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Recent developments in the study of the Takhtajan-Zograf metric

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Abstract

We will survey recent developments in the study of the Takhtajan-Zograf metric on the Teichmüller space. Main topics are the asymtotic behavior of the Takhtajan-Zograf metric near the boundary of moduli space of Riemann surfaces, which is the author's joint work with W.-K. To and L. Weng ([OTW]), and the asymtotic behavior of the Weil-Petersson metric near the boundary of moduli space of Riemann surfaces, which is the author's joint work with S.A. Wolpert ([OW]).

§0. Introduction

We consider the Teichmüller space $T_{g,n}$ and the associated Teichmüller curve $\mathcal{T}_{g,n}$ of Riemann surfaces of type (g, n) (i.e., Riemann surfaces of genus g and with $n > 0$ punctures). We will assume that $2g-2+n > 0$, so that each fiber of the holomorphic projection map $\pi : \mathcal{T}_{g,n} \to T_{g,n}$ is stable or equivalently, it admits the complete hyperbolic metric of constant sectional curvature -1. The kernel of the differential $T\mathcal{T}_{g,n} \to TT_{g,n}$ forms the so-called vertical tangent bundle over $\mathcal{T}_{g,n}$, which is denoted

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by $T^V\mathcal{T}_{g,n}$. The hyperbolic metrics on the fibers induce naturally a Hermitian metric on $T^V\mathcal{T}_{g,n}$.

In the study of the family of $\bar{\partial}_k$-operators acting on the k-differentials on Riemann surfaces (i.e., cross-sections of $\left(T^V\mathcal{T}_{g,n}\right)^{-k}\big|_{\pi^{-1}(s)} \to \pi^{-1}(s)$, $s \in T_{g,n}$), Takhtajan and Zograf introduced in [TZ1], [TZ2] a Kähler metric on $T_{g,n}$, which is known as the Takhtajan-Zograf metric. In [TZ1], [TZ2], they showed that the Takhtajan-Zograf metric is invariant under the natural action of the Teichmüller modular group $\mathrm{Mod}_{g,n}$ and it satisfies the following remarkable identity on $T_{g,n}$:

$$c_1(\lambda_k, \|\cdot\|_k) = \frac{6k^2-6k+1}{12\pi^2}\,\omega_{\mathrm{WP}} - \frac{1}{9}\omega_{\mathrm{TZ}}.$$

Here $\lambda_k = \det(\operatorname{ind}\bar{\partial}_k) = \bigwedge^{\max}\operatorname{Ker}\bar{\partial}_k \otimes (\bigwedge^{\max}\operatorname{Coker}\bar{\partial}_k)^{-1}$ denotes the determinant line bundle on $T_{g,n}$, $\|\cdot\|_k$ denotes the Quillen metric on λ_k, and ω_{WP}, ω_{TZ} denote the Kähler form of the Weil-Petersson metric, the Takhtajan-Zograf metric on $T_{g,n}$ respectively. In [We], Weng studied the Takhtajan-Zograf metric in terms of Arakelov intersection, and he proved that $\frac{4}{3}\omega_{\mathrm{TZ}}$ coincides with the first Chern form of an associated metrized Takhtajan-Zograf line bundle over the moduli space $\mathcal{M}_{g,n} = T_{g,n}/\mathrm{Mod}_{g,n}$. Recently, Wolpert [Wo5] gave a natural definition of a Hermitian metric on the Takhtajan-Zograf line bundle whose first Chern form gives $\frac{4}{3}\omega_{\mathrm{TZ}}$.

The first of main topics in this article is to present the asymptotic behavior of the Takhtajan-Zograf metric near the boundary of $T_{g,n}$ ([OTW]), which we describe heuristically as follows. Near the boundary of $T_{g,n}$, the tangent space at any point in $T_{g,n}$ can be roughly considered as the direct sum of the pinching directions and the non-pinching directions (that are 'parallel' to the boundary). Roughly speaking, our result shows that the Takhtajan-Zograf metric is smaller than the Weil-Petersson metric by an additional factor of $1/|\log|t||$ along each pinching tangential direction, i.e. it is essentially of the order of growth $1/|t|^2(\log|t|)^4$ along the pinching direction corresponding to a pinching coordinate t. Also, we show that the Takhtajan-Zograf metric extends continuously along the non-pinching tangential directions to the "nodally-depleted Takhtajan-Zograf metrics" on the boundary Teichmüller spaces, which, unlike the case of the Weil-Petersson metric, are only positive semi-definite on the boundary Teichmüller spaces.

The second of main topics in this article is to present a new formula for the asymptotic behavior of the Weil-Petersson metric near the boundary of $T_{g,n}$ ([OW]). Masur [Ma] first found that the Weil-Petersson metric extends continuously along the non-pinching tangential directions to the "nodally-depleted Weil-Petersson metrics" on the boundary Teichmüller spaces. Furthermore, Yamada [Y] gave an order estimate for the second term of the asymptotic expansion of the Weil-Petersson metric along the non-pinching tangential directions. In §3, we will succeed to determine the the second term of the asymptotic expansion of the Weil-Petersson metric along the non-pinching tangential directions, which is exactly the Takhtajan-Zograf metrics on the boundary Teichmüller spaces. It should be remarked that Mirzakhani [Mi] proved essentially the same formula in the context of symplectic geometry by the symplectic reduction technique, which is totally different from our method of the proof.

§1. Notation and The First Theorem

(1.1) For $g \geq 0$ and $n > 0$, we denote by $T_{g,n}$ the Teichmüller space of Riemann surfaces of type (g, n). Each point of $T_{g,n}$ is a Riemann surface X of type (g, n), i.e., $X = \bar{X} \backslash \{p_1, \cdots, p_n\}$, where X is a compact Riemann surface of genus g, and the punctures $p_1, \cdots, p_n$ of X are n distinct points in $\bar{X}$. We will always assume that $2g - 2 + n > 0$, so that X admits the complete hyperbolic metric of constant sectional curvature -1. By the uniformization theorem, X can be represented as a quotient $\mathbb{H}/\Gamma$ of the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ by the natural action of Fuchsian group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ of the first kind. Γ is generated by $2g$ hyperbolic transformations $A_1, B_1, \cdots, A_g, B_g$ and n parabolic transformations

$P_1, \cdots, P_n$ satisfying the relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} P_1 P_2 \cdots P_n = \mathrm{Id}.$$

Let $z_1, \cdots, z_n \in \mathbb{R} \cup \{\infty\}$ be the fixed points of the parabolic transformations $P_1, \cdots, P_n$ respectively, which are also called cusps. The cusps $z_1, \cdots, z_n$ correspond to the punctures $p_1, \cdots, p_n$ of X under the projection $\mathbb{H} \to \mathbb{H}/\Gamma \simeq X$ respectively. For each $i = 1, 2, \ldots, n$, it is well-known that P_i generates an infinite cyclic subgroup of Γ, and we can select $\sigma_i \in \mathrm{PSL}(2, \mathbb{R})$ so that $\sigma_i(\infty) = z_i$ and $\sigma_i^{-1} P_i \sigma_i$ is the transformation $z \mapsto z + 1$ on $\mathbb{H}$. For each $i = 1, 2, \cdots, n$ and $s \in \mathbb{C}$, the Eisenstein series $E_i(z, s)$ attached to the cusp z_i is given by

$$E_i(z, s) := \sum_{\gamma \in <P_i> \backslash \Gamma} \mathrm{Im}(\sigma_i^{-1} \gamma z)^s, \quad z \in \mathbb{H}. \tag{1.1.1}$$

If $\mathrm{Re}\, s > 1$, then the above series is uniformly convergent on compact subsets of $\mathbb{H}$. Moreover, $E_i(z, s)$ is invariant under Γ, and thus it descends to a function on X, which we denote by the same symbol. Furthermore, it is well-known that

$$\Delta E_j = s(s-1) E_j \quad \text{on } X, \tag{1.1.2}$$

where Δ denotes the negative hyperbolic Laplacian on X (see e.g. [Ku]).

The Teichmüller space $T_{g,n}$ is naturally a complex manifold of dimension $3g-3+n$. To describe its tangent and cotangent spaces at a point X, we first denote by $Q(X)$ the space of holomorphic quadratic differentials $\phi = \phi(z)\, dz^2$ on X with finite L^1 norm, i.e., $\int_X |\phi| < \infty$. Also, we denote by $B(X)$ the space of L^∞ measurable Beltrami differentials $\mu = \mu(z)\, d\bar{z}/dz$ on X (i.e., $\|\mu\|_\infty := \text{ess. sup}_{z \in X} |\mu(z)| < \infty$). Let $HB(X)$ be the subspace of $B(X)$ consisting of elements of the form $\overline{\phi}/\rho$ for some $\phi \in Q(X)$. Here $\rho = \rho(z)\, dz\, d\bar{z}$ denotes the hyperbolic metric on X. Elements of $HB(X)$ are called harmonic Beltrami differentials. There is a natural Kodaira-Serre pairing $\langle\ ,\ \rangle : B(X) \times Q(X) \to \mathbb{C}$ given by

$$\langle \mu, \phi \rangle = \int_X \mu(z) \phi(z)\, dz d\bar{z} \tag{1.1.3}$$

for $\mu \in B(X)$ and $\phi \in Q(X)$. Let $Q(X)^{\perp} \subset B(X)$ be the annihilator of $Q(X)$ under the above pairing. Then one has the decomposition $B(X) = HB(X) \oplus Q(X)^{\perp}$. It is well-known that one has the following natural isomorphism

$$\begin{aligned} T_X T_{g,n} \simeq& B(X)/Q(X)^{\perp} \simeq HB(X), \quad \text{and} \\ T_X^* T_{g,n} \simeq& Q(X) \end{aligned} \tag{1.1.4}$$

with the duality between $T_X T_{g,n}$ and $T_X^* T_{g,n}$ given by (1.1.3). It should be remarked that Bers was responsible for many of the concepts described above (see [Be]).

The Weil-Petersson metric g^{WP} and the Takhtajan-Zograf metric g^{TZ} on $T_{g,n}$ (the latter being introduced in [TZ1] and [TZ2]) are defined as follows (see e.g. [IT], [Wo2] and the references therein for background materials on g^{WP}): for $X \in T_{g,n}$ and $\mu, \nu \in HB(X)$, one has

$$\begin{aligned} g^{\mathrm{WP}}(\mu, \nu) =& \int_X \mu \bar{\nu} \rho, \\ g^{\mathrm{TZ}}(\mu, \nu) =& \sum_{i=1}^{n} g^{(i)}(\mu, \nu), \quad \text{where} \\ g^{(i)}(\mu, \nu) =& \int_X E_i(\cdot, 2) \mu \bar{\nu} \rho, \quad i = 1, 2, \cdots, n \end{aligned} \tag{1.1.5}$$

(see (1.1.1)). It follows from results in [A], [Ch], [Wo1], [TZ2], [O1], [O2] that the metrics g^{WP}, $g^{(i)}$, g^{TZ} are all Kählerian and non-complete. Note that g^{TZ} is well-defined only when $n > 0$. Moreover, each $g^{(i)}$ is intrinsic to the corresponding cusp p_i in the sense that if an element γ in the Teichmüller modular group $\mathrm{Mod}_{g,n}$ carries the cusp p_i to another cusp p_j, then γ also carries $g^{(i)}$ to $g^{(j)}$. To facilitate subsequent discussion, we will call $g^{(i)}$ the Takhtajan-Zograf cuspidal metric on $T_{g,n}$ associated to the cusp z_i (or the puncture p_i).

The moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of type (g, n) is obtained as the quotient of $T_{g,n}$ by the Teichmüller modular group $\mathrm{Mod}_{g,n}$, i.e., $\mathcal{M}_{g,n} \simeq T_{g,n}/\mathrm{Mod}_{g,n}$ (see e.g. [N]). As such, $\mathcal{M}_{g,n}$ is naturally endowed with the structure of a complex

V-manifold ([Ba]). The metrics g^{WP} and g^{TZ} (but not each individual $g^{(i)}$ unless $n = 1$) are invariant under $\mathrm{Mod}_{g,n}$ and thus they descend to Kähler metrics on (the smooth points of) $\mathcal{M}_{g,n}$, which we denote by the same names and symbols.

(1.2) To facilitate ensuing discussion, we consider some related pseudo-metrics on the associated boundary Teichmüller spaces of $T_{g,n}$.

As in [Ma] (in the case of $T_{g,0}$), we denote by $\delta_{\gamma_1,\cdots,\gamma_m}T_{g,n}$ the boundary Teichmüller space of $T_{g,n}$ arising from pinching m distinct points. Take a point $X_0 \in \delta_{\gamma_1,\cdots,\gamma_m}T_{g,n}$. Then X_0 is a Riemann surface with n punctures $p_1, \cdots, p_n$ and m nodes $q_1, \cdots, q_m$. Observe that $X_0^o := X\backslash\{q_1, \cdots, q_m\}$ is a non-singular Riemann surface with $n + 2m$ punctures. Each node q_i corresponds to two punctures on X_0^o (other than $p_1, \cdots, p_n$). Denote the components of X_0^o by S_α, $\alpha = 1, 2, \ldots, d$. Each S_α is a Riemann surface of genus g_α and with n_α punctures, i.e., S_α is of type (g_α, n_α). It will be clear in (1.3) that we will only need to consider the case where $2g_\alpha - 2 + n_\alpha > 0$ for each α, so that each S_α also admits the complete hyperbolic metric of constant sectional curvature -1. It is easy to see that $\sum_{\alpha=1}^{d}(3g_\alpha - 3 + n_\alpha) + m = 3g - 3 + n$. With respect to the disjoint union $X_0^o = \cup_{\alpha=1}^{d} S_\alpha$, one easily sees that $\delta_{\gamma_1,\cdots,\gamma_m}T_{g,n}$ is a product of lower dimensional Teichmüller spaces given by

$$\delta_{\gamma_1,\cdots,\gamma_m}T_{g,n} = T_{g_1,n_1} \times T_{g_2,n_2} \times \cdots \times T_{g_d,n_d} \tag{1.2.1}$$

with each $S_\alpha \in T_{g_\alpha,n_\alpha}$, $\alpha = 1, 2, \cdots, d$. Recall that the punctures of S_α arise from either the punctures or the nodes of X_0, and for simplicity, they will be called old cusps and new cusps of S_α respectively. Denote the number of old cusps (resp. new cusps) of S_α by n'_α (resp. n''_α), so that $n_\alpha = n'_\alpha + n''_\alpha$. We index the punctures of S_α such that $\{p_{\alpha,i}\}_{1 \le i \le n'_\alpha}$ denotes the set of old cusps, and $\{p_{\alpha,i}\}_{n'_\alpha+1 \le i \le n_\alpha}$ denotes the set of new cusps. For each α and i, we denote by $g^{(\alpha,i)}$ the Takhtajan-Zograf cuspidal metric on T_{g_α,n_α} with respect to the puncture $p_{\alpha,i}$ (cf. (1.1.5)). Now we define a pseudo-metric $\hat{g}^{\mathrm{TZ},\alpha}$ on T_{g_α,n_α} by summing the $g^{(\alpha,i)}$'s over the old cusps, i.e.,

$$\hat{g}^{\mathrm{TZ},\alpha} := \sum_{1\le i\le n'_\alpha} g^{(\alpha,i)}. \tag{1.2.2}$$

If none of the punctures of S_α are old cusps, then $\hat{g}^{\mathrm{TZ},\alpha}$ is simply defined to be zero identically. As such, $\hat{g}^{\mathrm{TZ},\alpha}$ is positive definite precisely when S_α possesses at least one old cusp. Note that by contrast, the Takhtajan-Zograf metric $g^{\mathrm{TZ},\alpha}$ on T_{g_α,n_α} is given by $g^{\mathrm{TZ},\alpha} := \sum_{1\le i\le n_\alpha} g^{(\alpha,i)}$, and $g^{\mathrm{TZ},\alpha}$ is always positive definite.

Definition 1.2.1. The *nodally depleted Takhtajan-Zograf pseudo-metric* $\hat{g}^{\mathrm{TZ},(\gamma_1,\cdots,\gamma_m)}$ on $\delta_{\gamma_1,\cdots,\gamma_m}T_{g,n}$ is defined to be the product pseudo-metric of the $\hat{g}^{\mathrm{TZ},\alpha}$'s on the T_{g_α,n_α}'s, i.e.,

$$\left(\delta_{\gamma_1,\cdots,\gamma_m}T_{g,n}, \hat{g}^{\mathrm{TZ},(\gamma_1,\cdots,\gamma_n)}\right) = \prod_{i=1}^{d} \left(T_{g_\alpha,n_\alpha}, \hat{g}^{\mathrm{TZ},\alpha}\right). \tag{1.2.3}$$

(1.3) Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces of type (g,n) as in (1.1), and let $\overline{\mathcal{M}}_{g,n}$ denote the Knudsen-Deligne-Mumford stable curve compactification of $\mathcal{M}_{g,n}$ ([KM], [Kn]). Like $\mathcal{M}_{g,n}$, $\overline{\mathcal{M}}_{g,n}$ admits a V-manifold structure, which we describe as follows. Similar description for $\overline{\mathcal{M}}_g$ (i.e., when $n=0$) can be found in [Ma] or [Wo3].

Take a point $X_0 \in \overline{\mathcal{M}}_{g,n}\backslash\mathcal{M}_{g,n}$. Then X_0 is a stable Riemann surface with n punctures $p_1,\cdots,p_n$ and m nodes $q_1,\cdots,q_m$ for some $m>0$. Thus we may regard X_0 as a point in $\delta_{\gamma_1,\cdots,\gamma_m}T_{g,n}$ (cf. (1.2)). Write $X_0\backslash\{q_1,\cdots,q_m\} = \cup_{1\le\alpha\le d}S_\alpha$ and write $\delta_{\gamma_1,\cdots,\gamma_m}T_{g,n} = \prod_{\alpha=1}^{d} T_{g_\alpha,n_\alpha}$ with each component $S_\alpha \in T_{g_\alpha,n_\alpha}$ as in (1.2). Note that since X_0 is stable, each S_α admits the complete hyperbolic metric of constant sectional curvature -1. Also, for some $0<r<1$, each node q_j in X_0 admits an open neighborhood

$$N_j = \{(z_j,w_j)\in\mathbb{C}^2 : |z_j|,\ |w_j|<r,\ z_j\cdot w_j = 0\} \tag{1.3.1}$$

so that $N_j = N_j^1 \cup N_j^2$, where $N_j^1 = \{(z_j,0)\in\mathbb{C}^2 : |z_j|<r\}$ and $N_j^2 = \{(0,w_j)\in$

$\mathbb{C}^2 : |w_j| < r\}$ are the coordinate discs in $\mathbb{C}^2$. Without loss of generality, we will assume that r is independent of j, upon shrinking r if necessary. For each α, we choose $3g_\alpha - 3 + n_\alpha$ linearly independent Beltrami differentials $\nu_i^{(\alpha)}, 1 \leq i \leq 3g_\alpha - 3 + n_\alpha$, which are supported on $S_\alpha \backslash \cup_{j=1}^n N_j$, so that their harmonic projections form a basis of $T_{S_\alpha} T_{g_\alpha, n_\alpha}$ (cf. (1.1.4)). For simplicity, we rewrite $\{v_i^{(\alpha)}\}_{1 \leq \alpha \leq d, 1 \leq i \leq 3g_\alpha - 3 + n_\alpha}$ as $\{v_i\}_{1 \leq i \leq 3g-3+n-m}$. Then one has an associated local coordinate neighborhood V of X_0 in $\delta_{\gamma_1, \cdots, \gamma_m} T_{g,n}$ with holomorphic coordinates $\tau = (\tau_1, \cdots, \tau_{3g-3+n-m})$ such that X_0 corresponds to 0. Shrinking and reparametrizing V if necessary, we may assume $V \simeq \Delta^{3g-3+n-m}$, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ denotes the unit disc in $\mathbb{C}$. For a point $\tau \in V$, one has the associated Beltrami differential $\mu(\tau) = \sum_{i=1}^{3g-3+n-m} \tau_i v_i$ and a quasi-conformal homeomorphism $w^{\mu(\tau)} : X_0 \to X_\tau$ onto a Riemann surface X_τ satisfying

$$\frac{\partial w^{\mu(\tau)}}{\partial \bar{z}} = \mu(z) \frac{\partial w^{\mu(\tau)}}{\partial z}. \tag{1.3.2}$$

The map $w^{\mu(\tau)}$ is conformal on each N_j, $j = 1, \cdots, m$, so that we may regard $N_j \subset X_\tau$ for each j. Then for each $t = (t_1, \cdots, t_m)$ with each $|t_j| < r$, we obtain a new Riemann surface $X_{t,\tau}$ for X_τ by removing the disks $\{z_j \in N_j^1 : |z_j| < |t_j|\}$ and $\{w_j \in N_j^2 : |w_j| < |t_j|\}$ and identifying $z_j \in N_j^1$ with $w_j = t_j / z_j \in N_j^2$, $j = 1, \cdots, m$. Then one obtains a holomorphic family of noded Riemann surfaces $\{X_{t,\tau}\}$ parametrized by the coordinates $(t; \tau) = (t_1, \cdots, t_m, \tau_1, \cdots, \tau_{3g-3+n-m})$ of $\Delta^m(r) \times V \simeq \Delta^m(r) \times \Delta^{3g-3+n-m}$, where $\Delta^m(r)$ denotes the m-fold Cartesian product of the disc $\Delta(r) = \{z \in \mathbb{C} : |z| < r\}$ in $\mathbb{C}$. Moreover, the Riemann surfaces $X_{t,\tau}$ with $(t, \tau) \in (\Delta^*(r))^m \times V$ are of type (g, n), where $\Delta^*(r) = \Delta(r) \backslash \{0\}$. The coordinates $t = (t_1, \cdots, t_m)$ will be called pinching coordinates, and $\tau = (t_1, \cdots, t_{3g-3+n-m})$ will be called boundary coordinates. For $1 \leq j \leq m$, let α_j denote the simple closed curve $|z_j| = |w_j| = |t_j|^{\frac{1}{2}}$ on $X_{t,\tau}$. Shrinking $\Delta^m(r)$ and V if necessary, it is known that the universal cover of $(\Delta^*(r))^m \times V$ is naturally a domain in $T_{g,n}$ and the corresponding covering transformations are generated by Dehn twist about the α_j's. Since Dehn twists are elements of $\mathrm{Mod}_{g,n}$, the $\mathrm{Mod}_{g,n}$-invariant metrics g^{WP} and g^{TZ} descend to metrics on $(\Delta^*(r))^m \times V$, which we denote by the same symbols and names. It is well-known that each $X_0 \in \overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}$ admits an open neighborhood $\hat{U}$ in $\overline{\mathcal{M}}_{g,n}$ together with a local uniformizing chart $\chi : U \simeq \Delta^m(r) \times V \to \hat{U}$ for some $\Delta^m(r) \times V$ as described above, where χ is a finite ramified cover. Obviously the metrics g^{WP} and g^{TZ} on $(\Delta^*(r))^m \times V \subset U$ may also be regarded as extensions of the pull-back of the corresponding metrics on the smooth points of $\hat{U} \cap \mathcal{M}_{g,n}$ via the map χ.

(1.4) Before we state our main result, we first need to make the following definition.

Definition 1.4.1. Let X_0 be a Riemann surface with n punctures $p_1, \cdots, p_n$ and m nodes $q_1, \cdots, q_m$. A node q_i is said to be *adjacent to punctures* (*resp. a puncture* p_j) if the component of $X_0 \backslash \{q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_m\}$ containing q_i also contains at least one of the p_j's (resp. the puncture p_j). Otherwise, it is said to be *non-adjacent to punctures* (*resp. the puncture* p_j).

Now we are ready to state the first main result in the following

Theorem 1. *For $g \geq 0$ and $n > 0$, let $X_0 \in \overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}$ be a stable Riemann surface with n punctures $p_1, \cdots, p_n$ and m nodes $q_1, \cdots, q_m$ arranged in such a way that q_i is adjacent (resp. non-adjacent) to punctures for $1 \leq i \leq m'$ (resp. $m'+1 \leq i \leq m$). Let $\hat{U}$ be an open neighborhood of X_0 in $\overline{\mathcal{M}}_{g,n}$, together with a local uniformizing chart $\psi : U \simeq \Delta^m(r) \times V \to \hat{U}$, where $V \simeq \Delta^{3g-3+n-m}$ is a domain in the boundary Teichmüller space $\delta_{\gamma_1, \cdots, \gamma_m} T_{g,n}$ corresponding to X_0 and with each γ_i corresponding to q_i. Let $(s_1, \cdots, s_{3g-3+n}) = (t_1, \cdots, t_m, \tau_1, \cdots, \tau_{3g-3+n-m}) = (t, \tau)$ be the pinching and boundary coordinates of U, and let the components of the Takhtajan-Zograf metric g^{TZ} be given by*

$$g_{i\bar{j}}^{TZ} = g^{TZ}\left(\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j}\right), \quad 1 \leq i, j \leq 3g-3+n, \tag{1.4.1}$$

on $U^* := (\Delta^*(r))^m \times V \subset U$. Then the following statements hold:

(i) For each $1 \le j \le m$ and any $\varepsilon > 0$, one has

$$\limsup_{(t,\tau)\in U^* \to (0,0)} |t_j|^2(-\log|t_j|)^{4-\varepsilon} g_{j\bar{j}}^{TZ}(t,\tau) = 0. \tag{1.4.2}$$

(ii) For each $1 \le j \le m'$ and any $\varepsilon > 0$, one has

$$\liminf_{(t,\tau)\in U^* \to (0,0)} |t_j|^2(-\log|t_j|)^{4+\varepsilon} g_{j\bar{j}}^{TZ}(t,\tau) = +\infty. \tag{1.4.3}$$

(iii) For each $1 \le j,\ k \le m$ with $j \ne k$, one has

$$|g_{j\bar{k}}^{TZ}(t,\tau)| = O\Big(\frac{1}{|t_j|\,|t_k|\,(\log|t_j|)^3(\log|t_k|)^3}\Big) \quad \text{as } (t,\tau) \in U^* \to (0,0). \tag{1.4.4}$$

(iv) For each $j\,,k \ge m+1$, one has

$$\lim_{(t,\tau)\in U^* \to (0,0)} g_{j\bar{k}}^{TZ}(t,\tau) = \hat{g}_{j\bar{k}}^{TZ,(\gamma_1,\cdots,\gamma_m)}(0,0). \tag{1.4.5}$$

(v) For each $j \le m$ and $k \ge m+1$, one has

$$|g_{j\bar{k}}^{TZ}(t,\tau)| = O\Big(\frac{1}{|t_j|(-\log|t_j|)^3}\Big) \quad \text{as } (t,\tau) \in U^* \to (0,0). \tag{1.4.6}$$

Here in (1.4.5), $\hat{g}_{j\bar{k}}^{TZ,(\gamma_1,\cdots,\gamma_m)}$ denotes the (j,k)-th component of the nodally depleted Takhtajan-Zograf pseudo-metric on $\delta_{\gamma_1,\cdots,\gamma_m}T_{g,n}$ (cf. Definition 1.2.1).

Remark 1.4.2. (i) Theorem 1(i) is equivalent to the following statement: For each $1 \le j \le m$ and any $\varepsilon > 0$, there exists a constant $C_{1,\varepsilon} > 0$ (depending on ϵ) such that

$$g_{j\bar{j}}^{\mathrm{TZ}}(t,\tau) \le \frac{C_{1,\varepsilon}}{|t_j|^2(-\log|t_j|)^{4-\varepsilon}} \quad \text{for all } (t,\tau) \in U^*. \tag{1.4.7}$$

Similarly, Theorem 1(ii) is equivalent to the following statement: For each $1 \le j \le m'$ and any $\varepsilon > 0$, there exists a constant $C_{2,\varepsilon} > 0$ (depending on ϵ) such that

$$g_{j\bar{j}}^{\mathrm{TZ}}(t,\tau) \ge \frac{C_{2,\varepsilon}}{|t_j|^2(-\log|t_j|)^{4+\varepsilon}} \quad \text{for all } (t,\tau) \in U^*. \tag{1.4.8}$$

(ii) In view of Theorem 1(i) and (ii), it is natural to ask the following question: Does the stronger estimate

$$g_{j\bar{j}}^{\mathrm{TZ}}(t,\tau) \sim \frac{1}{|t_j|^2(-\log|t_j|)^4} \text{ hold for } 1 \le j \le m' \text{ and } (t,\tau) \in U^*? \tag{1.4.9}$$

§2. Some Modifications and The Second Theorem

(2.1) In this section, we will present the second theorem. For that, we need a slight modification of local pinching parameters in §1. Let us remember the settings in (1.3).

The Beltrami differentials (1.3.2) can be modified a small amount so that in terms of each *cusp coordinate* the diffeomorphisms $w^{\hat{\mu}(\tau)}$ are simply rotations (Lemma 1.1, [Wo4]); $w^{\hat{\mu}(\tau)}$ is a hyperbolic isometry in a neighborhood of the cusps; $w^{\hat{\mu}(\tau)}$ cannot be complex analytic in τ, but is real analytic. We note that for τ small the τ-derivatives of $\mu(\tau)$ and $\hat{\mu}(\tau)$ are close. We say that $w^{\hat{\mu}(\tau)}$ *preserves cusp coordinates.* The parameterization provides a key ingredient for obtaining simplified estimates of the degeneration of hyperbolic metrics and an improved expansion for the Weil-Petersson metric.

We describe a *local manifold cover* of the compactified moduli space $\overline{\mathcal{M}}_{g,n}$. The quasiconformal deformation space of X_0 in (1.3), $Def(X_0)$, is the product of the Teichmüller spaces of the components of X_0. As above for $3g-3+n-m=$ dim $Def(X_0)$ there is a real analytic family of Beltrami differentials $\hat{\mu}(\tau)$, τ in a neighborhood of the origin in $\mathbb{C}^{3g-3+n-m}$, such that $\tau \rightarrow X_\tau = X^{\hat{\mu}(\tau)}$ is a coordinate parameterization of a neighborhood of X_0 in $Def(X_0)$ and the prescribed mappings $w^{\hat{\mu}(\tau)} : X_0 \rightarrow X^{\hat{\mu}(\tau)}$ preserve the cusp coordinates at each puncture. For X_0 with m nodes we prescribe the plumbing data $(N_j^1, N_j^2, z_j, w_j, t_j)$, $j=1,\dots,m$, for $X^{\hat{\mu}(\tau)}$. The parameter t_j parameterizes opening the j-th node. For all t_j suitably small, perform the m prescribed plumbings to obtain the family $X_{t,\tau} = X^{\hat{\mu}(\tau)}_{t_1,\dots,t_m}$. The tuple $(t,\tau) = (t_1,\dots,t_m,\tau_1,\dots,\tau_{3g-3+n-m})$ provides real analytic local coordinates, the *hyperbolic metric plumbing coordinates*, for the local manifold cover of $\overline{\mathcal{M}}_{g,n}$ at X_0, [Ma] and [Wo3, Secs. 2.3, 2.4]. The coordinates have a special property: for τ fixed the parameterization is holomorphic in t. The property is a basic feature of the plumbing construction. The family $X_{t,\tau}$ parameterizes the small deformations of the marked noded surface with punctures X_0.

(2.2) We review the geometry of the local manifold covers. For a complex manifold M the complexification $\boldsymbol{T}^{\mathbb{C}}M$ of the $\mathbb{R}$-tangent bundle is decomposed into the subspaces of *holomorphic* and *antiholomorphic* tangent vectors. A Hermitian metric g is prescribed on the holomorphic subspace. For a general complex parameterization $s = u + iv$ the coordinate $\mathbb{R}$-tangents are expressed as $\frac{\partial}{\partial u} = \frac{\partial}{\partial s} + \frac{\partial}{\partial \bar{s}}$ and $\frac{\partial}{\partial v} = i\frac{\partial}{\partial s} - i\frac{\partial}{\partial \bar{s}}$. For the $X_{t,\tau}$ parameterization in (2.1), the τ-parameters are not holomorphic while for τ-parameters fixed the t-parameters are holomorphic; $\{\frac{\partial}{\partial \tau_k} + \frac{\partial}{\partial \bar{\tau}_k}, i\frac{\partial}{\partial \tau_k} - i\frac{\partial}{\partial \bar{\tau}_k}, \frac{\partial}{\partial t_j}, i\frac{\partial}{\partial t_j}\}$ is a basis over $\mathbb{R}$ for the tangent space of the local manifold cover. For a smooth Riemann surface the dual of the space of holomorphic tangents is the space of quadratic differentials with at most simple poles at punctures. The following is a modification of Masur's result [Ma, Prop. 7.1].

Lemma 1. *The hyperbolic metric plumbing coordinates (t, τ) are real analytic and for τ fixed the parameterization is holomorphic in t. Provided the modification $\hat{\mu}$ is small, for a neighborhood of the origin there are families in (t, τ) of regular 2-differentials φ_k, ψ_k, $k = 1, \ldots, 3g - 3 + n - m$ and η_j, $j = 1, \ldots, m$ such that:*
(i) Each regular 2-differential has an expansion of the form $\varphi(s,t) = \varphi(s,0) + O(t)$ locally away from the nodes of R.
(ii) For $X_{t,\tau}$ with $t_j \neq 0$, all j, $\{\varphi_k, \psi_k, \eta_j, i\eta_j\}$ forms the dual basis to $\{\frac{\partial \hat{\mu}(\tau)}{\partial \tau_k} + \frac{\partial \hat{\mu}(\tau)}{\partial \bar{\tau}_k}, i\frac{\partial \hat{\mu}(\tau)}{\partial \tau_k} - i\frac{\partial \hat{\mu}(\tau)}{\partial \bar{\tau}_k}, \frac{\partial}{\partial t_j}, i\frac{\partial}{\partial t_j}\}$ over $\mathbb{R}$.
(iii) For $X_{t,\tau}$ with $t_j = 0$, all j, the η_j, $j = 1, \ldots, m$, are trivial and the $\{\varphi_k, \psi_k\}$ span the dual of the holomorphic subspace $\boldsymbol{T}Def(X_0)$.

(2.3) Now we are ready to state the second main theorem in the following

Theorem 2. *For a noded Riemann surface X_0 with punctures the hyperbolic metric plumbing coordinates for $X_{t,\tau}$ provide real analytic coordinates for a local manifold cover neighborhood for $\overline{\mathcal{M}}_{g,n}$. The parameterization is holomorphic in t for τ fixed. On the local manifold cover the Weil-Petersson metric is formally Hermitian satisfying:*

(i) For $t_j = 0$, $j = 1, \dots, m$, *the restriction of the metric is a smooth Kähler metric, isometric to the Weil-Petersson product metric for a product of Teichmüller spaces* $\delta_{\gamma_1, \cdots, \gamma_m} T_{g,n}$.

(ii) For the tangents $\{\frac{\partial}{\partial \tau_k}, \frac{\partial}{\partial \bar{\tau}_k}, \frac{\partial}{\partial t_j}\}$ *and the quantity* $\sigma = \sum_{j=1}^{m} (\log |t_j|)^{-2}$ *then:*

$$g^{WP}\Big(\frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_j}\Big)(t, \tau) = \frac{\pi^3}{|t_j|^2(-\log^3 |t_j|)}(1 + O(\sigma)), \tag{2.3.1}$$

$$g^{WP}\Big(\frac{\partial}{\partial t_k}, \frac{\partial}{\partial t_\ell}\Big)(t, \tau) = O((|t_k t_\ell| \log^3 |t_k| \log^3 |t_\ell|)^{-1}) \ \mathit{for}\ k \neq \ell, \tag{2.3.2}$$

$$g^{WP}\Big(\frac{\partial}{\partial t_j}, \mathfrak{u}\Big)(t, \tau) = O((|t_j|(-\log^3 |t_j|))^{-1}), \ \mathit{for}\ \mathfrak{u} = \frac{\partial}{\partial s_k}, \frac{\partial}{\partial \bar{s}_k}. \tag{2.3.3}$$

(iii) For $\mathfrak{u} = \frac{\partial}{\partial \tau_k}, \frac{\partial}{\partial \bar{\tau}_k}$, *represented at* $X_{0,\tau}$ *by* μ_k *and* $\mathfrak{v} = \frac{\partial}{\partial \tau_\ell}, \frac{\partial}{\partial \bar{\tau}_\ell}$ *represented at* $X_{0,\tau}$ *by* μ_ℓ *then:*

$$g^{WP}(\mathfrak{u}, \mathfrak{v})(t, \tau) = g^{WP}(\mathfrak{u}, \mathfrak{v})(0, \tau) + \tfrac{4\pi^4}{3} \sum_{j=1}^{m} (\log |t_j|)^{-2} \langle \mu_k, \mu_\ell (E_{j,1} + E_{j,2}) \rangle_{WP}(0, \tau)$$
$$+ O\Big(\sum_{j=1}^{m} (-\log |t_j|)^{-3}\Big), \tag{2.3.4}$$

where the Eisenstein series $E_{j,1}, E_{j,2}$ *are for the pair of punctures representing the* j*-th node.*

Remark 2.3.1. (i) Theorem 2(iii) is an improvement of Masur's formula [Ma], i.e., the Takhtajan-Zograf metrics corresponding to the nodes appear in the second term. (ii) It should be noted that Yamada [Y] has proved before that the second term in (2.3.4) is $O\big(\sum_{j=1}^{m} (-\log |t_j|)^{-2}\big)$.

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