数理解析研究所講究録
第 1617 巻 2008 年 11-17

SPECTRAL SQUARE MEANS FOR PERIOD INTEGRALS OF WAVE FUNCTIONS

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1. Introduction

Let G be a connected semisimple Lie group with finite center of non-compact type, and $\mathfrak{D} = G/K$ the corresponding symmetric space with K a maximal compact subgroup of G. Set $d = \dim_{\mathbb{R}}(\mathfrak{D})$. By fixing a G-invariant $\mathbb{R}$-bilinear form proportional to the Killing form on the Lie algebra of G, we endow the manifold $\mathfrak{D}$ with a G-invariant Riemannian metric $\mathrm{d}s^2$. Given an arithmetic subgroup Γ of G, let $L^2(\Gamma\backslash\mathfrak{D})$ be the Hilbert space of all the complex valued measurable functions $\phi(\tau)$ on $\mathfrak{D}$ such that $\phi(\gamma\tau) = \phi(\tau)$ for all $\gamma \in \Gamma$ with the finite L^2-norm

$$\|\phi\| = \{\int_{\Gamma\backslash G} |\phi(\tau)|^2 \,\mathrm{d}\mu_{\mathfrak{D}}(\tau)\}^{1/2}.$$

where $\mathrm{d}\mu_{\mathfrak{D}}$ is the volume form of $(\mathfrak{D}, \mathrm{d}s^2)$. Let $\bar{\Delta}_\Gamma$ be the self-adjoint extension of the Laplacian of $(\mathfrak{D}, \mathrm{d}s^2)$ with the domain $\{L^2(\Gamma\backslash G)^\infty\}^K$, where $L^2(\Gamma\backslash G)^\infty$ means the smooth vectors of the right regular representation of G on $L^2(\Gamma\backslash G)$. Then the space of L^2-wave forms on Γ of eigenvalue λ is defined by

$$\mathcal{A}(\Gamma\backslash\mathfrak{D};\lambda) \stackrel{\mathrm{def}}{=} \{\phi \in \mathrm{Dom}(\bar{\Delta}_\Gamma) |\, \bar{\Delta}_\Gamma\phi = \lambda\,\phi\,\},$$

and the set of eigenvalues of $\bar{\Delta}_\Gamma$ by

$$\Lambda_\Gamma \stackrel{\mathrm{def}}{=} \{\lambda \in \mathbb{C} |\, \mathcal{A}(\Gamma\backslash\mathfrak{D};\lambda) \neq \{0\}\,\}.$$

It is known that the space $\mathcal{A}(\Gamma\backslash\mathfrak{D};\lambda)$ is a finite dimensional space consisting of automorphic forms in the sense of Harish-Chandra and that the set Λ_Γ is a subset of non-negative real numbers such that $\sharp(\Lambda_\Gamma \cap [0, x)) < +\infty$ for any $x > 0$ ([1]). Note that 0 is the minimal element of Λ_Γ with the corresponding normalized eigenfunction $\phi_0 = (\mathrm{vol}(\mathcal{F}_\Gamma))^{-1/2}$ (constant).

In order to study the distribution of eigenvalues counted with multiplicities, it is common to introduce the counting function

$$N_\Gamma(x) := \sum_{\lambda \in \Lambda_\Gamma \cap (0,x)} \dim_{\mathbb{C}} \mathcal{A}(\Gamma\backslash\mathfrak{D};\lambda), \quad x > 0.$$

Then, by Selberg's trace formula, one can show that the non-Euclidean analogue of Weyl's law for the asymptotic distribution of the eigenvalues of the Laplacian takes the form

$$N_\Gamma(x) \sim \frac{\mathrm{vol}(\Gamma\backslash\mathfrak{D})}{(4\pi)^{d/2}\Gamma(d/2+1)}\, x^{d/2}, \quad x \to +\infty$$

at least when the lattice Γ is uniform or of real rank one ([4], [10]). For a non-uniform Γ of higher rank, it gets harder to establish a similar formula. A weak form of Weyl's law for cuspidal spectrum is obtained by H. Donnelly ([2]) for a general setting. True

Weyl's law for cusp forms on $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ $(n \geqslant 3)$ is proved by Müller ([8]); a refined formula with error term is obtained by Lapid-Müller quite recently ([9]). These asymptotic formula yield infinitely many cusp forms belonging to different eigenvalues of the Laplacian.

Let $H \subset G$ be a closed subgroup and $\mathfrak{D}_H$ an H-orbit in $\mathfrak{D}$. The integral of an automorphic form f on $\Gamma\backslash\mathfrak{D}$ along the quotient $\Gamma \cap H\backslash\mathfrak{D}_H$ is often called the H-period integral of f, probably by abuse of terminology. In recent years, through an active research by many people, it is observed that this kind of period integrals sometimes are closely related with the special values of certain automorphic L-functions. In [11], we introduce yet another counting function by taking an average of norm square of H-periods of L^2-wave forms for a symmetric subgroup $H \subset G$, and derive its asymptotic law similar to Weyl's law for several examples. By our formula, we can show the existence of infinitely many L^2-wave forms with non-vanishing H-periods by assuming a subconvexity bound of certain automorphic L-functions. This article contain a brief summary of results in [11].

The author would like to thank the organizer of the workshop, Professor Kaoru Hiraga, for giving him an opportunity of talk.

2. Results

2.1. Let G be a reductive algebraic group defined over $\mathbb{Q}$ and G the identity component of the real Lie group $\mathsf{G}(\mathbb{R})$. Let σ be an involutive $\mathbb{Q}$-automorphism of G and $H = G^\sigma$ the fixed point subgroup of σ on G. Let $\Gamma \subset \mathsf{G}(\mathbb{Q})$ be an arithmetic lattice in G such that $\Gamma_H = \Gamma \cap H$ yields a lattice of H. In particular, $\mathrm{vol}(\Gamma_H\backslash H) < +\infty$. We suppose, for simplicity, the base point K of $\mathfrak{D}$ is taken so that $K_H = H \cap K$ is a maximal compact subgroup of H. Thus, $\mathfrak{D}_H = H/K_H$ is a symmetric space of H with a natural inclusion $\iota : \mathfrak{D}_H \hookrightarrow \mathfrak{D}$. Let $\mathrm{d}s_H^2$ be the pull back of $\mathrm{d}s^2$ by ι, and $\mathrm{d}\mu_{\mathfrak{D}_H}$ the volume form of $(\mathfrak{D}_H, \mathrm{d}s_H^2)$. Fix an L^2-wave form $\phi \in \mathcal{A}(\Gamma_H\backslash\mathfrak{D}_H;\mu)$ with the Laplace eigenvalue μ. Then we make the following definition.

Definition: For a Γ-invariant continuous function $F : \mathfrak{D} \to \mathbb{C}$, define the period integral along $(\Gamma_H\backslash\mathfrak{D}_H, \phi)$ by

$$\mathcal{P}_H^\phi(F) = \int_{\Gamma_H\backslash\mathfrak{D}_H} \phi \cdot (F|_{\mathfrak{D}_H})\, \mathrm{d}\mu_{\mathfrak{D}_\mathrm{H}}$$

if convergent. □

From the fact we stated in the introduction, the set of eigenvalues Λ_Γ of $\tilde{\Delta}_\Gamma$ can be enumerated in a non-decreasing sequence

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$$

so that each $\lambda \in \Lambda_\Gamma$ occurs with its multiplicity $\dim_\mathbb{C} \mathcal{A}(\Gamma\backslash\mathfrak{D};\lambda)$. Fix an orthonormal system $\{F_n\}_{n=0}^\infty$ of L^2-wave forms such that $\Delta F_n = \lambda_n F_n$ for any n. Then, our new counting function is defined as follows.

Definition:

$$N_H^\phi(\Gamma;x) \stackrel{\mathrm{def}}{=} \sum_{\lambda_n \leqslant x} \left| \mathcal{P}_H^\phi(F_n) \right|^2, \qquad x > 0.$$

2.2. Let $\mathbb{Q} \subset F \subset E$ be field extensions of finite degree. We suppose that F is totally real over $\mathbb{Q}$ of degree d_F and that
(i) $E = F$, or
(ii) E is a quadratic extension of F such that E is totally imaginary over $\mathbb{Q}$.

Let $\iota_\alpha : F \hookrightarrow \mathbb{R}$ $(1 \leqslant \alpha \leqslant d_F)$ be the set of all the embeddings of F into $\mathbb{R}$; when $E \neq F$, each ι_α can be extended to embeddings $E \hookrightarrow \mathbb{C}$ in exactly two ways, one of which we choose once and for all and denote it by ι_α also.

Let $S = (s_{ij}) \in \mathrm{GL}_m(E)$ be a hermitian matrix (i.e., ${}^t\bar{S} = S$) such that $S^{(\alpha)} := (s_{ij}^{\iota_\alpha})$ is positive definite unless $\alpha = 1$ in which case the signature of $S^{(1)}$ is $(p+, q-)$ with $p \geqslant 2$, $q \geqslant 1$ and $p + q = m$.

Let $\mathsf{G} = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}(S)$ be the restriction of scalars of the 'unitary group' of S over F, i.e.,

$$\mathsf{G}(\mathbb{Q}) = \{g \in \mathrm{GL}_m(E) \,|\, {}^t\bar{g}Sg = S\}.$$

Note that G is an orthogonal group in the usual sense when $F = E$. Let $G^{(\alpha)}$ be the unitary group of $S^{(\alpha)}$. Then the $\mathbb{R}$-valued points $\mathsf{G}(\mathbb{R})$ is decomposed as the product $\prod_{\alpha=1}^{d_F} G^{(\alpha)}$. By the assumption on S, the Lie group $G^{(\alpha)}$ for $2 \leqslant \alpha \leqslant d_F$ is compact and the group $G := G^{(1)}$ is isomorphic to (i) $\mathrm{O}(p,q)$ if $F = E$, or to (ii) $\mathrm{U}(p,q)$ if $F \neq E$. Let $\mathrm{pr}_1 : \mathsf{G}(\mathbb{R}) \to G^{(1)}$ be the first projection in the decomposition $\mathsf{G}(\mathbb{R}) \cong \prod_{\alpha=1}^{d_F} G^{(\alpha)}$.

Let $\mathfrak{o}$ be the integer ring of E and $\mathcal{L} = \mathfrak{o}^m$ the standard $\mathfrak{o}$-lattice in E^m, the space of column vectors with entries in E. Define $\mathsf{G}_{\mathbf{Z}} = \{\gamma \in \mathsf{G}(\mathbb{Q}) \,|\, \gamma\mathcal{L} = \mathcal{L}\}$. Then, the first projection $G_{\mathbf{Z}} := \mathrm{pr}_1\mathsf{G}_{\mathbf{Z}}$ is a lattice in G which is uniform unless $d_F = 1$. Let $\mathfrak{L}$ be the set of lattices in G commensurable to $G_{\mathbf{Z}}$. For an $\mathfrak{o}$-ideal $I \subset \mathfrak{o}$, the principal congruence subgroup of level I, denoted by $\Gamma(I)$, is defined to be the kernel of the reduction homomorphism $\mathsf{G}_{\mathbf{Z}} \to \mathrm{GL}(\mathcal{L}/I\mathcal{L})$. Then $\Gamma(I) \in \mathfrak{L}$.

Fix a non-zero vector $\mathsf{v} \in \mathcal{L}$ such that $S[\mathsf{v}]^{\iota_1} > 0$ and denote by H the stabilizer of v in G. Set $H = \mathrm{pr}_1\mathsf{H}(\mathbb{R})$.

Fix a positive definite subspace U_1 of maximal dimension for $S^{(1)}$ such that $\mathsf{v}^{\iota_1} \in U_1$. Then $K = \{k \in G \,|\, kU_1 = U_1\}$ is a maximal compact subgroup of G such that $H \cap K$ is maximally compact in H.

For an $\mathfrak{o}$-ideal $I \subset \mathfrak{o}$, let $\delta(I)$ be the minimal norm of the vectors $(\xi^{\iota_\alpha}) \in \mathbb{C}^{d_F}$ $(\xi \in I - \{0\})$.

2.3. **Results for uniform lattices.** Let us state our first result on the counting function $N_H^\phi(\Gamma; x)$ with ϕ being the constant function 1 and $\Gamma \in \mathfrak{L}$ being cocompact.

Theorem 1. *In the above settings, suppose $d_F > 1$ further. Let $\{I_n\}$ be any sequence of $\mathfrak{o}$-ideals such that $\delta(I_n) \to 0$. Then there exists some number n_0 such that the following holds. For any $\Gamma \in \mathfrak{L}$ such that $\Gamma \subset \Gamma(I_n)$ $(\exists n \geqslant n_0)$,*

$$N_H^1(\Gamma; x) \sim \frac{\mathrm{vol}(\Gamma_H \backslash \mathfrak{D}_H)}{(4\pi)^d \Gamma(d+1)} x^d, \quad x \to +\infty.$$

Here $d = \frac{1}{2}\{\dim_{\mathbb{R}} \mathfrak{D} - \dim_{\mathbb{R}} \mathfrak{D}_H\}$.

2.4. **Results for non-uniform lattices.** Our second result concerns a non-uniform lattice inside $G = \mathrm{O}(p+, 1-)$. Let $F = E = \mathbb{Q}$ and $q = 1$ in the notation of 2.2, and take

$$S = \begin{bmatrix} & & -2 \\ & 1_{p-1} & \\ -2 & & \end{bmatrix}.$$

Let $\mathcal{C}$ be the set of all the one dimensional S-isotropic $\mathbb{Q}$-subspaces $\ell \subset \mathbb{Q}^{p+1}$. For $\ell \in \mathcal{C}$, let P^ℓ be the stabilizer of ℓ in G. Then P^ℓ is a $\mathbb{Q}$-parabolic subgroup of G. Let N^ℓ be the unipotent radical of P^ℓ. Fix a basis $\mathrm{e}_\ell \in \ell$ and choose an S-isotropic vector $\mathrm{e}'_\ell \in \mathbb{Q}^{p+1}$ such that $S(\mathrm{e}_\ell, \mathrm{e}'_\ell) = +1$. Define the torus A^ℓ to be the set of all the elements $\mathrm{a}_l(t)$, $t > 0$ such that $\mathrm{a}_\ell(t)\,\mathrm{e}_l = t\,\mathrm{e}_\ell$, $\mathrm{a}_\ell(t)\,\mathrm{e}'_\ell = t^{-1}\,\mathrm{e}'_\ell$ and $\mathrm{a}_\ell(t)$ is identity on the orthogonal complement of $\mathbb{Q}\mathrm{e}_\ell + \mathbb{Q}\mathrm{e}'_\ell$ in $\mathbb{Q}^{p+1}$. Then, A^ℓ is a $\mathbb{Q}$-split component of P^ℓ and we have an Iwasawa decomposition $G = N^\ell A^\ell K$. For $g \in G$, let us define the number $t_\ell(g)(> 0)$ by the relation $g \in N^\ell\,\mathrm{a}_\ell(t_\ell(g))\,K$.

Let $\Gamma \in \mathfrak{L}$. Then, the orbit space $\Gamma\backslash\mathcal{C}$ is a finite set. Fix a complete set of representatives $\ell_j\,(1 \leqslant j \leqslant h)$ for $\Gamma\backslash\mathcal{C}$. For each j, the Eisenstein series $E^{(j)}(s;\tau)$ is defined by the series

$$E^{(j)}(s;\tau) = \sum_{\gamma \in \Gamma \cap N^{\ell_j}\backslash\Gamma} t_{\ell_j}(\gamma g)^{s+(p-1)/2}, \qquad \tau = gK \in \mathfrak{D}.$$

which is absolutely convergent on $\mathrm{Re}(s) > (p-1)/2$. It is known that the function $s \mapsto E^{(j)}(s;\tau)$ has a meromorphic continuation to the whole complex plane so that $E^{(j)}(s;\tau)$ is holomorphic on the imaginary axis. Moreover, for a fixed $t \in \mathbb{R}$, the function $\tau \mapsto E^{(j)}(\sqrt{-1}t;\tau)$ is an automorphic form on $\Gamma\backslash\mathfrak{D}$.

Theorem 2. *([11]) Let $\Gamma \in \mathfrak{L}$. We assume $p > 3$ unless $\Gamma_H\backslash\mathfrak{D}_H$ is compact. Moreover, suppose ϕ is a cusp form or the constant function* 1. *Then, the period integrals $\mathcal{P}_H^\phi(F_n)$, $(n \in \mathbb{N})$ and $\mathcal{P}_H^\phi(E^{(j)}(\sqrt{-1}t))$, $(1 \leqslant j \leqslant h,\ t \in \mathbb{R})$ converge absolutely. We have the asymptotic law*

$$N_H^\phi(\Gamma;x) + \frac{1}{4\pi}\int_{-\sqrt{x}}^{\sqrt{x}} \sum_{j=1}^{h} \left|\,\mathcal{P}_H^\phi(E^{(j)}(\sqrt{-1}t))\,\right|^2 \mathrm{d}t \sim \frac{\|\phi\|^2}{\pi}\,x^{1/2}, \qquad x \to +\infty.$$

2.4.1. *Spectral zeta function with period integrals.* We consider a Dirichlet series associated with a system of periods $\{\mathcal{P}_H^\phi(F_n)\}_{n=1}^\infty$ and $\{\mathcal{P}_H^\phi(E^{(j)}(\sqrt{-1}t))\}_{t\in\mathbb{R}}$:

$$Z_{H,\phi}^\Gamma(s) \stackrel{\mathrm{def}}{=} \sum_{n=1}^\infty \frac{1}{\lambda_n{}^s}\left|\,\mathcal{P}_H^\phi(F_n)\,\right|^2 + \frac{1}{4\pi}\int_{\mathbb{R}}\Big\{\sum_{j=1}^h \left|\,\mathcal{P}_H^\phi(E^{(j)}(it))\,\right|^2\Big\}\frac{\mathrm{d}t}{(t^2+\rho^2)^s}.$$

Theorem 3. *([11]) The series $Z_{H,\phi}^\Gamma(s)$ converges absolutely on the half-plane* $\mathrm{Re}(s) > 2$. *The holomorphic function $Z_{H,\phi}^\Gamma(s)$ on* $\mathrm{Re}(s) > 2$ *has a meromorphic continuation to the whole s-plane. It has possible simple poles at $s = \frac{1}{2} - n\,(n \in \mathbb{Z}_{\geqslant 0})$ and possible double poles at $s = -m\,(m \in \mathbb{Z}_{\geqslant 0})$. We have*

$$\mathrm{Res}_{s=1/2} Z_{H,\phi}^\Gamma(s) = (2\sqrt{\pi})^{-1}\|\phi\|^2.$$

3. The case of $\mathrm{PSL}_2(\mathbb{R})$

An element $\eta \in \mathrm{PSL}_2(\mathbb{R})$ is called hyperbolic if there exists $R_\eta \in \mathrm{PSL}_2(\mathbb{R})$ and $N(\eta) > 1$ such that

$$\eta = \pm R_\eta \begin{bmatrix} N(\eta)^{1/2} & 0 \\ 0 & N(\eta)^{-1/2} \end{bmatrix} R_\eta^{-1}.$$

The number $N(\eta)$ is called the norm of η. Let $C_\eta \subset \mathfrak{H}$ be the geodesic curve in $\mathfrak{H}$ joining the two fixed points $\theta_+(\eta) = R_\eta\langle i\infty\rangle$ and $\theta_-(\eta) = R_\eta\langle 0\rangle$ of η in $\mathbb{R}$, or explicitly $C_\eta = \{R_\eta\langle it\rangle \,|\, 1 < t < N(\eta)\}$. From now on, we fix a lattice Γ commensurable with $\mathrm{PSL}_2(\mathbb{Z})$. A hyperbolic element $\eta \in \Gamma$ is called to be primitive in Γ if the centralizer of

η in Γ is a cyclic group $\langle\eta\rangle$ generated by η. The group $\langle\eta\rangle$ preserves the curve C_η; its quotient $\langle\eta\rangle\backslash C_\eta$, denoted by C_η^Γ, is regarded as a simple geodesic of $\Gamma\backslash\mathfrak{H}$. The period integral of a continuous function $f:\Gamma\backslash\mathfrak{H}\to\mathbb{C}$ along C_η^Γ is defined by

$$\int_{C_\eta^\Gamma} f\,\mathrm{d}s = \int_0^{\log \mathrm{N}(\eta)} f(R_\eta\langle ie^t\rangle)\,\mathrm{d}t. \tag{3.1}$$

We fix a complete set of Γ-inequivalent cusps $\{c_j\}$ of Γ and a family of elements $\{\sigma_j\}$ in SO(2) such that $\sigma_j\langle\infty\rangle = c_j$. Then the Eisenstein series at the cusp c_j is defined by the series

$$\mathfrak{e}^{(j)}(s;\tau) = \sum_{\gamma\in\Gamma_{c_j}\backslash\Gamma} \mathrm{Im}(\sigma_j^{-1}\gamma\tau)^s, \quad \mathrm{Re}(s)>1,\ \tau\in\mathfrak{H}.$$

Let $\{\lambda_n\}$ be the non-decreasing sequence of eigenvalues counted with multiplicity of hyperbolic Laplacian $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}\right)$ acting on $L^2(\Gamma\backslash\mathfrak{H})$, and $\{f_n\}$ an orthonormal system of eigenforms, i.e.,

$$-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}\right) f_n = \lambda_n f_n.$$

Then from Theorem 2, we can deduce the following theorem.

Theorem 4. *([11])*

$$\sum_{\lambda_n\leqslant x}\left|\int_{C_\eta^\Gamma} f_n\,\mathrm{d}s\right|^2 + \frac{1}{4\pi}\int_{-\sqrt{x}}^{\sqrt{x}}\sum_{j=1}^{h}\left|\int_{C_\eta^\Gamma}\mathfrak{e}^{(j)}\left(\tfrac12+it\right)\mathrm{d}s\right|^2 \mathrm{d}t \sim \frac{\log\mathrm{N}(\eta)}{\pi}x^{1/2}, \quad x\to+\infty.$$

3.0.2. The projective modular group $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$ has a unique cusp ∞ up to $\mathrm{PSL}_2(\mathbb{Z})$-equivalence. The Eisenstein series is

$$\mathfrak{e}(\nu,\tau) = \sum_{(c,d)=1}\frac{(\mathrm{Im}(\tau))^\nu}{|c\tau+d|^{2\nu}}, \quad \mathrm{Re}(\nu)>1.$$

To each primitive hyperbolic element $\eta = \left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]$ of $\mathrm{PSL}_2(\mathbb{Z})$, we associate an integral binary quadratic form $Q_\eta(X,Y) = cX^2+(d-a)XY-bY^2$. The number $D=(\mathrm{tr}(\eta))^2-4\,(>0)$ is the discriminant of Q_η. For $n\in\mathbb{Z}-\{0\}$, the representation number of n by Q_η is

$$\mathfrak{R}(Q_\eta;n)=\sharp(\{(x,y)\in\mathbb{Z}^2 \mid Q_\eta(x,y)=n\}/E(Q_\eta)),$$

with $E(Q_\eta)=\{\gamma\in\mathrm{SL}_2(\mathbb{Z}) \mid {}^t\gamma Q_\eta\gamma = Q_\eta\}$ the unit group of Q_η. Then define the zeta function of Q_η by

$$\zeta(Q_\eta;\nu)=\sum_{n\in\mathbb{Z}-\{0\}}\frac{\mathfrak{R}(Q_\eta;n)}{|n|^\nu},$$

which is absolutely convergent on $\mathrm{Re}(\nu)>1$. The computation of the period integral of $\mathfrak{e}(\nu)$ along C_η^Γ is due to Hecke. In our case, the formula is

$$\int_{C_\eta^\Gamma}\mathfrak{e}(\nu)\,\mathrm{d}s = \frac18\hat{\zeta}(Q_\eta;\nu)\,\hat{\zeta}(2\nu)^{-1} \tag{3.2}$$

where $\hat{\zeta}(Q_\eta;\nu) = D^{\nu/4}\Gamma_{\mathbb{R}}(\nu)^2\zeta(Q_\eta;\nu)$. The functional equation $\hat{\zeta}(2\nu)\,\mathfrak{e}(\nu) = \hat{\zeta}(2-2\nu)\,\mathfrak{e}(1-\nu)$, combined with (3.2) yilelds the functional equation $\hat{\zeta}(Q_\eta;1-\nu)=\hat{\zeta}(Q_\eta;\nu)$.

Hence, by the usual technique, we obtain the convexity bound of the zeta function $\zeta(Q_\eta; s)$ on the critical line:

$$\zeta\left(Q_\eta; \tfrac{1}{2} + it\right) \prec (1 + |t|)^{1/2+\epsilon}, \quad t \in \mathbb{R}$$

for any $\epsilon > 0$.

Proposition 5. *([11]) Suppose the subconvexity bound of $\zeta(Q_\eta; s)$ on the critical line*

$$|\zeta(Q_\eta; 1/2 + it)| \prec (1 + |t|)^\delta,\ t \in \mathbb{R}$$

holds for some $\delta < 1/2$. Then,

$$\sum_{\lambda_n \leqslant x} \left| \int_{C_\eta^\Gamma} f_n \, \mathrm{d}s \right|^2 \sim \frac{\log \mathrm{N}(\eta)}{\pi} x^{1/2}, \quad x \to +\infty.$$

4. Concluding remarks and problems

4.1. Observations.

- Theorem 2 yields the estimation of the mean value of the Eisenstein period:

$$\int_{\sqrt{x}}^{\sqrt{x}} \sum_{j=1}^{h} \left| \mathcal{P}_{H,\phi}(E^{(j)}(\sqrt{-1}t)) \right|^2 \mathrm{d}t \prec x^{1/2}, \quad (x \to +\infty).$$

  By combining this with the integral representation of the standard L-functions of orthogonal groups by Murase-Sugano ([7]), we obtain some bound of the square mean value

$$\int_0^x \left| L\left(\tfrac{1}{2} + it\,; \phi\right) \right|^2 dt$$

  for the Hecke eigen wave cuspform ϕ. This seems yield a better bound than the convexity bound for general ϕ not necessarily belonging to the images of liftings from other groups.
- In the situation of the paragraph 2.1, we further suppose that the symmetric space $H\backslash G$ is of split rank one. From the experience of several concrete examples, we can guess what the asymptotic formula of $N_H^\phi(\Gamma; x)$ should look like. Under the convergence of relevant period integrals of automorphic forms, the following formula is plausible.

$$\sum_{\lambda_n \leqslant x} \left| \mathcal{P}_H^\phi(F_n) \right|^2 + \int |\text{'Eisenstein period '}|^2 \sim \frac{\|\phi\|^2}{(4\pi)^d \Gamma(1+d)} x^d, \quad x \to +\infty$$

  with $d = \frac{1}{2}\{\dim_{\mathbb{R}} \mathfrak{D} - \dim_{\mathbb{R}} \mathfrak{D}_H\}$.

4.2. Problems. Here are some problems on the asymptotic formula of $N_H^\phi(\Gamma; x)$.

- Theorem 1 and Theorem 2 yields the main term of the asymptotic $N_H^\phi(\Gamma; x)$ as $x \to +\infty$. It seems interesting to obtain an error term estimate.
- To formulate our problem, we assumed convergence of several integrals, for example the L^2-norm of ϕ, the finiteness of the volume $\mathrm{vol}(\Gamma_H \backslash \mathfrak{D}_H)$ and the period of wave forms $\mathcal{P}_H^\phi(F_n)$. It may be interesting drop these conditions, replacing the relevant integrals by properly regularized ones. In this aspect, we should mention the work of Zagier ([12]).

- By a technical reason, all the spaces $H\backslash G$ we consider so far are of real-rank-one. To drop this condition and prove an asymptotic formula for $N_H^{\phi}(\Gamma;x)$ of full generality, the relative trace formula of Jacquet ([5]) should be the most promising tool.

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