数理解析研究所講究録
第 1612 巻 2008 年 99-100

THE Nil-Nil THEOREM IN ALGEBRAIC K-THEORY

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The reduced Nil-groups are certain reduced K-theory groups defined by Friedhelm Waldhausen for pure amalgams and tensor algebras [1]. They measure the defect in his Mayer–Vietoris sequence in the algebraic K-theory of rings. In the main paper [2], we recently showed that the apparently more complicated amalgam Nil can be computed in terms of the apparently simpler tensor Nil.

Theorem 1. *Let R be a (unital, associative) ring. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be R-bimodules. Suppose I is a small, filtered category and $\mathscr{B}_2 = \operatorname{colim}_{\alpha\in I} \mathscr{B}_2^\alpha$ is a direct limit of R-bimodules such that the left R-module structure of each $\mathscr{B}_2^\alpha$ is finitely generated and projective. Then, for every $n \in \mathbb{Z}$, there is an induced isomorphism*

$$\widetilde{K}_n(j) : \widetilde{\mathrm{Nil}}_n(R;\mathscr{B}_1,\mathscr{B}_2) \longrightarrow \widetilde{\mathrm{Nil}}_n(R;\mathscr{B}_1\otimes_R \mathscr{B}_2)\ .$$

An important special case are those amalgams of group rings which are induced by an epimorphism onto the infinite dihedral group $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2 = \mathbb{Z} \rtimes_{-1} \mathbb{Z}/2$.

Corollary 2. *Suppose G is a group with an epimorphism $p : G \to D_\infty$. Denote the p-induced injective amalgamated product decomposition $G = G_1 *_F G_2$. Consider the index-two subgroup $\overline{G} := p^{-1}(\mathbb{Z})$ of G. Denote the p-induced injective HNN-extension $\overline{G} = F \rtimes_\alpha \mathbb{Z}$. Then, for all rings R and for all $n \in \mathbb{Z}$, there is an isomorphism of abelian groups:*

$$\widetilde{\mathrm{Nil}}_n(R[F];R[G_1-F],R[G_2-F]) \cong NK_{n+1}(R[F],\alpha)\ .$$

The right-hand side of the isomorphism is the twisted Bass Nil-group [3] of F.T. Farrell and W.C. Hsiang [4]. These are more readily computable since they involve the Wang sequence in K-theory of the twisted polynomial ring $R[F]_\alpha[x]$.

The following application [2] of the above corollary is a sharpening of the fibered isomorphism conjecture of F. T. Farrell and L. E. Jones in algebraic K-theory. Given a group G, denote vc as the class of virtually cyclic subgroups and fbc as the subclass of finite-by-cyclic subgroups. The elements of the complement vc $-$ fbc are exactly those subgroups of G which are finite-by-D_∞.

Theorem 3. *Let $\varphi : \Gamma \to G$ be an epimorphism of groups. Then, for all rings R and for all $n \in \mathbb{Z}$, the following induced map is an isomorphism:*

$$H_n^\Gamma(E_{\varphi^*\mathrm{fbc}}\Gamma;\mathbf{K}_R) \longrightarrow H_n^\Gamma(E_{\varphi^*\mathrm{vc}}\Gamma;\mathbf{K}_R)\ .$$

Both sides are equivariant homology groups, whose coefficients are given by the spectrum-valued functor $\mathbf{K}_R : \mathrm{Or}\,\Gamma \to \mathrm{SPECTRA}$ of the Bredon orbit category [5].

Recently, the Farrell–Jones conjecture has been proven by A. Bartels, W. Lück, and H. Reich for a large class of infinite groups with torsion [6]. Consider the non-fibered case of $\varphi = \mathrm{id}_\Gamma$.

Corollary 4. *Let Γ be a word-hyperbolic group. Then, for all rings R and for all $n \in \mathbb{Z}$, the algebraic K-theory assembly map is an isomorphism:*

$$H_n^\Gamma(E_{\mathrm{fbc}}\Gamma; \mathbf{K}_R) \longrightarrow K_n(R[\Gamma]) \; .$$

This isomorphism yields specific fruit. In the following calculation [2], the Bass NK-groups vanish if R is a regular Noetherian ring, such as if $R = \mathbb{Z}$. Here $K_n(R[x]) = K_n(R) \oplus NK_n(R)$ by definition.

Theorem 5. *Consider the modular group $\Gamma = PSL(2,\mathbb{Z})$. Then, for any ring R and integer n, we have*

$$K_n(R[\Gamma]) = (\, K_n(R[\mathbb{Z}/2]) \oplus K_n(R[\mathbb{Z}/3])\,) \,/\, K_n(R) \, \oplus \, \bigoplus_{\aleph_0} NK_n(R) \; .$$

Finally, [2] provides the first example of a non-vanishing amalgam Nil-group.

Example 6. Consider the group $G_0 := \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}$. Then

$$\widetilde{\mathrm{Nil}}_0(\mathbb{Z}[G_0]; \mathbb{Z}[G_0], \mathbb{Z}[G_0]) = NK_1(\mathbb{Z}[G_0])$$

is a non-zero abelian group (see [3]), which is a summand of the Whitehead group $\mathrm{Wh}(G_0 \times D_\infty)$. Therefore we obtain the following topological consequence. Consider the finite CW-complexes

$$\begin{aligned} W &:= \mathbb{RP}^2 \times \mathbb{RP}^2 \times S^1 \\ X &:= W \times S^2 \\ Y &:= W \times (\mathbb{RP}^3 - \mathrm{int}\, D^3) \; . \end{aligned}$$

Given any non-zero element of the above amalgam Nil-group, one can construct [7] the first known example of a homotopy equivalence $h : K \to Y \cup_X Y$, where K is a certain finite CW-complex, such that h is **not** splittable along X. Here, we say h is *splittable along* X if there exist a simple homotopy equivalence $s : K' \to K$ of finite CW-complexes and a homotopy equivalence $h' : K' \to Y \cup_X Y$ such that $h \circ s \simeq h'$ and the cellular restriction $(h')^{-1}(X) \to X$ is a homotopy equivalence.

References

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