A generalization of Hardy spaces on spaces of homogeneous type

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1. INTRODUCTION

This is an announcement of my recent work [10].

Let \( X = (X, d, \mu) \) be a space of homogeneous type in the sense of Coifman and Weiss [1, 2] (see the next section for the definition). Using atoms, Coifman and Weiss [2] introduced the Hardy space \( H^p(X) \). The purpose of this report is to generalize the definition of Hardy space \( H^p(X) \) and prove that the generalized Hardy spaces have the same property as \( H^p(X) \). Our definition includes a kind of Hardy spaces with variable exponent. The results are new even for the \( \mathbb{R}^n \) case.

First we state definitions of Campanato and Hölder spaces. Let \( 1 \leq p < \infty \) and \( \phi : X \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \), where \( \mathbb{R}_+ = (0, \infty) \). For a ball \( B = B(x, r) \), we shall write \( \phi(B) \) in place of \( \phi(x, r) \). For a function \( f \in L^1_{\text{loc}}(X) \) and for a ball \( B \), let \( f_B = \frac{1}{\mu(B)} \int_B f(x) \, d\mu(x) \). Then the Campanato spaces \( \mathcal{L}_{p, \phi}(X) \) and the Hölder spaces \( \Lambda_{\phi}(X) \) are defined to be the sets of all \( f \) such that \( \|f\|_{\mathcal{L}_{p, \phi}} < \infty \) and \( \|f\|_{\Lambda_{\phi}} < \infty \), respectively, where

\[
\|f\|_{\mathcal{L}_{p, \phi}} = \sup_B \frac{1}{\phi(B)} \left( \frac{1}{\mu(B)} \int_B |f(x) - f_B|^p \, d\mu(x) \right)^{1/p},
\]

\[
\|f\|_{\Lambda_{\phi}} = \sup_{x, y \in X, x \neq y} \frac{2|f(x) - f(y)|}{\phi(x, d(x, y)) + \phi(y, d(y, x))}.
\]

Let \( C \) be the space of all constant functions. Then \( \mathcal{L}_{p, \phi}(X)/C \) and \( \Lambda_{\phi}(X)/C \) are Banach spaces with the norm \( \|f\|_{\mathcal{L}_{p, \phi}} \) and \( \|f\|_{\Lambda_{\phi}} \), respectively. Campanato spaces of these type were studied in [11, 7, 8, 12, 9]. See [9] for relations among these spaces. When \( p = 1 \), we denote \( \mathcal{L}_{1, \phi}(X) \) by \( \text{BMO}_\phi(X) \). If \( \phi \equiv 1 \), then \( \mathcal{L}_{1, \phi}(X) = \text{BMO}(X) \).

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For \( \phi(x, r) = r^{\alpha(x)} \), \( \alpha(x) > 0 \), we denote \( \Lambda_{\phi}(X) \) by \( \text{Lip}_{\alpha()}(X) \). Then
\[
\|f\|_{\text{Lip}_{\alpha()}} = \sup_{x, y \in X, x \neq y} \frac{2|f(x) - f(y)|}{d(x, y)^{\alpha(x)} + d(y, x)^{\alpha(y)}}.
\]
If \( \alpha() \) satisfies a certain condition, then \( \text{Lip}_{\alpha()}(X) = \mathcal{L}_{p, \phi}(X) \) for all \( p \in [1, \infty) \).

Using atoms, Coifman and Weiss [2] defined the Hardy space \( H^p(X) \) as a subspace of the dual of \( \text{Lip}_{\alpha}(X) \) and they proved that \( \text{Lip}_{\alpha}(X) \) is the dual of \( H^p(X) \). Their results are generalization of the case \( X = \mathbb{R}^n \).

In [2], \( \text{Lip}_{\alpha}(X) \) was regarded as the space of functions modulo constants. Therefore, we denote by \( (H^p(X))^* = \text{Lip}_{\alpha}(X)/C \) the fact above.

In this report, using \([\phi, q]\)-atoms, we define a generalized Hardy space \( H_{U}^{\phi,q}(X) \) as a subspace of the dual of \( \mathcal{L}_{q', \phi}(X)/C \) and prove that \( \mathcal{L}_{q', \phi}(X)/C \) is the dual of \( H_{U}^{\phi,q}(X) \), i.e. \( \left( H_{U}^{\phi,q}(X) \right)^* = \mathcal{L}_{q', \phi}(X)/C \), where \( 1 < q \leq \infty, 1/q + 1/q' = 1, U \) is a concave strictly increasing function from \([0, \infty)\) to itself and \( U(0) = 0 \) (see the third section for the precise definition of \( H_{U}^{\phi,q}(X) \)). The definition of \( H^p(X) \) in [2], \( 0 < p \leq 1 \), is a special case of ours, since \( \text{Lip}_{\alpha}(X) \) is a special case of \( \mathcal{L}_{q', \phi}(X) \).

Coifman and Weiss [2] first defined \( H^{p,q}(X) \), and then proved \( H^{p,q}(X) = H^{p,\infty}(X) \), which was denoted by \( H^p(X) \). We will prove that \( H_{U}^{\phi,q}(X) = H_{U}^{\phi,\infty}(X) \) under a certain condition. In particular, for Hardy spaces with variable exponent \( p(x) \), we use the condition that \( p(x) \) is log-Hölder continuous (see Corollary 4.2).

The log-Hölder continuity was used to prove boundedness of the Hardy-Littlewood maximal operator on \( L^{p(x)} \), Lebesgue spaces with variable exponent, as follows.

Let \( G \subset \mathbb{R}^n \) be bounded. For a function \( p : G \to [1, \infty) \), let
\[
L^{p(x)}(G) = \left\{ f \in L^1(G) : \int_G (c |f(x)|^{p(x)} )^x dx < \infty \text{ for some } c > 0 \right\}.
\]
For \( f \in L^{p(x)}(G) \), let
\[
\|f\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_G \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.
\]
Then \( \| \cdot \|_{p(x)} \) is a norm and thereby \( L^{p(x)}(G) \) is a Banach space. For a function \( f \) on \( G \), the Hardy-Littlewood maximal function of \( f \) is defined by
\[
Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B \cap G} |f(y)| dy,
\]
where the supremum is taken over all balls \( B \) containing \( x \). By the definition we have
\[
\|Mf\|_{\infty} \leq \|f\|_{\infty}.
\]
We say that $p(x)$ is log-Hölder continuous if

$$|p(x) - p(y)| \leq \frac{c}{|\log |x - y||} \quad \text{for } |x - y| \leq \frac{1}{2}.$$ 

**Theorem 1.1** (Diening [3]). If $p(x)$ is log-Hölder continuous, then the operator $M$ is bounded on $L^{p(x)}(G)$.

**Remark 1.1.** Let

$$p(x) = \begin{cases} 4 & (-1 < x \leq 0) \\ 2 & (0 < x < 1). \end{cases}$$

If $f(x) = \begin{cases} 0 & (-1 < x \leq 0) \\ x^{-1/3} & (0 < x < 1), \end{cases}$ then $Mf(x) \geq c|x|^{-1/3}$. In this case $f \in \mathcal{L}^{p(x)}(-1, 1)$ and $Mf \notin \mathcal{L}^{p(x)}(-1, 1)$.

2. **Space of homogeneous type**

Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e. $X$ is a topological space endowed with a quasi-distance $d$ and a nonnegative measure $\mu$ such that

$$d(x, y) \geq 0 \quad \text{and} \quad d(x, y) = 0 \text{ if and only if } x = y,$$

$$d(x, y) = d(y, x),$$

$$d(x, y) \leq K_1(d(x, z) + d(z, y)), \quad (2.1)$$

the balls (d-balls) $B(x, r) = B^d(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$, form a basis of neighborhoods of the point $x$, $\mu$ is defined on a $\sigma$-algebra of subsets of $X$ which contains the balls, and

$$0 < \mu(B(x, 2r)) \leq K_2 \mu(B(x, r)) < \infty, \quad (2.2)$$

If there are constants $\theta$ ($0 < \theta \leq 1$) and $K_3 \geq 1$ such that

$$|d(x, z) - d(y, z)| \leq K_3 (d(x, z) + d(y, z))^{1-\theta} d(x, y)^\theta, \quad x, y, z \in X, \quad (2.3)$$

then the balls are open sets. Note that (2.1) for some $K_1 \geq 1$ follows from (2.3) (Lemarié [4]). Conversely, from (2.1) it follows that there exist $\theta > 0$, $K_3 \geq 1$ and a quasi-distance which is equivalent to the original $d$ such that (2.3) holds (Máfias and Segovia [5]). Therefore We always assume (2.3) in this report.

It is known that, if $\mu(X) < +\infty$, then there is a constant $R_0 > 0$ such that

$$x = B(x, R_0) \quad \text{for all } x \in X \quad (2.4)$$

(see [12, Lemma 5.1]).
3. Definitions

**Definition 3.1** ($[\phi, q]$-atom (resp. $(p(\cdot), q)$-atom)). Let $\phi : X \times (0, \infty) \to (0, \infty)$ and $1 < q \leq \infty$. A function $a$ on $X$ is called a $[\phi, q]$-atom (resp. $(p(\cdot), q)$-atom) if there exists a ball $B$ such that

(i) $\text{supp } a \subset B,$

(ii) $\|a\|_q \leq \frac{1}{\mu(B)^{1/q'} \phi(B)}$

(resp. $\|a\|_q \leq \mu(B)^{1/q} \theta + Q$, where $x$ is the center of $B$),

(iii) $\int_X a(x) \, d\mu(x) = 0,$

where $\|a\|_q$ is the $L^q$ norm of $a$ and $1/q + 1/q' = 1$. We denote by $A[\phi, q]$ the set of all $[\phi, q]$-atoms. (We denote by $A(p(\cdot), q)$ the set of all $(p(\cdot), q)$-atoms.)

We note that $(p(\cdot), q)$-atoms are special cases of $[\phi, q]$-atoms. If $p(x) \equiv p$, then the $(p(\cdot), q)$-atom is the usual $(p, q)$-atom. Let $p_- = \inf p(x)$ and $p_+ = \sup p(x)$.

**Remark 3.1.** Assume that $\mu(B(x, r)) \sim r^Q$ ($Q > 0$) for $x \in X$ and $0 < r < \infty$ ($0 < r \leq R_0$ if $\mu(X) < \infty$). Let $\alpha(x) = Q(1/p(x) - 1)$. If $Q/(\theta + Q) \leq p_- \leq p_+ < 1$, then $0 < \alpha_- \leq \alpha_+ \leq \theta$ and $\text{Lip}_{\alpha(\cdot)}(X) = \mathcal{L}_{q', \phi}(X)$ for all $q' \in [1, \infty)$.

If $a$ is a $[\phi, q]$-atom and a ball $B$ satisfies (i)-(iii), then

$$ |\int_X a(x)g(x) \, d\mu(x)| = |\int_B a(x)(g(x) - g_B) \, d\mu(x)| $$

$$ \leq \|a\|_q \left( \int_B |g(x) - g_B|^{q'} \, d\mu(x) \right)^{1/q'} $$

$$ \leq \frac{1}{\phi(B)} \left( \frac{1}{\mu(B)} \int_B |g(x) - g_B|^{q'} \, d\mu(x) \right)^{1/q'} $$

$$ \leq \|g\|_{\mathcal{L}_{q', \phi}}.$$

That is, the mapping $g \mapsto \int_X a g \, d\mu$ is a bounded linear functional on $\mathcal{L}_{q', \phi}(X)/C$ with norm not exceeding 1.

**Definition 3.2** ($H^{[\phi, q]}_{U}(X)$). Let $\phi : X \times \mathbb{R}_+ \to \mathbb{R}_+$, $1 < q \leq \infty$ and $1/q + 1/q' = 1$. Let $U$ be a continuous, concave, increasing and bijective function from $[0, +\infty)$ to itself. Assume that $\mathcal{L}_{q', \phi}(X)/C \neq \{0\}$. We define the space $H^{[\phi, q]}_{U}(X) \subset (\mathcal{L}_{q', \phi}(X)/C)^*$ as follows:
$f \in H_{U}^{[\phi, q]}(X)$ if and only if there exist sequences \( \{a_{j}\} \subset A[\phi, q] \) and positive numbers \( \{\lambda_{j}\} \) such that

\[
(3.2) \quad f = \sum_{j} \lambda_{j} a_{j} \text{ in } (\mathcal{L}_{q', \phi}(X)/C)^{*} \quad \text{and} \quad \sum_{j} U(\lambda_{j}) < \infty.
\]

From \( U(0) = 0 \) and the concavity of \( U \) it follows that

\[
(3.3) \quad U(Cr) \leq CU(r), \quad 1 \leq C < \infty, \quad 0 \leq r < \infty,
\]

\[
(3.4) \quad U(r + s) \leq U(r) + U(s), \quad 0 \leq r, s < \infty.
\]

Then \( H_{U}^{[\phi, q]}(X) \) is a linear space. (3.4) implies

\[
(3.5) \quad \sum_{j} \lambda_{j} \leq U^{-1}\left(\sum_{j} U(\lambda_{j})\right).
\]

Therefore, if \( \sum_{j} U(\lambda_{j}) < \infty \), then \( \sum_{j} \lambda_{j} < \infty \) and \( \sum_{j} \lambda_{j} a_{j} \) converges in \( (\mathcal{L}_{q', \phi}(X)/C)^{*} \).

In general, the expression (3.2) is not unique. We define

\[
\|f\|_{H_{U}^{[\phi, q]}} = \inf \left\{ U^{-1}\left(\sum_{j} U(\lambda_{j})\right) \right\},
\]

where the infimum is taken over all expressions as in (3.2). We note that \( \|f\|_{H_{U}^{[\phi, q]}} \) is not a norm in general. Let \( d(f, g) = U(\|f - g\|_{H_{U}^{[\phi, q]}}) \) for \( f, g \in H_{U}^{[\phi, q]}(X) \). Then \( d(f, g) \) is a metric and \( H_{U}^{[\phi, q]}(X) \) is complete with respect to this metric. If \( I(r) = r \), then \( \|f\|_{H_{U}^{[\phi, q]}} \) is a norm and \( H_{U}^{[\phi, q]}(X) \) is a Banach space.

In the case of \( (p(\cdot), q) \)-atoms instead of \([\phi, q] \)-atoms, we denote \( H_{U}^{[\phi, q]}(X) \) by \( H_{U}^{p(\cdot), q}(X) \).

4. Results

**Theorem 4.1.** If there exists a constant \( C_{*} > 0 \) such that

\[
(4.1) \quad U(rs) \leq C_{*}U(r)U(s) \quad \text{for} \quad 0 < r, s \leq 1,
\]

\[
(4.2) \quad U\left(\frac{\mu(B_{1})\phi(B_{1})}{\mu(B_{2})\phi(B_{2})}\right) \leq C_{*} \frac{\mu(B_{1})}{\mu(B_{2})} \quad \text{for all balls } B_{1} \text{ and } B_{2} \text{ with } B_{1} \subset B_{2},
\]

then

\[
H_{U}^{[\phi, q]}(X) = H_{U}^{[\phi, \infty]}(X),
\]

with equivalent topologies.
Corollary 4.2. Let $Q > 0$. Assume that $\mu(X) < \infty$ and that $\mu(B(x, r)) \sim r^Q$ for all $x \in X$ and $0 < r < R_0$, where $R_0$ is the constant in (2.4). Let $U(r) = r^{p_+}$ with $0 < p_- \leq p_+ \leq 1$, where $p_- = \inf p(x)$ and $p_+ = \sup p(x)$. If there exists a constant $C_0 > 0$ such that

$$|p(x) - p(y)| \leq \frac{C_0}{\log(1/d(x, y))} \text{ for } d(x, y) < 1/2,$$

then

$$H_U^{p(+),q}(X) = H_U^{p(+),\infty}(X),$$

with equivalent topologies.

In this case we denote $H_U^{p(+),q}(X)$ by $H^{p(+)}(X)$ simply, which is a kind of Hardy spaces with variable exponent.

Proof of Corollary 4.2. The inequality (4.1) holds clearly. We show (4.2).

For $B(x, r) \subset B(y, s)$,

$$\frac{U\left(\frac{\mu(B(x, r))}{\mu(B(y, s))}\right)}{\frac{\mu(B(x, r))}{\mu(B(y, s))}} \sim \left(\frac{r}{s}\right)^{Qp_+(1/p(x)-1/p_+)} s^{Qp_+(1/p(x)-1/p(y))} \leq s^{Qp_+(1/p(x)-1/p(y))},$$

since $r/s \leq 1$. If $1/2 < s < R_0$, then

$$s^{Qp_+(1/p(x)-1/p(y))} \leq R_0^{Qp_+/p_-}.$$

If $s \leq 1/2$, then $d(x, y) < s$ and

$$\log s^{Qp_+(1/p(x)-1/p(y))} \leq Qp_+ \left|\frac{1}{p(y)} - \frac{1}{p(x)}\right| \log(1/s) \leq Qp_+ \left|\frac{p(x) - p(y)}{p(x)p(y)}\right| \log(1/d(x, y)) \leq \frac{C_0 Qp_+}{p_-^2}. \quad \square$$

Lemma 4.3. Let $E = H_U^{p,q}(X)$. If

$$\sup_{0 < s \leq 1} \frac{U(rs)}{U(s)} \to 0 \quad (r \to 0),$$

then

$$\|\ell\|_{E^*} = \sup \{ |\ell(f)| : \|f\|_E \leq 1 \}$$

is finite for all $\ell \in E^*$, and $\|\ell\|_{E^*}$ is a norm.

Remark 4.1. If (4.1) holds, then (4.4) holds. If (4.4) holds, then there exist constants $C > 0$ and $p > 0$ such that $U(r) \leq Cr^p$ for $r \in (0, 1]$. If $\alpha > 0$ and $U(r) = (\log(1/r))^{-\alpha}$ for small $r > 0$, then $U$ does not satisfy (4.4).
Let $L_c^q(X)$ be the set of all $L^q$-functions with bounded support, and let

$$L_c^{q,0}(X) = \left\{ f \in L_c^q(X) : \int_X f \, d\mu = 0 \right\}.$$

Then, for $1 < q \leq \infty$, $L_c^{q,0}(X)$ is dense in $H_U^{[\phi,q]}(X)$.

**Theorem 4.4.** If $U$ satisfies (4.4), then

$$\left( H_U^{[\phi,q]}(X) \right)^* = L_{q',\phi}(X)/C.$$

More precisely, if $g \in L_{q',\phi}(X)/C$, then the mapping $\ell : f \mapsto \int_X f(g + c) \, d\mu$, for $f \in L_c^{q,0}(X)$, can be extended to a continuous linear functional on $H_U^{[\phi,q]}(X)$. Conversely, if $\ell$ is a continuous linear functional on $H_U^{[\phi,q]}(X)$, then there exists $g \in L_{q',\phi}(X)/C$ such that $\ell(f) = \int_X f(g + c) \, d\mu$ for $f \in L_c^{q,0}(X)$. The norm $\|\ell\|$ is equivalent to $\|g\|_{L_{q',\phi}}$.

**Corollary 4.5.** Assume the conditions in Remark 3.1 and Corollary 4.2. Then

$$\left( H^{p(\cdot)}(X) \right)^* = \text{Lip}_{\alpha(\cdot)}(X)/C.$$

**REFERENCES**


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