setminus\{0\})^{N-1}$, containing the point $\Big(0, (-1)^{a_2}, \dots, (-1)^{a_N}\Big)$. The r-orientation of the total space of the vector bundle $\underbrace{\mathbb{I}\oplus\dots\oplus\mathbb{I}}_{I}$ over the set $(\mathbb{R}\setminus\{0\})^N$ can be extended to the total space of this bundle over the set $(\mathbb{R}\setminus\{0\})^N \cup W^{a_2,\dots,a_N}$ if and only if it is the same at the points $\big(-1, (-1)^{a_2}, \dots, (-1)^{a_N}\big) \in \mathbb{R}^1 \times (\mathbb{R}\setminus\{0\})^{N-1}$ and $\big(1, (-1)^{a_2}, \dots, (-1)^{a_N}\big) \in \mathbb{R}^1 \times (\mathbb{R}\setminus\{0\})^{N-1}$. The Σ_i-orientation of the line bundle $\mathbb{I}$ over $(\mathbb{R}\setminus\{0\})^N$ is the same at these two points if and only if $\Sigma_i(\gamma)$ is even (this number is the multiplcity of the hypersurface W in the divisor of the section τ_{Σ_i}, that defines the Σ_i-orientation). The r-orientation of $(\mathbb{R}\setminus\{0\})^N$ is the same at these two points if and only if $r(0, a_2, \dots, a_N) - r(1, a_2, \dots, a_N) = 1$. Thus, the r-orientation on the total space of the vector bundle $\underbrace{\mathbb{I}\oplus\ldots\oplus\mathbb{I}}_{I}$ is the same at these two points if and only if $r(0,a_2,\ldots,a_N)-r(1,a_2,\ldots,a_N)-1+\sum_i\Sigma_i(\gamma)=0$ modulo 2, i.e. $r(a)-r(a+\gamma)-1+\sum_i\Sigma_i(\gamma)=0$ modulo 2, where $a=(0,a_2,\ldots,a_n)$ or $a=(1,a_2,\ldots,a_n)$. This condition for all a and γ gives the definition of the set $O_{\Sigma_1,\ldots,\Sigma_I}$. □

We recall that the function s_i on the space $(\mathbb{R}\setminus\{0\})^N\times(\mathbb{R}[\Sigma_1]\oplus\ldots\oplus\mathbb{R}[\Sigma_I])$ is defined by the equality $s_i(t,\varphi_1,\ldots,\varphi_I)=\varphi_i(t)$. The section $s_i\tau_{\Sigma_i}$ of the pullback of the bundle $\mathbb{R}\mathcal{I}_{\Sigma_i}$ to the product $(\mathbb{R}\setminus\{0\})^N\times(\mathbb{R}[\Sigma_1]\oplus\ldots\oplus\mathbb{R}[\Sigma_I])$ can be extended to a certain section $\tilde{s}_i$ of the pullback of this bundle to $\mathbb{R}\mathbb{T}^\Gamma\times(\mathbb{R}[\Sigma_1]\oplus\ldots\oplus\mathbb{R}[\Sigma_I])$. The divisors of zeros of the sections $\tilde{s}_1,\ldots,\tilde{s}_I$ intersect transversally on the smooth variety $\mathbb{R}\mathbb{T}^\Gamma_1$, and their intersection equals U_1. For every point $(t,x)\in U_1\subset\mathbb{R}\mathbb{T}^\Gamma_1\times(\mathbb{R}[\Sigma_1]\oplus\ldots\oplus\mathbb{R}[\Sigma_I])$, the differential $(d\tilde{s}_1,\ldots,d\tilde{s}_I)$ identifies the fiber of the vector bundle $\bigoplus_i\mathbb{R}\mathcal{I}_{\Sigma_i}$ at t with the fiber of the normal bundle of U_1 at (t,x). Equipping the fiber $\left(\bigoplus_i\mathbb{R}\mathcal{I}_{\Sigma_i}\right)_t$ and the fiber of the normal bundle $\mathrm{N}_{(t,x)}U_1$ with the same orientation, and equipping M at the point $t\in\mathbb{R}\mathbb{T}^\Gamma_1\subset M$ with the r-orientation (see the formulation of Lemma 10 for this notation), we can equip the smooth varieties $\mathbb{R}[\Sigma_1]\oplus\ldots\oplus\mathbb{R}[\Sigma_I]$ and U_1 with orientations O_1 and O_2 that induce the same orientation of the two sides of the identity

$$(\oplus_i\mathbb{R}\mathcal{I}_{\Sigma_i})_t+\mathrm{N}_{(t,x)}U_1+\mathrm{T}_{(t,x)}U_1=\mathrm{T}_tM+\mathrm{T}_x(\mathbb{R}[\Sigma_1]\oplus\ldots\oplus\mathbb{R}[\Sigma_I]).$$

Since the projection p establishes a diffeomorphism between $\tilde{D}$ and an open dense subset of U_1, we can consider O_2 as an orientation on $\tilde{D}$, and the pair (O_1,O_2) induces a coorientation on this set. By Lemma 9, this coorientation extends to a coorientation of the resultant variety $R_{\mathbb{R}}(\Sigma_1,\ldots,\Sigma_I)$. On the other hand, as a coorientation on D, it is induced by the r-orientation of $(\mathbb{R}\setminus\{0\})^N$.

4 Intersection numbers of real resultant varieties

A key observation in [6] is Proposition 5, which relates intersection numbers of complex resultant varieties and intersection numbers of divisors on toric varieties. To formulate the real version of this result, we need the following definition.

Suppose that the codimension of the singular locus of a closed m-dimensional subanalytic set M is at least 2, suppose that the total space S of an m-dimensionsl real vector bundle E on M is oriented, and suppose that the zero locus of a continuous section w of E is compact. Then we can consider a smooth perturbation $\tilde{w}$ of this section, such that its zero locus consists of finitely many nondegenerate isolated points p_i in the smooth locus of M near the zero locus of w. The differential $d\tilde{w}$ at the point p_i identifies the tangent spaces T_0 and $T_{\tilde{w}}$ to the graphs of the sections 0 and $\tilde{w}$ of the bundle E. Since the tangent space to the total space S at p_i equals the direct sum $T_0 + T_{\tilde{w}}$, an arbitrary orientation of T_0 and the corresponding orientation of $T_{\tilde{w}}$ define an orientation of S at p_i, which differs from the given orientation of S by ± 1. We denote the latter number by sgn_i.

DEFINITION **11.** The sum $\sum_i \mathrm{sgn}_i$ does not depend on the choice of the perturbation $\tilde{w}$, and is called *the index of* w.

Let n be the codimension of the resultant variety $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$, and let $f : \mathbb{R}^n \to \mathbb{R}[\Sigma_1] \oplus \ldots \oplus \mathbb{R}[\Sigma_I]$ be a continuous mapping such that the preimage of $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ is compact. We adopt the notation $\mathbb{R}\mathcal{I}_{\Sigma_i}$, $\tilde{s}_i$ and M from the proof of Lemma 8. In this notation, let S be the total space of the poolback of the vector bundle $\bigoplus_i \mathbb{R}\mathcal{I}_{\Sigma_i}$ under the projection $\mathbb{R}\mathbb{T}^\Gamma \times \mathbb{R}^n \to \mathbb{R}\mathbb{T}^\Gamma$. The pullback of the section $(\tilde{s}_1, \ldots, \tilde{s}_I)$ under the map $(\mathrm{id}, f) : \mathbb{R}\mathbb{T}^\Gamma \times \mathbb{R}^n \to \mathbb{R}\mathbb{T}^\Gamma \times (\mathbb{R}[\Sigma_1] \oplus \ldots \oplus \mathbb{R}[\Sigma_I])$ is a section s of this bundle.

We equip $\mathbb{R}^n$ with the standard orientation, and equip the resultant variety $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ with a coorientation, induced by the r-orientation of $(\mathbb{R} \setminus \{0\})^N$ for some $r \in O_{\Sigma_1, \ldots, \Sigma_I}$. The r-orientation of the total space M and the standard orientation of $\mathbb{C}^n$ define an orientation of S.

THEOREM **12.** *For the specified choice of orientations and coorientations, the intersection number of* $f(\mathbb{R}^n)$ *and* $R_{\mathbb{R}}(\Sigma_1, \ldots, \Sigma_I)$ *equals the index of* s.

Proof. If the image of f intersects the resultant variety transversally at a single point, then the statement is obvious from definitions. The general case can be reduced to this one by a small perturbation of the map f. □

References

[1] V. I. Danilov; The geometry of toric varieties; Russian Math. Surveys, 33 (1978), 2, 97-154.

[2] A. N. Varchenko; Newton polyhedra and estimation of oscillating integrals; Funct. Anal. Appl , 10 (1976), 3, 175-196.

[3] I. M. Gelfand, M. M. Kapranov, A.V.Zelevinsky; Discriminants, Resultants, and Miltidimensional Determinants. Birkhäuser, Boston Basel Berlin, 1994.

[4] B. Sturmfels; On the Newton polytope of the resultant; Journal of Algebraic Combinatorics, 3 (1994), 207-236.

[5] A. Esterov; Indices of 1-forms, resultants, and Newton polyhedra; Russian Math. Surveys 60 (2005), 2, 352–353.

[6] A. Esterov; Indices of 1-forms, intersection indices, and Newton polyhedra; Sb. Math., 197 (2006), 7, 1085–1108.

[7] A. Esterov; Determinantal Singularities and Newton Polyhedra; Trudy MIAN, 259 (2007), 1, 20-38.