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Calculation of 2-adic surgery obstruction of complex projective spaces

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ABSTRACT

Improving the result of last year's workshop at RIMS, we prove that the Kervaire spheres Σ_K^{4k+1} where $k+1$ is not a power of 2, do not admit any free S^1-actions if k is not divisible by 16.

1 Introduction

We have been trying to prove the following conjecture:

Conjecture: The Kervaire sphere Σ_K^{4k+1}, where $k+1$ is not a power of two, does not admit any smooth free S^1-action.

This problem goes back to the work of Brumfiel [1] where he proved that the conjecture is true for $k=2$. Later Igarashi ([4]) verified the conjecture for $k \leq 32$. In last year's workshop, we proved the conjecture for the case where the 2-order of k $\nu_2(k) \leq 2$. The purpose of this note is to give the proof for the case $\nu_2(k) = 3$ and at the same time we obtain some directions to get to the complete solution.

Let's fix some notations that will be used in this note. Let p be a prime. For a nonzero integer n, the exponent of p in the prime factorization of n is called the p-order of n and is denoted by $\nu_p(n)$. We state our main theorem which is a one-step improvement of the last year's result [6].

Theorem. Let k be an integer such that $k+1$ is not a power of two. Then the Kervaire sphere Σ_K^{4k+1} does not admit any smooth free S^1-action if $\nu_2(k) \leq 3$.

In the following description, sections 2 and 4 are completely new. Other sections 3 and 5 are essentially the same, but are included to make this note self-contained.

2 Elementary number theory and formal power series

Let p be a prime. The notion of p-order can be extended to nonzero rational numbers by defining $\nu_p(n/d) = \nu_p(n) - \nu_p(d)$ for a nonzero rational n/d.

Let a nonzero integer n be expressed in an p-adic form $n = \sum_{i=0}^{r} t_i p^i$, then the sum of all digits $\sum_{i=0}^{r} t_i$ is denoted by $\kappa_p(n)$.

Lemma 2.1. Let n be a nonnegative integer. Then we have the folloing.

(a) $\nu_p(n!) = \dfrac{n - \kappa_p(n)}{p-1}$.

(b) $\nu_p\left(\binom{n}{k}\right) = \dfrac{\kappa_p(k) + \kappa_p(n-k) - \kappa_p(n)}{p-1}$.

(c) Let $n = \sum_i n_i p^i$ and $k = \sum_i k_i p^i$ be p-adic expansions of nonneggative integers n and k. Then the binomial coefficient $\binom{n}{k}$ is divisible by p if and only if there exists an i such that $n_i < k_i$.

Proof. It is not difficult to see that both $q_n = \nu_p(n!)$ and $q_n = (n - \kappa_p(n))/(p-1)$ satisfy the same inductive formula

$$q_0 = 0 \quad \text{and} \quad q_n = [n/p] + q_{[n/p]},$$

where $[t]$ denotes the greatest integer not exceeding t. This formula uniquely determines the sequence $\{q_n\}$ and we get (a). (b) follows immediately from (a). To show (c), if such column position i exists, then in the addition process of k and $n-k$ in p-adic forms, there is a column where digit addition carries 1 to the next column. Then the total sum of digits decreases and we have $\kappa_p(k) + \kappa_p(n-k) > \kappa_p(n)$. ■

Lemma 2.2. Let n be an odd natural number.

(a) If m is odd, $\nu_2(n^{qm}+1) = \nu_2(n^q+1)$ and $\nu_2(n^{qm}-1) = \nu_2(n^q-1)$.

(b) $\nu_2(n^{2i}-1) = \nu_2(n^2-1) + \nu_2(i)$.

(c) $\nu_2(n^i - (-1)^i) = \begin{cases} \nu_2(n+1), & \text{if } i \text{ is odd} \\ \nu_2(n^2-1) + \nu_2(i) - 1, & \text{if } i \text{ is even.} \end{cases}$

Proof. (a) follows immediately from the factorization

$$n^{qm} - 1 = (n^q - 1)(n^{(m-1)q} + n^{(m-2)q} + \cdots + n^q + 1).$$

To show (b), in view of (a), without loss of generality we may assume that $i = 2^e$. Then from the factorization

$$n^{2i} - 1 = (n^2-1)(n^2+1)(n^{2^2}+1)\cdots(n^{2^e}+1),$$

we have (b) since $\nu_2(n^{2^r}+1) = 1$. When i is odd, (c) follows from (a). When i is even, (c) is included in (b). ■

We shall consider formal power series with rational coefficients. The quotient field of $\mathbb{Q}[[x]]$ is the ring of formal Laurent series with only a finite number of terms with negative powers of x. We shall only consider such Laurent series.

Given a power series or a Laurent series $F(x)$ we shall denote the coefficient of x^i in $F(x)$ by $(F(x))_i$. For a Laurent series $F(x)$, the coefficient of x^{-1} in $F(x)$ is called the residue of $F(x)$ is denoted by $\mathrm{Res}_x(F(x))$. Let $G(x) = \sum_{i\geq 0} \gamma_i x^i$ be a power series with $\gamma_0 = 0$ and $\gamma_1 \neq 0$. Taking residues are subject to the following change of variables formula:

Proposition 2.3. ([5]) For a Laurent series $F(y)$, we have

$$\mathrm{Res}_y(F(y)) = \mathrm{Res}_x(F(G(x))G'(x)),$$

where $G'(x) = \sum i\gamma_i x^{i-1}$ is the formal derivative of $G(x)$. ■

Bernoulli numbers B_i are rational numbers characterized by

$$\text{(1)} \qquad \frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \sum_{i\geq 1} \frac{(-1)^{k-1}B_i}{(2i)!}x^{2i}.$$

Let us consider the power series of Hirzebruch's index formula:

$$\text{(2)} \qquad h(x) = \frac{x}{\tanh x} = 1 + \sum_{i\geq 1} \frac{(-1)^{i+1}2^{2i}B_i}{(2i)!}x^{2i}.$$

To simplify our notation, we shall put $a_i = (h(x))_{2i} = (-1)^{i+1}2^{2i}B_i/(2i)!$.

Let $\mathbb{Z}_{(2)}$ be the ring of integers localized at 2, that is, $\mathbb{Z}_{(2)}$ is the set of all rational numbers with odd denominator.

Lemma 2.4. As to the 2-order of the coefficients of $h(x)$, we have $\nu_2(a_i) = \kappa_2(i) - 1$ for $i \geq 1$. Therefore all the coefficients of $h(x)$ belong to $\mathbb{Z}_{(2)}$.

Proof. From the theorem of von Staudt and Clausen ([3], 7.10), $\nu_2(B_i) = -1$. Thus we have, $\nu_2(a_i) = 2i - 1 - \nu_2(2i) = 2i - 1 - (2i - \kappa_2(2i)) = \kappa_2(i) - 1$. ■

Later we shall consider the power series $1 + g(x) = h(3x)/h(x)$. Let us write $g(x) = \sum_{i\geq 1} b_i x^{2i}$. From the equality $h(x)g(x) = h(3x) - h(x)$, we have

$$\sum_{i\geq 1} b_i x^{2i}\left(1 + \sum_{i\geq 1} a_i x^{2i}\right) = \sum_{i\geq 1}(3^{2i} - 1)a_i x^{2i}.$$

And by comparing the coefficients, we have

$$\text{(3)} \qquad b_n + \sum_{i=1}^{n-1} a_i b_{n-i} = (3^{2n} - 1)a_n.$$

Lemma 2.5. $\nu_2(b_n) \geq 3$ and $\nu_2(b_n) = 3$ holds if and only if n is a power of 2.

Proof. The assertion is true for $b_1 = 8/3$. We assume that our claim is true for all b_i with $i < n$. If n is not a power of two, $\nu_2((3^{2n} - 1)a_n) = \nu_2(3^2 - 1) + \nu_2(i) + \nu_2(a_n) \geq 3 + 1 = 4$

since $\nu_2(a_n) \geq 1$. From the inductive assumption $\nu_2(a_i b_{n-i}) \geq 3$ and the equality holds if and only if boh i and $n-i$ are of the form $i = 2^r$ and $n - i = 2^s$. However, $r \neq s$ since n is not a power of 2. And if there is such term $a_i b_{n-i}$, then the term $a_{n-i} b_i$ also has 2-order 3. This shows that $\nu_2(a_i b_{n-i} + a_{n-i} b_i) \geq 4$. And we have $\nu_2(b_n) \geq 4$. If n is a power of two, then only the term $a_i b_{n-i}$ with $i = n/2$ has 2-order 3, and since $\nu_2((3^{2n} - 1)a_n) = \nu_2(3^{2n} - 1) = 3 + \nu_2(n) \geq 4$, we have $\nu_2(b_n) = 3$.

3 Surgery Obstruction

This section contains nothing new compared to last year's article. However this section is included to make this article self-contained and thus give the readers the knowledge about the geometric aspects of our subject.

We shall translate the statement concerning group actions to the one about surgery obstructions.

Lemma 3.1. The following two statements are equivalent.
(a) The Kervaire sphere Σ_K^{4k+1} does not admit any free S^1-action.
(b) If the normal map

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M^{4k+2} & \xrightarrow{f} & \mathbb{C}P(2k+1) \end{array} \tag{4}$$

has zero $4k$-dimensional surgery obstruction $s_{4k} = 0$ for the surgery data

$$f|f^{-1}(\mathbb{C}P(2k)) : f^{-1}(\mathbb{C}P(2k)) \to \mathbb{C}P(2k)$$

obtained by restriction to the codimension 2 subspace, then the $(4k+2)$-dimensional surgery obstruction s_{4k+2} of f must also vanish.

Proof. Let us prove that (a) implies (b). Suppose there exists a normal map $f : M^{4k+2} \to \mathbb{C}P(2k+1)$ such that the surgery obstruction s_{4k+2} of f is nonzero and the restricted surgery problem to $\mathbb{C}P(2k)$ has zero surgery obstruction $s_{4k} = 0$. Then we can perform surgery on $f^{-1}(\mathbb{C}P(2k))$ and within the normal cobordism class we may assume that $X = f^{-1}(\mathbb{C}P(2k)) \to \mathbb{C}P(2k)$ is a homotopy equivalence. The tubular neighborhood N of X is homotopy equivalent to $\mathbb{C}P(2k+1)_0 = \mathbb{C}P(2k+1) - \mathrm{int}D^{4k+2}$ and its boundary ∂N is homotopy equivalent to S^{4k+1}. But the remaining part $W = M - \mathrm{int}(N)$ is a parallelizable manifold and its surgery obstruction for the normal map $W \to D^{4k+2}$ rel. ∂W is nonzero. Therefore W has nonzero Kervaire obstruction and its boundary $\partial W = \partial N$ is the Kervaire sphere. Since ∂N is the total space of an S^1-bundle, this implies that the Kervaire sphere admits a free S^1-action.

Conversely, suppose that (b) holds, but (a) does not hold. If the Kervaire sphere Σ_K^{4k+1} admits a free S^1-action, the quotient space of the S^1-action $X^{4k} = \Sigma^{4k+1}/S^1$ is homotopy equivalent to the complex projective space $\mathbb{C}P(2k)$ and the associated D^2-bundle $N^{4k+2} = (\Sigma_K^{4k+1} \times D^2)/S^1$ is homotopy equivalent to $\mathbb{C}P(2k+1)_0 = (S^{4k+1} \times D^2)/S^1$ where the $S^1 \subset \mathbb{C}$ acts on $S^{4k+1} \subset \mathbb{C}^{2k+1}$ and on $D^2 \subset \mathbb{C}$ by complex number multiplication. Let W^{4k+2} be a smooth parallelizable manifold with $\partial W = \Sigma_K^{4k+1}$ and Kervaire invariant $c(W) = 1$. Then by gluing N and W along the common boundary Σ_K, we obtain a normal map $f : M^{4k+2} = N \cup_{\Sigma_K} W \to \mathbb{C}P(2k+1)$ with an appropriate vector bundle ξ, and its surgery obstruction s_{4k+2} is equal to $c(W) = 1$. Hence we have a normal map f with target space $\mathbb{C}P(2k+1)$ with nonzero Kervaire surgery obstruction, but the codimension 2 surgery problem obtained by restricting the target manifold to $\mathbb{C}P(2k)$ has zero surgery obstruction $s_{4k} = 0$, since $f|X^{4k} : X^{4k} \to \mathbb{C}P(2k)$ is a homotopy equivalence. This contradicts the assumption (b). This completes the proof of Lemma 3.1. ■

Our objective of this note is to show that the statement (b) in Lemma 3.1 is true. To do so, we must deal with all possible vector bundles that appear in (4). We point out the following four items that needs consideration:

Bundle data The stable bundle difference $\zeta = \nu_{\mathbb{C}P(2k+1)} - \xi$ is fiber homotopically trivial, namely it belongs to the kernel of the J-homomorphism $J : \widetilde{KO}(\mathbb{C}P(2k+1)) \to \tilde{J}(\mathbb{C}P(2k+1))$. The generators of the kernel can be expressed by Adams operations in KO-theory. The solution of the Adams conjecture imply that 2-local generators are given by the images of $\psi^3_{\mathbb{R}} - 1$ ([10], Theorem 11.4.1).

The surgery obstruction s_{4k} in dimension $4k$ In dimension $4k$, the surgery obstruction is given by the index obstruction, which can be computed using Hirzebruch's L classes. However, the exact form of the obstruction gets complicated and requires simplified treatment.

Surgery obstruction s_{4k+2} in dimension $4k+2$ The surgery obstruction s_{4k+2} in dimension $4k+2$ can be dealt with by the results of [7],[8], [9]. In fact, the obstruction s_{4k+2} is equal to the two dimensional obstruction s_2 for the surgery data s_2, which is essentially the 2-dimensional Kervaire class K_2.

Relation of K_2 and the first Pontrjagin class p_1 From the result originally due to Sullivan, the blacksquare of K_2 for the bundle data ζ is equal to $p_1(\zeta)/8 \mod 2$ (see [11], 14C). This fact gives us a bridge connecting the integral index obstruction and the mod 2 Kervaire obstruction.

4 Index obstruction in dimension $4k$

The kernel of the 2-local J-homomorphism $J : \widetilde{KO}(\mathbb{C}P(2k+1)) \to \tilde{J}(\mathbb{C}P(2k+1))$ is generated by Image$(\psi^q_{\mathbb{R}} - 1)$ (q odd), where $\psi^q_{\mathbb{R}}$ is the Adams operation in KO-theory and we may take $q = 3$. The additive generators of $\widetilde{KO}(\mathbb{C}P(2k+1))$ are given by ω^j $(1 \le j \le k+1)$

where ω is the realification of the complex virtual vector bundle $\eta_{\mathbb{C}} - 1_{\mathbb{C}}$, where $\eta_{\mathbb{C}}$ is complex Hopf line bundle. The Adams operation $\psi^j_{\mathbb{R}}$ on ω is given by the formula

$$\psi^j_{\mathbb{R}}(\omega) = T_j(\omega) \tag{5}$$

where $T_j(z)$ is a polynomial of degree j characterized by

$$T_j(t + t^{-1} - 2) = t^j + t^{-j} - 2\,. \tag{6}$$

Since the coefficient of z^j in $T_j(z)$ is one, we may consider $T_j(\omega)$ $(1 \le j \le k+1)$ as generators of $\widetilde{KO}(\mathbb{C}P(2k+1))$. However, when restricted on $\mathbb{C}P(2k)$, we have $\omega^{k+1} = 0$ and we may safely discard ω^{k+1} in the actual computation. In our argument, we do not necessarily need to know the kernel of $J : \widetilde{KO}(\mathbb{C}P(2k+1)) \to \tilde{J}(\mathbb{C}P(2k+1))$. Later computation shows that we can ignore odd multiples of elements and we have only to know 2-local generators of the kernel. The 2-local generators of the kernel of J are

$$\zeta_j = (\psi^3_{\mathbb{R}} - 1)\psi^j_{\mathbb{R}}(\omega) \qquad (j = 1, 2, \ldots, k) \tag{7}$$

and an element of the 2-local kernel of the J-homomorphism has the form

$$\zeta = \sum_{j=1}^{k} m_j \zeta_j \tag{8}$$

where m_j belong to $\mathbb{Z}_{(2)}$, the ring of integers localized at 2.

The surgery obstruction s_{4k} of the surgery data (4) when restricted on $\mathbb{C}P(2k)$ is given by

$$8s_{4k} = (\mathrm{Index}(M) - \mathrm{Index}(\mathbb{C}P(2k))) = ((\mathcal{L}(\zeta) - 1)\mathcal{L}(\mathbb{C}P(2k)))\,[\mathbb{C}P(2k)] \tag{9}$$

where $\mathcal{L}$ is the multiplicative class associated to the power series

$$h(x) = \frac{x}{\tanh x} = 1 + \sum_{i \ge 1} \frac{(-1)^{i+1} 2^{2i} B_i}{(2i)!} x^{2i}. \tag{10}$$

If the total Pontrjagin class of a bundle ξ is given by $p(\xi) = \prod_i (1 + x_i^2)$, $\mathcal{L}(\xi)$ is given by $\prod_i h(x_i)$ and when M is a manifold, we define $\mathcal{L}(M) = \mathcal{L}(\tau_M)$. To calculate the Pontrjagin class of $\psi^j_{\mathbb{R}}(\omega)$, we note that

$$\begin{aligned} \psi^j_{\mathbb{R}}(\omega) \otimes \mathbb{C} &= \psi^j_{\mathbb{C}}(\omega \otimes \mathbb{C}) = \psi^j_{\mathbb{C}}(\eta_{\mathbb{C}} \oplus \bar{\eta}_{\mathbb{C}} - 2_{\mathbb{C}}) \\ &= \psi^j_{\mathbb{C}}(\eta_{\mathbb{C}}) + \psi^j_{\mathbb{C}}(\bar{\eta}_{\mathbb{C}}) - 2_{\mathbb{C}} = \eta^j_{\mathbb{C}} + \bar{\eta}^j_{\mathbb{C}} - 2_{\mathbb{C}}, \end{aligned}$$

whose total Chern class is $(1 + jx)(1 - jx) = 1 - j^2x^2$, where x is the generator of $H^2(\mathbb{C}P(2k+1))$. Hence the total Pontrjagin class of $\psi^j_{\mathbb{R}}(\omega)$ is $1 + j^2x^2$. For the virtual bundle ζ in (8), we have

$$\mathcal{L}(\zeta) = \prod_{j=1}^{k} \left(\frac{h(3jx)}{h(jx)}\right)^{m_j}. \tag{11}$$

Given a power series $f(x)$ in x, let us express the the coefficient of x^n in $f(x)$ by $(f(x))_n$. The $4k$-dimensional obstruction s_{4k} is given by

$$s_{4k} = \left((\mathcal{L}(\zeta) - 1)h(x)^{2k+1}\right)_{2k} \ / \ 8. \tag{12}$$

We now calculate the $\mathcal{L}$ class:

$$\begin{aligned}
\mathcal{L}(\zeta) - 1 &= \prod_j (1 + g(jx))^{m_j} - 1 \\
&= \prod_j \left(1 + \binom{m_j}{1} g(jx) + \binom{m_j}{2} (g(jx))^2 + \cdots \right) - 1 \\
&= \sum_{i_1+i_2+\cdots+i_k \geq 1} \binom{m_1}{i_1}\binom{m_2}{i_2} \cdots \binom{m_k}{i_k} g(x)^{i_1} g(2x)^{i_2} \cdots g(kx)^{i_k} \\
&\equiv \sum_j \binom{m_j}{1} g(jx) + \sum_j \binom{m_j}{2} g(jx)^2 + \sum_{i<j} \binom{m_i}{1}\binom{m_j}{1} g(ix)g(jx) \quad \bmod 512
\end{aligned}$$

Let us write

$$A_j = \left(g(jx)h(x)^{2k+1}\right)_{2k}, \qquad B_{i,j} = \left(g(ix)g(jx)^2 h(x)^{2k+1}\right)_{2k}.$$

The $4k$-dimensional surgery obstruction s_{4k} is calculated as

$$\begin{aligned}
8s_{4k} &= \left((\mathcal{L}(\zeta) - 1)h(x)^{2k+1}\right)_{2k} \\
&\equiv \sum_j m_j A_j + \sum_j \binom{m_j}{2} B_{j,j} + \sum_{i<j} \binom{m_i}{1}\binom{m_j}{1} B_{i,j} \quad \bmod 512
\end{aligned} \tag{13}$$

Lemma 4.1.

(a) $A_1 = \dfrac{2(3^k - (-1)^k)}{3^k}, \qquad \nu_2(A_1) = \nu_2(k) + 3.$

(b) $B_{1,1} = \dfrac{4(3^k - (-1)^k + (-1)^k 4k)}{3^k}, \qquad \nu_2(B_{1,1}) \geq \nu_2(k) + 5.$

Proof. We first prove (a) : From

$$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

and

$$g(x) = \frac{h(3x)}{h(x)} - 1 = \frac{8\tanh^2 x}{3 + \tanh^2 x},$$

we have

$$\begin{aligned}
A_1 &= \left(\frac{8\tanh^2 x}{3 + \tanh^2 x}\left(\frac{x}{\tanh x}\right)^{2k+1}\right)_{2k} \\
&= \mathrm{Res}_x \left(\frac{8\tanh^2 x}{(3 + \tanh^2 x)\tanh^{2k+1} x}\right)_{2k}
\end{aligned}$$

by change of variables $y = \tanh x$,

$$
\begin{aligned}
&= \operatorname{Res}_y \left(\frac{8}{y^{2k-1}(3+y^2)(1-y^2)} \right) \\
&= \left(\frac{8}{(3+y^2)(1-y^2)} \right)_{2k-2} \\
&= \left(\frac{8}{(3+z)(1-z)} \right)_{k-1} \\
&= 2 \left(\frac{1}{3+z} + \frac{1}{1-z} \right)_{k-1} \\
&= 2 \left(\frac{1}{3} \left(-\frac{1}{3} \right)^{k-1} + 1 \right) \\
&= \frac{2(3^k - (-1)^k)}{3^k}.
\end{aligned}
$$

From Lemma 2.2 (c), we have $\nu_2(3^k - (-1)^k) = \nu_2(k) + 3$. This proves (a).

(b) : In a similar manner, we can calculate $B_{1,1}$.

$$
\begin{aligned}
B_{1,1} &= \left(g(x)^2 h(x)^{2k+1} \right)_{2k} \\
&= \left(\left(\frac{8 \tanh^2 x}{3 + \tanh^2 x} \right)^2 \left(\frac{x}{\tanh x} \right)^{2k+1} \right)_{2k} \\
&= 64 \operatorname{Res}_y \left(\frac{1}{y^{2k-3}(3+y^2)(1-y^2)} \right) \\
&= 64 \left(\frac{1}{(3+y^2)^2(1-y^2)} \right)_{2k-4} \\
&= 64 \left(\frac{1}{(3+z)^2(1-z)} \right)_{k-2} \\
&= 64 \left(\frac{1}{48} \sum_{i \geq 0} \left(\left(-\frac{1}{3} \right)^i + \frac{1}{36} \left(-\frac{1}{3} \right)^i (i+1) + \frac{1}{16} \right) z^i \right)_{k-2} \\
&= 64 \left(\left(-\frac{1}{3} \right)^{k-2} \left(\frac{1}{48} + \frac{k-1}{36} \right) + \frac{1}{16} \right) \\
&= \frac{4(3^k - (-1)^k + (-1)^k 4k)}{3^k}.
\end{aligned}
$$

Since $\nu_2(4(3^k - (-1)^k)) = \nu_2(k) + 4$ and $\nu_2(16k) = \nu_2(k) + 4$, we see that their sum $B_{1,1}$ satisfies $\nu_2(B_{1,1}) \geq \nu_2(k) + 5$. ■

Lemma 4.2. Suppose $m \leq k$, then we have

(a) $$\left(x^{2m} h(x)^{2k+1} \right)_{2k} \equiv \binom{k+m}{2m} \mod 2,$$

(b) $$\left(x^{2m} h(x)^{2k+1} \right)_{2k} \equiv \frac{1}{2^{2m}} \left(1 + (-1)^{k-m} \sum_{i=0}^{2m-1} \binom{k-m+i}{i} 2^i \right) \mod 4.$$

In particular, we have

(c) $$\left(x^2h(x)^{2k+1}\right)_{2k} \equiv \binom{k+1}{2} \mod 2.$$

(d) $$\left(x^2h(x)^{2k+1}\right)_{2k} \equiv \frac{(-1)^{k+1}(2k+1-(-1)^k)}{4} \mod 4.$$

(e) $$\left(x^{2^{r+1}}h(x)^{2k+1}\right)_{2k} \equiv \binom{k+2^r}{2^{r+1}} \mod 2.$$

(f) If $\nu_2(k) \geq r+2$ then $(x^{2^{r+1}}h(x)^{2k+1})_{2k}$ is even.

Proof. (a) : We have

$$\begin{aligned}\left(x^{2m}h(x)^{2k+1}\right)_{2k} &= \mathrm{Res}_x\left(\frac{x^{2m}}{\tanh^{2k+1}x}\right) = \mathrm{Res}_y\left(\frac{(\mathrm{arctan}y)^{2m}}{y^{2k+1}(1-y^2)}\right)\\ &= \mathrm{Res}_y\left(\frac{(y+y^3/3+y^5/5+\cdots)^{2m}}{y^{2k+1}(1-y^2)}\right) = \mathrm{Res}_y\left(\frac{\varphi(y)^{2m}}{y^{2k+1-2m}(1-y^2)}\right)\end{aligned}$$

where $\varphi(y) = 1 + y^2/3 + y^4/5 + \cdots$,

$$\begin{aligned}&\equiv \mathrm{Res}_y\left(\frac{(1+y^2+y^4+y^6+\cdots)^{2m}}{y^{2k+1-2m}(1-y^2)}\right) \mod 2\\ &= \mathrm{Res}_y\left(\frac{1}{y^{2k+1-2m}(1-y^2)^{2m+1}}\right) = \left(\frac{1}{(1-y^2)^{2m+1}}\right)_{2k-2m}\\ &= \left(\frac{1}{(1-z)^{2m+1}}\right)_{k-m} = \left(\sum_i \binom{2m+i}{i} z^i\right)_{k-m}\\ &= \binom{k+m}{k-m} = \binom{k+m}{2m}.\end{aligned}$$

(b) : By similar calculation, we have

$$\begin{aligned}(x^{2m}h(x))_{2k} &\equiv \mathrm{Res}_y\left(\frac{(1-y^2+y^4-y^6+\cdots)^{2m}}{y^{2k+1-2m}(1-y^2)}\right) \mod 4\\ &= \mathrm{Res}_y\left(\frac{1}{y^{2k+1-2m}(1-y^2)(1+y^2)^{2m}}\right)\\ &= \left(\frac{1}{(1-y^2)(1+y^2)^{2m}}\right)^{2k-2m} = \left(\frac{1}{(1-z)(1+z)^{2m}}\right)_{k-m}.\end{aligned}$$

On the other, we have the following expansion

(14) $$\frac{1}{(1-z)(1+z)^n} = \frac{1}{2^n}\left(\frac{1}{1-z} + \frac{1}{2}\sum_{j=1}^{n}\left(\frac{2}{1+z}\right)^i\right)$$

In view of this formular, we have

$$(x^{2m}h(x))_{2k} \equiv \frac{1}{2^{2m}}\left(1 + (-1)^{k-m}\sum_{i=0}^{2m-1}\binom{k-m+i}{i}2^i\right) \mod 4.$$

(c) and (d) follow from (a) and (b) respectively. (e) is a special case of (a) . Last, we prove (f). If k is divisible by 2^{r+2}, then note that the dyadic expansion of $k+2^r$ does not contain 2^{r+2}, therefore by Lemma 2.1 (c), the binominal coefficient $\binom{k+2^r}{2^{r+1}}$ is even. ■

Lemma 4.3 (a) If j is even then $\nu_2(A_j) \geq 5$. In addition if $2 \leq \nu_2(k) \leq 3$, then $\nu_2(A_j) \geq \nu_2(k)+4$.

(b) If j is odd, then $\nu_2(A_j - A_1) \geq 6$ and if in addition $\nu_2(k) \geq 2$, then $\nu_2(A_j - A_1) \geq 7$.

(c) If either i or j is even, then $\nu_2(B_{i,j}) \geq 8$.

(d) If both i and j are odd, then $\nu_2(B_{i,j} - B_{1,1}) \geq 9$.

Proof. (a): In the expression $A_j = \sum_i b_i j^{2i}(x^{2i}h(x)^{2k+1})_{2k}$, $\nu_2(b_i j^{2i}(x^{2i}h(x)^{2k+1})_{2k} = \nu_2(b_i)+2i\nu_2(j)+\nu_2(x^{2i}h(x)^{2k+1})_{2k} \geq 3+2i+\nu_2(x^{2i}h(x)^{2k+1})_{2k}$. If $i>1$, then $\nu_2(b_i j^{2i}) \geq 7$ holds. If $i=1$, then by Lemma 4.2, if $\nu_2(k)=2$ then $(x^2h(x)^{2k+1})_{2k}$ is even. We also have $\nu_2(b_i j^2(x^2h(x)^{2k+1}))_{2k} \geq 6 = \nu_2(k)+4$. If $\nu_2(k)=3$ then from Lemma 4.2 (d) we see that $(x^2h(x)^{2k+1})_{2k} \equiv -k/2 \mod 4$, which is congruent to $0 \mod 4$. This shows that $\nu_2((x^2h(x)^{2k+1})_{2k}) \geq 2$. This proves $\nu_2(A_j) \geq \nu_2(k)+4$.
(b): Let us turn to the case where j is odd. In the expression, $A_j - A_1 = \sum_{i\geq 1} b_i(j^{2i} - 1)(x^{2i}h(x)^{2k+1})_{2k}$, let us consider

$$\begin{aligned} N_i &= \nu_2(b_i(j^2-1)(x^{2i}h(x)^{2k+1})_{2i}) \\ &= \nu_2(b_i)+\nu_2(j^{2i}-1)+\nu_2(x^{2i}h(x)^{2k+1})_{2k} \end{aligned}$$

Here since j is odd, we have $\nu_2(j^2-1) \geq 3$. Thus we have $\nu_2(N_i) \geq 6$. If i is even, then $\nu_2(i) \geq 1$ and we have $N_i \geq 7$. If i is odd then, $\nu_2(b_i) \geq 4$ except for $i=1$. However when $i=1$, if $\nu_2(k) \geq 2$, we see that $(x^2h(x)^{2k+1})_{2k}$ is even by Lemma 4.2 (c). Therefore $N_i \geq 7$.
(c) follows immediately from the fact that $\nu_2(b_i) \geq 3$. To show (d), in the expression

$$g(ix)g(jx) - g(x)^2 = g(ix)(g(jx)-g(x)) + g(x)(g(ix)-g(x),$$

we note that all the coefficients of $g(ix)$ and $g(jx)$ are divisible by 2^3 and that those of $g(jx)-g(x)$ and $g(ix)-g(x)$ are divisible by 2^6. From these facts, we conclude that all the coefficients of $(g(ix)g(jx)-g(x)^2)h(x)^{2k+1}$ are divisible by 2^9. ■

Now we are ready to prove our key lemma:

Lemma 4.4. Suppose that $\nu_2(k) \leq 3$. Then as to the surgery obstruction s_{4k}, we have

$$s_{4k} \equiv 2^{\nu_2(k)} \sum_{j:odd} m_j \mod 2^{\nu_2(k)+1}$$

Proof. We shall examine the 2-orders of the terms on the right hand side of (13).
We first note that from Lemma 4.1 that

$$\nu_2(A_1) = \nu_2(k)+3, \text{ and } \nu_2(B_{1,1}) \geq \nu_2(k)+5.$$

When j is odd and $\nu_2(k) \leq 2$ then we have $\nu_2(A_j) = \nu_2(k)+3$ since by Lemma 4.3 (b), $\nu_2(A_j - A_1) \geq 6 \geq \nu_2(k)+4$. If j is odd and $\nu_2(k)=3$, then we also have $\nu_2(A_j) =$

$\nu_2(k) + 3$ since $\nu_2(A_j - A_1) \geq 7$. When j is even then from Lemma 4.3 (a), we have $\nu_2(A_j) \geq \nu_2(k) + 4$. From Lemma 4.3 (c),(d) and from the fact that $\nu_2(B_{1,1}) \geq \nu_2(k) + 5$, we see that for all i and j, $\nu_2(B_{i,j}) \geq \nu_2(k)+4$. Combining these facts, we get the conclusion. ■

Remark. This invariant $\sum_j m_j$ was called the μ-invariant of the surgery data in [2].

5 The first Kervaire class and the first Pontrjagin class

In the normal map (4), let $\zeta = \nu_{\mathbb{C}P(2k+1)} - \xi$, then it can be written (2-locally) $\zeta = \sum_{j=1}^{k} m_j \zeta_j$ where $\zeta_j = (\psi_{\mathbb{R}}^3 - 1)\psi_{\mathbb{R}}^j(\omega)$. The total Pontrjagin class of $\psi_{\mathbb{R}}^m(\omega)$ is given by

$$p(\psi_{\mathbb{R}}^m(\omega)) = 1 + m^2x^2 \tag{15}$$

and we have

$$p(\zeta_j) = \frac{1 + 9j^2x^2}{1 + j^2x^2} \tag{16}$$

$$p(\zeta) = \prod_j \left(\frac{1 + 9j^2x^2}{1 + j^2x^2}\right)^{m_j}. \tag{17}$$

For the first Pontrjagin class, we have

$$p_1(\zeta)/8 = \left(\sum_j j^2 m_j\right) x^2. \tag{18}$$

We know that the 2-dimensional surgery obstruction s_2 for $f|f^{-1}(\mathbb{C}P(1))$ is equal to $\sum_j j^2 m_j$ mod 2 since in the complex projective space surgery theory, the mod 2 reduction of $p_1(\zeta)$ coincides with the square of the 2-dimensional Kervaire class for the given normal map (see Wall's book [11, Chap 13.]). And it is known that if $k + 1$ is not a power of 2, then $(4k + 2)$-dimensional surgery obstruction coincides with 2-dimensional surgery obstruction ([9],[7],[8]). From these facts we get the following Proposition.

Proposition 5.1. If $\sum_{j:odd} m_j$ is even, then the surgery obstruction s_{4k+2} vanishes.

Proof of Theorem:

Let k is an integer such that $k + 1$ is not a power of two and assume that k is not divisible by 8. Then for the surgery problem of $\mathbb{C}P(2k + 1)$ with bundle data $\zeta = \sum_j m_j \zeta_j$, if the $4k$-dimensional surgery obstruction s_{4k} vanishes then $\sum_j m_j$ must be even from Lemma 4.4. Then by Proposition 5.1, the $(4k + 2)$-dimensional surgery obstruction s_{4k+2} should also vanish. In view of Lemma 3.1, this proves our assertion.

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