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# Generating the full mapping class group by involutions

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**Abstract**

Let $\Sigma_{g,b}$ denote a closed orientable surface of genus $g$ with $b$ punctures and let $\mathrm{Mod}(\Sigma_{g,b})$ denote its mapping class group. In [Luo] Luo proved that if the genus is at least 3, $\mathrm{Mod}(\Sigma_{g,b})$ is generated by involutions. He also asked if there exists a universal upper bound, independent of genus and the number of punctures, for the number of torsion elements/involutions needed to generate $\mathrm{Mod}(\Sigma_{g,b})$. Brendle and Farb [BF] gave an answer in the case of $g \geq 3, b = 0$ and $g \geq 4, b = 1$, by describing a generating set consisting of 6 involutions. Kassabov showed that for every $b$ $\mathrm{Mod}(\Sigma_{g,b})$ can be generated by 4 involutions if $g \geq 8$, 5 involutions if $g \geq 6$ and 6 involutions if $g \geq 4$. We proved that for every $b$ $\mathrm{Mod}(\Sigma_{g,b})$ can be generated by 4 involutions if $g \geq 7$ and 5 involutions if $g \geq 5$.

## 1 Introduction

Let $\Sigma_{g,b}$ be an closed orientable surface of genus $g \geq 1$ with arbitrarily chosen $b$ points (which we call punctures). Let $\mathrm{Mod}(\Sigma_{g,b})$ be the mapping class group of $\Sigma_{g,b}$, which is the group of homotopy classes of orientation-preserving homeomorphisms preserving the set of punctures. Let $\mathrm{Mod}^{\pm}(\Sigma_{g,b})$ be the extended mapping class group of $\Sigma_{g,b}$, which is the group of homotopy class of all (including orientation-reversing) homeomorphisms preserving the set of punctures. By $\mathrm{Mod}^0_{g,b}$ we will denote the subgroup of $\mathrm{Mod}_{g,b}$ which fixes the punctures pointwise.

In [MP], McCarthy and Papadopoulos proved that $\mathrm{Mod}(\Sigma_{g,0})$ is generated by infinitely many conjugetes of a single involution for $g \geq 3$. Luo, see [Luo], described the finite set of involutions which generate $\mathrm{Mod}(\Sigma_{g,b})$ for $g \geq 3$. He also proved that $\mathrm{Mod}(\Sigma_{g,b})$ is generated by torsion elements in all cases except $g = 2$ and $b = 5k + 4$, but this group is not generated by involutions if $g \square 2$. Brendle and Farb proved that $\mathrm{Mod}(\Sigma_{g,b})$ can be generated by 6involutions for $g \geq 3, b = 0$ and $g \geq 4, b \square 1$ (see [BF]). In [Ka], Kassabov proved that for every $b$ $\mathrm{Mod}(\Sigma_{g,b})$ can be generated by 4 involutions if $g \geq 8$, 5 involutions if $g \geq 6$ and 6 involutions if $g \geq 4$. He also proved in the case of $\mathrm{Mod}^{\pm}(\Sigma_{g,b})$.
Our main result is stronger than [Ka].

**Main Theorem.** *For all $g \geq 3$ and $b \geq 0$, the mapping class group* $\mathrm{Mod}(\Sigma_{g,b})$ *can be generated by:*

(*a*) 4 *involutions if* $g \geq 7$;
(*b*) 5 *involutions if* $g \geq 5$.

## 2 Preliminaries

Let $c$ be a simple closed curve on $\Sigma_{g,b}$. Then the (right hand) Dehn twist $T_c$ about $c$ is the homotopy class of the homeomorphism obtained by cutting $\Sigma_{g,p}$ along $c$, twisting one of the side by 360° to the right and gluing two sides of a back to each ohter. Figure 1 shows the Dehn twist about the curve $c$. We will

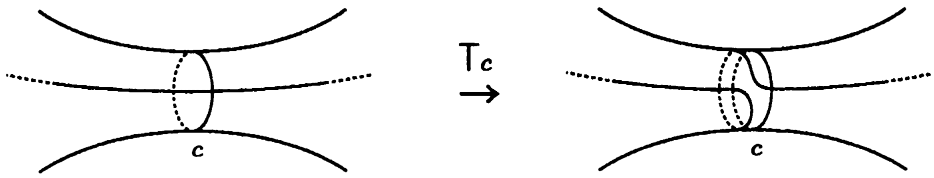


Figure 1: The Dehn twist

denote by $T_c$ the Dehn twist around the curve $c$.

We record the following lemmas.

**Lemma 1.** *For any homeomorphism $h$ of the surface $\Sigma_{g,b}$ the twists around the curves $c$ and $h(c)$ are conjugate in the mapping class group* $\mathrm{Mod}(\Sigma_{g,b})$,

$$T_{h(c)} = hT_ch^{-1}.$$

**Lemma 2.** *Let $c$ and $d$ be two simple closed curves on $\Sigma_{g,b}$. If $c$ is disjoint from $d$, then*

$$T_cT_d = T_dT_c$$

## 3 Proof of main theorem

In this section we proof maintheorem. The keypoints of proof are to generate $T_\alpha$ in 4 involutions by using lantern relation.

### 3.1 The policy of proof

We give the policy of proof of maintheorem.

**Lemma 3.** *Let $G$, $Q$ denote the groups and let $N, H$ denote the subgroups of $G$. Suppose that the group $G$ has the following exact sequence;*

$$1 \to N \xrightarrow{i} G \xrightarrow{\pi} Q \to 1.$$

*If $H$ contains $i(N)$ and has a surjection to $Q$ then we have that $H = G$.*

*Proof.* We suppose that there exists some $g \in G - H$. By the existence of surjection from $H$ to $Q$, we can see that there exists some $h \in H$ such that $\pi(h) = \pi(g)$. Therefore, since $\pi(g^{-1}h) = \pi(g)^{-1}\pi(h) = 1$, we can see that

$g^{-1}h \in \operatorname{Ker} \pi = \operatorname{Im} i$. Then there exists some $n \in N$ such that $i(n) = g^{-1}h$. By $i(N) \subset H$, since $i(n) \in H$ and $h \in H$, we have

$$g = h \cdot i(n)^{-1} \in H.$$

This is contradiction in $g \notin H$. Therefore, we can prove that $H = G$. □

It is clear that we have the exact sequence:

$$1 \to \mathrm{Mod}^0_{g,b} \to \mathrm{Mod}_{g,b} \to \mathrm{Sym}_b \to 1.$$

Therefore, we can see the following corollary;

**Corollary 4.** *Let $H$ denote the subgroup of* $\mathrm{Mod}(\Sigma_{g,b})$, *which contains* $\mathrm{Mod}^0(\Sigma_{g,b})$ *and has a surjection to* $\mathrm{Sym}_b$. *Then $H$ is equal to* $\mathrm{Mod}(\Sigma_{g,b})$.

We generate the subgroup $H$ which has the condition of corollary 4 by involutions.

Let us embed our surface $\Sigma_{g,b}$ in the Euclidian space in two different ways as shown on Figure 2. (In these pictures we will assume that genus $g = 2k + 1$ is odd and the number of punctures $b = 2l + 1$ is odd. In the case of even genus we only have to swap the top parts of the pictures, and in the case of even number of punctures we have to remove the last point.)

In Figure 2 we have also marked the puncture points as $x_1, \ldots, x_b$ and we have the curves $\alpha_i$, $\beta_i$, $\gamma_i$ and $\delta$. The curve $\alpha_i$, $\beta_i$, $\gamma_i$ are non separating curve and $\delta$ is separating curve.

Each embedding gives a natural involution of the surface—the half turn rotation around its axis of symmetry. Let us call these involutions $\rho_1$ and $\rho_2$.

Then we can get following lemma;

**Lemma 5** ([Mo]). *The subgroup of the mapping class group be generated by $\rho_1$, $\rho_2$ and 3 Dehn twists $T_\alpha$, $T_\beta$ and $T_\gamma$ around one of the curve in each family contains the subgroup* $\mathrm{Mod}^0(\Sigma_{g,b})$.

The existence a surjection from the subgroup $H$ of $\mathrm{Mod}(\Sigma_{g,b})$ to $\mathrm{Sym}_b$ is equivalent to showing taht the $\mathrm{Sym}_b$ can be generated by involutions;

$$\begin{aligned} r_1 &= (1, b-1)(2, b-2)\cdots(l, l+1)(b) \\ r_2 &= (2, b-1)(3, b-2)\cdots(l, l+2)(1)(l+1)(b) \\ r_3 &= (1, b)(2, b-1)(3, b-2)\cdots(l, l+2)(l+1) \end{aligned}$$

corresponding to 3 involutions in $H$.

**Lemma 6.** *The symmetric group* $\mathrm{Sym}_b$ *is generated by $r_1, r_2$ and $r_3$.*

*Proof.* The group generated by $r_i$ contains the long cycle $r_3r_1 = (1, 2, \ldots, b)$ and transposition $r_3r_2 = (1, b)$. These two elements generate the whole symmetric group, therefore the involutions $r_i$ generate $\mathrm{Sym}_b$. □

We note that the images of $\rho_1$ and $\rho_2$ to $\mathrm{Sym}_b$ are $r_1$ and $r_2$.

Therefore, by Lemma 1, Corollary 4, Lemma 5 and Lemma 6 we sufficient to generate $H$ by $\rho_1$, $\rho_2$ and involutions which have the following conditions;

⟨1⟩ involutions which genarate the Dehn twist around $\gamma$,

⟨2⟩ two of each involutions which exchange $\alpha$ and $\beta$, $\beta$ and $\gamma$, $\gamma$ and $\alpha$,

⟨3⟩ involution whose image is $r_3$.

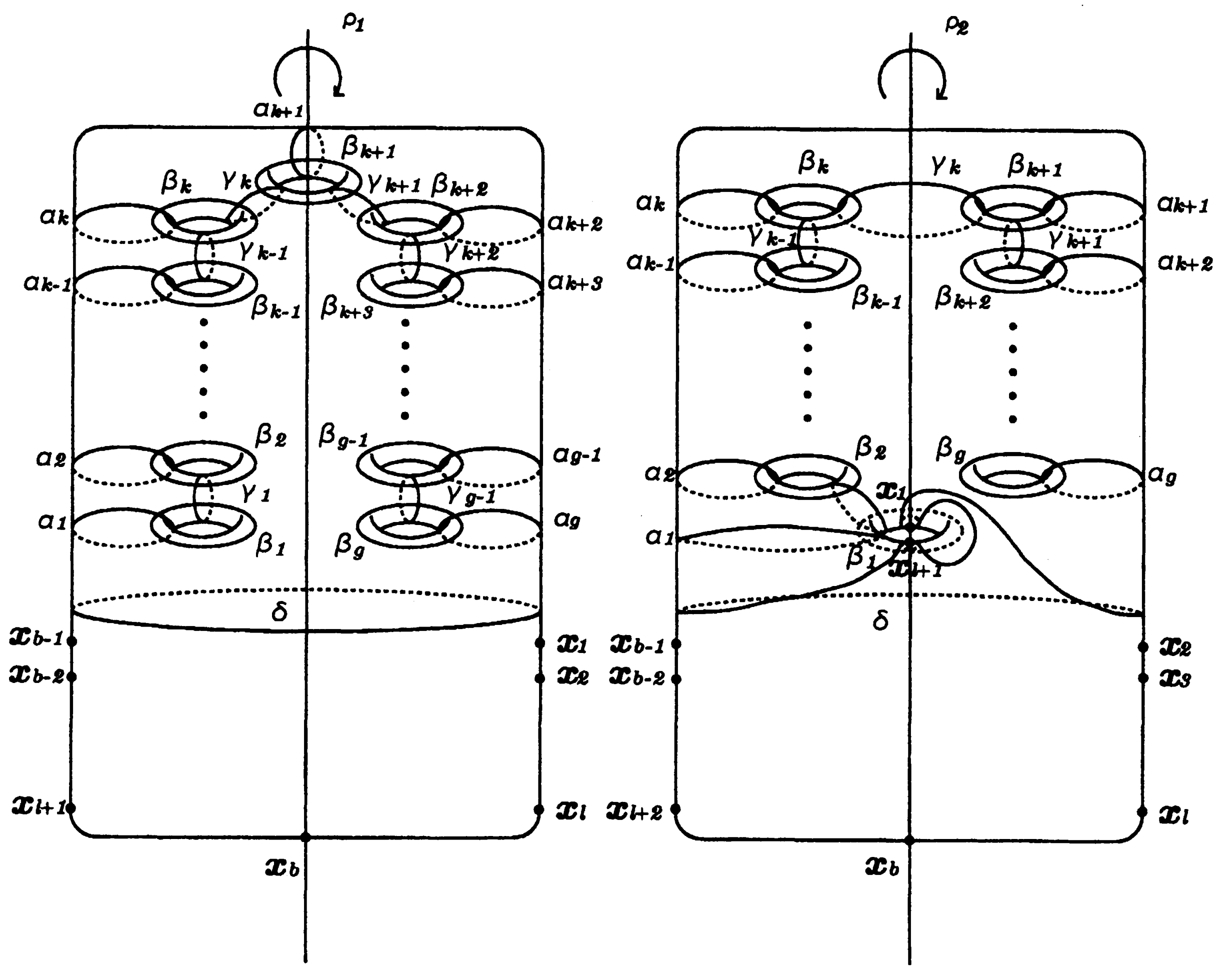


Figure 2: The embeddings of the surface $\Sigma_{g,b}$ in Euclidian space used to define the involutions $\rho_1$ and $\rho_2$.

### 3.2 Generating Dehn twists by 4 involutions

In this subsection, we argue about $\langle 1 \rangle$. Moreover, we generate Dehn twists by 4 involutions. The basic idea is to use the lantern relation.

We begin by recalling the lantern relation in the mapping class group. This relation was first discovered by Dehn and later rediscovered by Johnson.

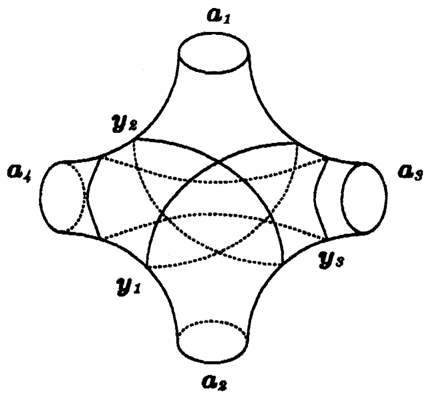


Figure 3: Lantern

From now on we will assume that the genus $g$ of the surface is at least 5.

Let the $S_{0,4}$ be a surface of genus 0 with 4 boundary components. Denote by $a_1, a_2, a_3$ and $a_4$ the four boundary curves of the surface $S_{0,4}$ and let the interior curves $y_1, y_2$ and $y_3$ be as shown in Figure 3.

The following relation:

$$T_{y_1}T_{y_2}T_{y_3} = T_{a_1}T_{a_2}T_{a_3}T_{a_4}. \tag{1}$$

among the Dehn twists around the curves $a_i$ and $y_i$ is known as the lantern relation. Notice that the curves $a_i$ do not intersect any other curve and that the Dehn twists $T_{a_i}$ commute with every twists in this relation. This allows us to rewrite the lantern relation as follows

$$T_{a_4} = (T_{y_1}T_{a_1}^{-1})(T_{y_2}T_{a_2}^{-1})(T_{y_3}T_{a_3}^{-1}). \tag{2}$$

Let $R$ denote the product $\rho_2\rho_1$. By Figure 2 we can see that $R = \rho_2\rho_1$ acts as follows:

$$\begin{aligned} R\alpha_i &= \alpha_{i+1}, \ (1 \square i < g) \\ R\beta_i &= \beta_{i+1}, \ (1 \square i < g) \\ R\gamma_i &= \gamma_{i+1}, \ (1 \square i < g-1). \end{aligned} \tag{3}$$

The lanterns $S$ and $R^{-2}S$ have a common boundary component $a_1 = R^{-2}a_2$ and their union is a surface $S_2$ homeomorphic to a sphere with 6 boundary components. By Figure 4 we can see that there exists an involution $\bar{J}$ of $S_2$ which takes $S$ to $R^{-2}S$.

Let us embed the surface $S_2$ in $\Sigma_{g,b}$ as shown on Figure 5. The boundary components of $S_2$ are $a_1 = \alpha_k$, $a_2 = \alpha_{k+2}$, $a_3 = \gamma_{k+1}$, $a_4 = \gamma_k$, $R^{-2}a_1 = \alpha_{k-2}$, $R^{-2}a_2 = \alpha_k$, $R^{-2}a_3 = \gamma_{k-1}$ and $R^{-2}a_4 = \gamma_{k-2}$; and the middle curve $y_1 =$

$\alpha_{k+1}$. The Figure 5 shows the existence of the involution $\tilde{J}$ on the complement of $S_2$ which is a surface of genus $g-5$ with 6 boundary components. Gluing together $\bar{J}$ and $\tilde{J}$ gives us the involution $J$ of the surface $\Sigma_{g,b}$. By Figure 4 $J$ acts as follows

$$J(a_1) = R^{-2}a_2,\ J(a_3) = R^{-2}a_1,\ J(y_1) = R^{-2}y_2,\ J(y_3) = R^{-2}y_1.$$

Therefore, we have

$$\begin{aligned} R^2J(a_1) = a_2,\ R^2J(y_1) = y_2 \\ JR^{-2}(a_1) = a_3,\ JR^{-2}(y_1) = y_3. \end{aligned} \tag{4}$$

Let $\rho_3$ denote $T_{a_1}\rho_2T_{a_1}^{-1}$.
By Lemma 1, (4) and that $\rho_2$ sends $a_1 = \alpha_k$ to $y_1 = \alpha_{k+1}$, we have

$$\begin{aligned} T_{y_1}T_{a_1}^{-1} = \rho_2T_{a_1}\rho_2T_{a_1}^{-1} = \rho_2\rho_3, \\ T_{y_2}T_{a_2}^{-1} = R^2J\rho_2\rho_3JR^{-2},\ T_{y_3}T_{a_3}^{-1} = JR^{-2}\rho_2\rho_3R^2J. \end{aligned} \tag{5}$$

By (2) and (5) we have

$$T_{\gamma_k} = (\rho_2\rho_3)(R^2J\rho_2\rho_3JR^{-2})(JR^{-2}\rho_2\rho_3R^2J). \tag{6}$$

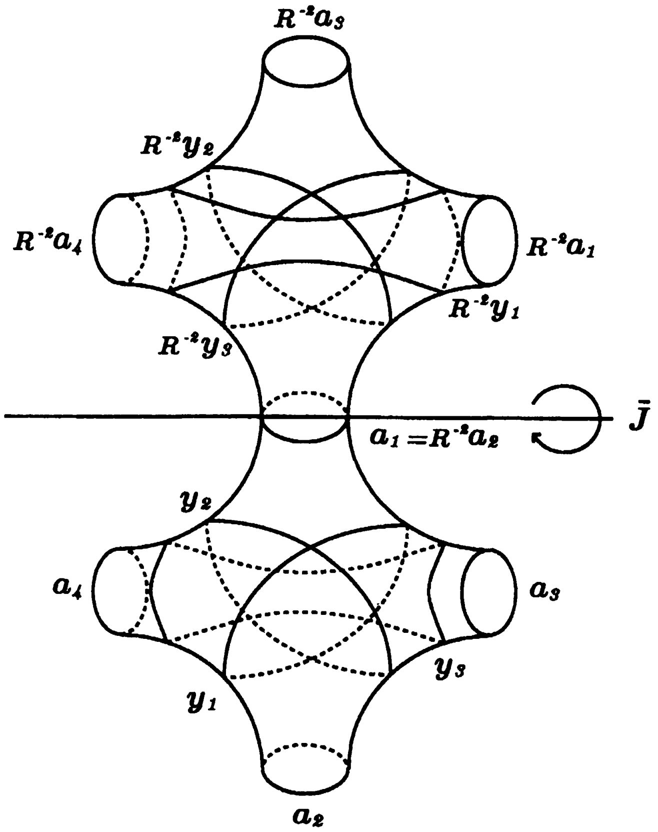


Figure 4: $S_2$ and the involution $\bar{J}$

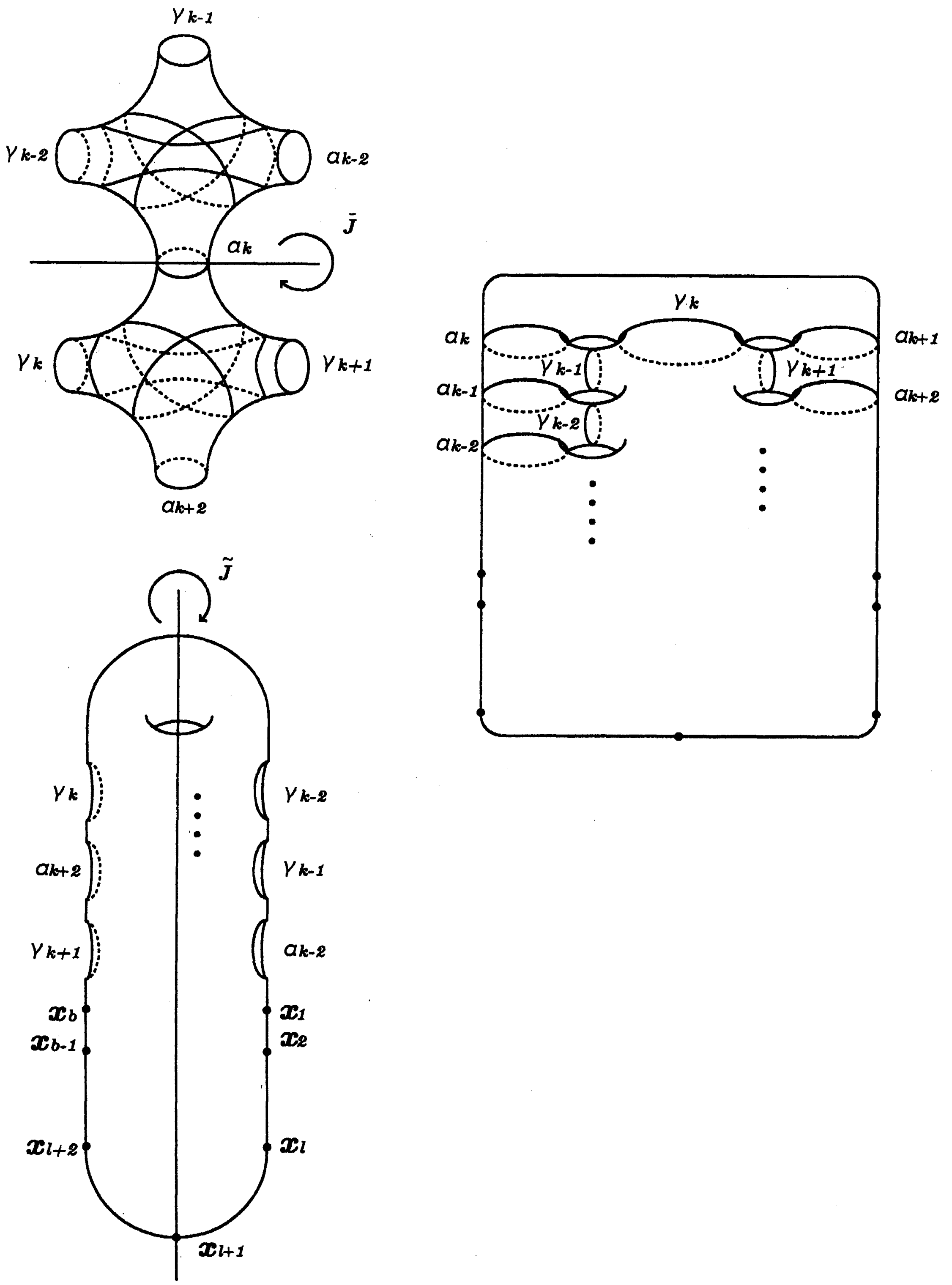


Figure 5: The involution $J$ on $\Sigma_{g,b}$

### 3.3 Genus at least 5

We proof that the mapping class group is generated by 5 involutions.

The five involutions are $\rho_1, \rho_2, \rho_3, J$ and another involution $I$. We construct involution $I$ in the same way as involution $J$ like Figure 6.

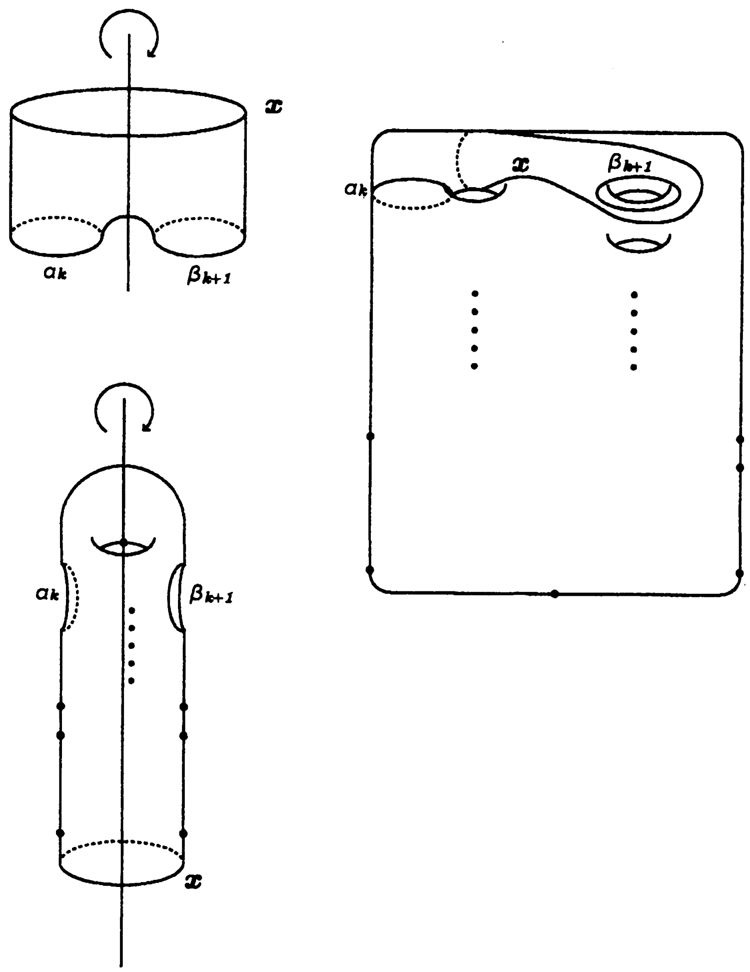


Figure 6: The involution $I$ on $\Sigma_{g,b}$

**Theorem 7.** *If $g \geq 5$, the group $G_3$ generated by $\rho_1$, $\rho_2$, $\rho_3$, $I$ and $J$ is the whole mapping class group* $\mathrm{Mod}(\Sigma_{g,b})$.

*Proof.* By the relation (6) we satisfy the condition $\langle 1 \rangle$. Since $J$ sends $\alpha_{k-2}$ to $\gamma_{k+1}$ and $I$ sends $\alpha_k$ to $\beta_{k+1}$, we consist the condition $\langle 2 \rangle$. We can also see that we satisfy the condition $\langle 3 \rangle$ from a way to the construction of the involution $J$.

Therefore, we can finish the proof of the theorem because we can satisfy the conditions in 3.1. $\square$

### 3.4 Genus at least 7

We want to improve the above argument and show that for the genus $g \geq 7$ we do not need the involution $I$ in order to generate the mapping class group. Assume that the genus of the surface is at least 7.

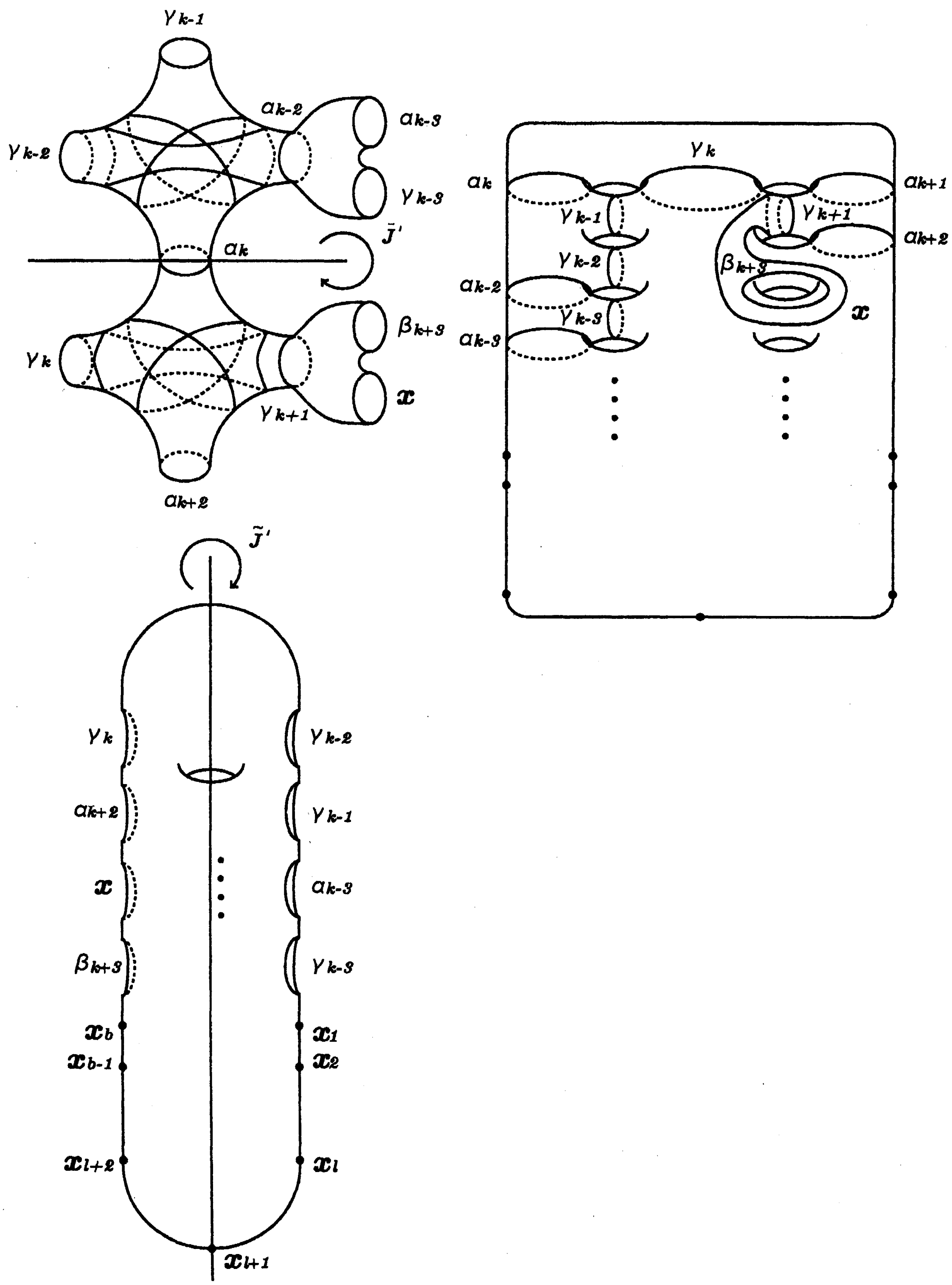


Figure 7: The involution $J'$ on $\Sigma_{g,b}$

The $S_2$ and two pairs of pants have common boundary components $R^{-2}a_1$ and $a_3$ and their union is a surface $S_3$ homeomorphic to a sphere with 8 boundary components. Figure 7 shows the existence of the involution $\bar{J}'$ on $S_3$ which extends the involution $\bar{J}$ on $S_2$.

Let us embed $S_3$ in the $\Sigma_{g,b}$ as shown on Figure 7. From Figure 7 we can find the involution $\tilde{J}'$ of the complement of $S_3$. Let $J'$ be the involution obtained by gluing together $\bar{J}'$ and $\tilde{J}'$. Moreover, from Figure 7 we can construct $J'$ which acts on the punctures as the involution $r_3$.

**Theorem 8.** *If $g \geq 7$, the group $G_4$ generated by $\rho_1$, $\rho_2$, $\rho_3$ and $J'$ is the whole mapping class group* $\mathrm{Mod}(\Sigma_{g,b})$.

*Proof.* From the construction of $J'$ we have

$$T_{\gamma_k} = (\rho_2\rho_3)(R^2J'\rho_2\rho_3J'R^{-2})(J'R^{-2}\rho_2\rho_3R^2J') \in G_4.$$

Therefore, we can see that we satisfy the condition $\langle 1 \rangle$. Since $J'$ can send $\alpha_{k-2}$ to $\gamma_{k+1}$ and $\beta_{k+3}$ to $\gamma_{k-3}$, we can satisfy the condition $\langle 2 \rangle$ only in $J'$. Moreover, By that $J'$ acts as $r_3$, we consist the condition $\langle 3 \rangle$. Therefore, the group $G_4$ is the whole mapping class group. □

# 4 Acknowledgement

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