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l-MODULAR LOCAL THETA CORRESPONDENCE : DUAL PAIRS OF TYPE II

by

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Abstract. — Let F be a non-Archimedean locally compact field, of residual characteristic p, and $(\mathrm{G}, \mathrm{G}')$ a reductive dual pair over F of type II. In this article we show how the results of [**Mi1**], [**Mi2**], [**Mi3**] and [**MS**] imply that the local theta correspondence is bijective for l-modular representations if $l \neq p$ is a *banal* prime for G and G'. Moreover, we give some counterexamples which show that the local theta correspondence can be non-bijective for l-modular representations if l is not banal.

Introduction

Let F be a non-Archimedean locally compact field, of residual characteristic p, and fix $\psi : \mathrm{F} \to \mathbb{C}$ a non-trivial additive character of F. Let W be a finite-dimensional symplectic vector space over F and denote by $\widetilde{\mathrm{Sp}}(\mathrm{W})$ the *metaplectic group* [**MVW**]: it is a group which fits in the short exact sequence

$$0 \to \mathbb{C} \to \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{Sp}(\mathrm{W}) \to 0,$$

where $\mathrm{Sp}(\mathrm{W})$ is the symplectic group. It is equipped with a complex representation, canonically attached to ψ, the *Weil representation*, also called the mataplectic representation, which, in this introduction, will be denoted by σ.

Let G and G' be two reductive subgroups of $\mathrm{Sp}(\mathrm{W})$, each one the centralizer of the other in $\mathrm{Sp}(\mathrm{W})$ (we say that they form a *dual (reductive) pair*). Dual pairs $(\mathrm{G}, \mathrm{G}')$ come in two types:

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(I) G, G' are unitary groups defined over F (or one is symplectic and the other orthogonal);

(II) G, G' are general linear groups over a p-adic division algebra D.

Denote by $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}'}$ their pre-images in $\widetilde{\mathrm{Sp}}(\mathrm{W})$. We are interested in the restriction of the Weil representation to the product $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{G}'}$. Its irreducible quotients are of the form $\pi \otimes \pi'$ where π and π' are irreducible smooth complex representations of $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}'}$ respectively. Roughly speaking, the local theta correspondence says that π' is uniquely determined by π.

More precisely, let π be an irreducible smooth representation of $\widetilde{\mathrm{G}}$. Consider the biggest π-isotypic quotient of σ. One proves that, as a $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{G}'}$-module, it is of the form $\pi \otimes \Theta(\pi)$, where $\Theta(\pi)$ is a finite length smooth representation of $\widetilde{\mathrm{G}'}$.

Howe and Waldspurger [**MVW**], [**Wal**] proved that, if the dual pair is of type I, $p \neq 2$ and $\Theta(\pi) \neq 0$, then $\Theta(\pi)$ has a unique irreducible quotient, denoted by $\theta(\pi)$. The map $\pi \mapsto \theta(\pi)$ is called the *local theta correspondence* (or the *Howe* correspondence).

The proofs of Howe and Waldspurger are non-constructive: they give the existence of the theta correspondence without explicitly describing the bijection or when $\Theta(\pi) \neq 0$. In [**Mi1**] a new method was given for proving the theta correspondence in the case of dual pairs of type II. This proof is valid for F of any (residual) characteristic (in particular, it is permitted $p = 2$) and allows the correspondence to be made explicit in terms of the Langlands classification.

So far we have only been concerned with *complex* representations. Recently, however, the applications of the representation theory of p-adic reductive groups in number theory have required considering l-modular representations also: that is, representations over an arbitrary algebraically closed field R of characteristic l.

The study of these representations has been developed by Vignéras (see [**Vig**]), and their behaviour is very different depending on whether $l = p$ or $l \neq p$. We will only be interested in the latter case, where Vignéras introduced the notion of *banal* characteristic: for example, if $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ then l is banal if and only if it is coprime to $|\mathrm{GL}_n(k_{\mathrm{F}})| = \prod_{i=0}^{n-1}(q_{\mathrm{F}}^n - q_{\mathrm{F}}^i)$, where q_{F} is the cardinality of the residue field k_{F} of F. In general, l is banal for a p-adic reductive group G if the l-modular representations of any compact open subgroup of G are all semisimple.

In this article we would like to answer to the following question: is the local theta correspondence still bijective for l-modular representations? We will show that the proof given in [**Mi1**] is also valid for l-modular representations in the banal case and even for banal representations (see Section 5). The main theorem we prove is:

***Theorem 0.1* (see Theorems 6.1 and 7.1).** — *Let* R *be an algebraically closed field of characteristic l different from p. Let n, m be a pair of integers such that $n \leq m$ and denote by $\sigma_{n,m}$ the restriction of the metaplectic* R*-representation to the dual pair* $\mathrm{GL}_n(\mathrm{D}) \times \mathrm{GL}_m(\mathrm{D})$.

Let π be a m-banal irreducible R*-representation of* $\mathrm{GL}_n(\mathrm{D})$ *(see 5.9). There exists a unique* R*-representation π' of* $\mathrm{GL}_m(\mathrm{D})$ *such that*

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathrm{D})\times\mathrm{GL}_m(\mathrm{D})}\left(\sigma_{n,m}, \pi \otimes \pi'\right) \neq 0.$$

Moreover, we have $\dim\left(\mathrm{Hom}_{\mathrm{GL}_n(\mathrm{D})\times\mathrm{GL}_m(\mathrm{D})}\left(\sigma_{n,m}, \pi\otimes\pi'\right)\right) = 1$.

Write $\pi' = \theta_m(\pi)$. The mapping $\pi \mapsto \theta_m(\pi)$ is a bijection between the set of m-banal irreducible R*-representations π of* $\mathrm{GL}_n(\mathrm{D})$ *such that* $\mathrm{Hom}_{\mathrm{GL}_n(\mathrm{D})}\left(\sigma_{n,m}, \pi\right) \neq 0$ *and the set of banal irreducible* R*-representations π' of* $\mathrm{GL}_m(\mathrm{D})$ *such that* $\mathrm{Hom}_{\mathrm{GL}_m(\mathrm{D})}\left(\sigma_{n,m}, \pi'\right) \neq 0$.

We deduce a formula (see Theorem 7.1 for more details) giving the "Zelevinsky" parameters of $\theta_m(\pi)$ in terms of those of π.

Intriguingly, however, the theta correspondence can be *non*-bijective when l is not banal – that is, given an irreducible R-representation π_1 of $\mathrm{GL}_{n_1}(\mathrm{F})$, there may be several inequivalent R-representations π_2 of $\mathrm{GL}_{n_2}(\mathrm{F})$ such that $\pi_1 \otimes \pi_2$ occurs as a quotient of the Weil representation.

We give now a brief account about the contents, section by section. In the first section we introduce notation and the theory of R-representations. We recall the theory of l-modular zeta functions of [**Mi3**] in Section 2: this theory provides us with an intertwining operator between the metaplectic representation restricted to the pair $(\mathrm{GL}_m(\mathrm{D}), \mathrm{GL}_m(\mathrm{D}))$ and $\pi \otimes \widetilde{\pi}$ for each irreducible R-representation π of $\mathrm{GL}_m(\mathrm{D})$, where $\widetilde{\pi}$ denotes the contragredient representation of π. In Sections 3 and 4, we recall the computations of [**Mi1**] which will allow us, in Section 6, to prove that $\Theta(\pi)$ has a unique irreducible quotient.

In Section 5, we recall the classification of [**MS**], in terms of segments, of the set of *banal* representations. With this classification in hand, we make the correspondence explicit in Section 7. Finally, in the last section we give some examples of the failure of the theta correspondence in the non-banal case.

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1. Notation

1.1. Let F be a non-Archimedean locally compact field, of residual characteristic p. We denote by $\mathcal{O}_F$ its ring of integers, $\mathfrak{p}_F$ its maximal ideal and k_F its residue field. We denote by q_F the cardinal of k_F.

1.2. Let R be an algebraically closed field of characteristic l different from p (eventually l can be 0) and let G be the group of rational points of a reductive group defined over F. By a smooth R-representation we understand a pair (π, V) where V is a vector space over R and π is a group morphism from G into GL(V) such that the stabilizer of every vector in V is an open subset of G. In this text all representations are supposed to be smooth.

A R-character of G is a R-representation of dimension 1, that is, a morphism from G into $R^\times$ with open kernel.

We denote by $\mathrm{Irr}_R(G)$ the set of all classes of irreducible R-representations of G. Given $\pi \in \mathrm{Irr}_R(G)$ we will denote by $\widetilde{\pi}$ the contragredient representation of π.

1.3. We suppose in this paragraph that R is an algebraic closure of a local field. We denote by $\mathcal{O}$ the ring of integers of R and by k its residue field which is algebraically closed and supposed of characteristic different from p.

A R-representation π of G in a R-vector space V is *integral* if it is admissible and it possesses an *integral structure*, that is, a sub-$\mathcal{O}$-module stable by G and generated by a basis of V over R. A R-representation π of G is integral if, and only if, its cuspidal support is integral.

Let π be an integral irreducible R-representation of G. Then, for every integral structure Γ of π, the k-representation of G in the k-vector space $\Gamma \otimes_{\mathcal{O}} k$ is of finite length and its semi-simplification does not depend on the choice of Γ. We will call it the reduction of π and denote it by $r_R(\pi)$.

An integral irreducible R-representation is *k-irreducible* if its reduction is an irreducible k-representation.

1.4. Let π and π' be two R-representations of G. We denote by

$$\mathrm{Hom}_{\mathrm{G}}(\pi, \pi')$$

the space of intertwining operators from π into π'. We will omit the index G if there is no confusion.

1.5. Let D be a division algebra over F of finite dimension over F. For any integers $n, m \geq 1$, we denote by $\mathscr{M}_{n,m}(\mathrm{D})$ the F-algebra of $n \times m$ matrices with coefficients in D, by $\mathscr{M}_m(\mathrm{D})$ the F-algebra of $m \times m$ matrices with coefficients in D and by $\mathrm{G}_m = \mathrm{GL}_m(\mathrm{D})$ its multiplicative group. For convenience, we denote by G_0 the trivial group.

Let N_m (resp. tr_m) be the reduced norm (resp. reduced trace) of $\mathscr{M}_m(\mathrm{D})$ over F and let $|\ |_{\mathrm{F}}$ be the normalized absolute value of F. We see it as a R-character of $\mathrm{F}^\times$. The map $g \mapsto |\mathrm{N}_m(g)|_{\mathrm{F}}$ is a R-character of G_m, which we simply denote by ν. Its order is the order of q_{F} in $\mathrm{R}^\times$.

1.6. To every partition $\alpha = (m_1, \ldots, m_r)$ of the integer m, we denote M_α the subgroup of G_n of invertible matrices which are diagonal by blocs of size m_i and P_α (resp. $\overline{\mathrm{P}}_\alpha$) the subgroup of upper (resp. lower) triangular matrices by blocs of size m_i.

1.7. We denote by $\natural - r^{\mathrm{G}_m}_{m_1,\ldots,m_r}$ the non-normalized Jacquet functor associated to the standard parabolic P_α and by $\natural - \overline{r}^{\mathrm{G}_m}_{m_1,\ldots,m_r}$ the Jacquet functor associated to $\overline{\mathrm{P}}_\alpha$.

Fix $q_{\mathrm{F}}^{\frac{1}{2}}$ a square root of q_{F} in R. We set

$$r^{\mathrm{G}_m}_{m_1,\ldots,m_r} = \delta_{\mathrm{P}_\alpha}^{-1/2} \natural - r^{\mathrm{G}_m}_{m_1,\ldots,m_r},$$
$$(\text{resp. } \ \overline{r}^{\mathrm{G}_m}_{m_1,\ldots,m_r} = \delta_{\overline{\mathrm{P}}_\alpha}^{-1/2} \natural - \overline{r}^{\mathrm{G}_m}_{m_1,\ldots,m_r}\ ),$$

the normalized Jacquet functor.

Given a R-representation ρ_i of each G_{m_i}, we denote by

$$\natural - \mathrm{Ind}_{\mathrm{P}_\alpha}^{\mathrm{G}_m}(\rho_1 \otimes \cdots \otimes \rho_r),$$

the non-normalized parabolically induced R-representation.

We denote also by $\rho_1 \times \cdots \times \rho_r$ the R-representation

$$\mathrm{Ind}_{\mathrm{P}_\alpha}^{\mathrm{G}_m}(\rho_1 \otimes \cdots \otimes \rho_r) = \delta_{\mathrm{P}_\alpha}^{1/2} \natural - \mathrm{Ind}_{\mathrm{P}_\alpha}^{\mathrm{G}_m}(\rho_1 \otimes \cdots \otimes \rho_r),$$

that is, the normalized parabolically induced R-representation.

1.8. Let n and m be some positive integers. We denote by $S_{\mathrm{R}}\left(\mathscr{M}_{n,m}(\mathrm{D})\right)$ the R-vector space of locally constant, compactly supported functions Φ from $\mathscr{M}_{n,m}(\mathrm{D})$ to R.

Set $\sigma_{n,m}$ the natural R-representation of $\mathrm{G}_n \times \mathrm{G}_m$ on $S_{\mathrm{R}}(\mathscr{M}_{n,m}(\mathrm{D}))$ defined by

$$\sigma_{n,m}(g,g')\,\Phi(x) = \Phi\left(g^{-1}xg'\right),$$

for $g \in \mathrm{G}_n,\ g' \in \mathrm{G}_m,\ x \in \mathscr{M}_{n,m}(\mathrm{D})$ and $\Phi \in S_{\mathrm{R}}(\mathscr{M}_{n,m}(\mathrm{D}))$.

Up to a character, this R-representation is isomorphic to the metaplectic representation restricted to the dual pair $\mathrm{G}_n \times \mathrm{G}_m$ (*cf.* [**MVW**, 2.II.6]).

1.9. We have two linear groups acting by multiplication on the left and on the right in a space of matrices. From now on, to distinguish these two actions, we will denote by G′ and P′ the linear and parabolic groups acting on the right and G and P the same groups acting on the left. If there might be confusion we will also denote by ν' the R-character ν when it acts on G′. This notation is very useful, though it may seem artificial or weird.

2. l-modular zeta functions

In this section, following [**Mi3**] and generalizing the results of [**GJ**], we associate to each irreducible R-representation π of $\mathrm{GL}_m(\mathrm{D})$, two invariants $\mathrm{L}(T,\pi)$, $\varepsilon(T,\pi,\psi)$, where T is an indeterminate and ψ is a non-trivial R-character of F. It allows us to construct an explicit intertwining operator between $\sigma_{m,m}$ and $\pi\otimes\widetilde{\pi}$ for each irreducible R-representation π of $\mathrm{GL}_m(\mathrm{D})$.

2.1. We fix F a non-Archimedean locally compact field, of residual characteristic p and D a division algebra over F of dimension d^2 over F. We also fix a positive integer m and set $n = md$.

Let ψ be a non-trivial additive R-character of F, $d\mu(x)$ a Haar measure on $\mathscr{M}_m(\mathrm{D})$ with values in R and $d\mu^\times(x)$ a Haar measure on $\mathrm{GL}_m(\mathrm{D})$ with values in R (see [**Vig**, I.2.4]).

For every function $\Phi \in S_{\mathrm{R}}\left(\mathscr{M}_m(\mathrm{D})\right)$, we denote by

$$\widehat{\Phi}(x) = \int_{\mathscr{M}_m(\mathrm{D})} \Phi(y)\,\psi\left(\mathrm{tr}_m(xy)\right)d\mu(y)$$

its Fourier transform. As usual, we suppose the Haar measure to be autodual.

Let π be an irreducible R-representation of G_m and f a coefficient of π. We denote by $\check{f}$ the coefficient of $\widetilde{\pi}$ defined by $\check{f}(g) = f(g^{-1})$. Let $\Phi \in S_{\mathrm{R}}\left(\mathscr{M}_m(\mathrm{D})\right)$ and $N \in \mathbb{Z}$. Then the integral

$$\int_{\mathrm{G}_m, \nu(x)=q_{\mathrm{F}}^{-N}} \Phi(x) f(x) \, d\mu^{\times}(x)$$

is well defined, as $\{x \in \mathrm{G}_m : \nu(x) = q_{\mathrm{F}}^{-N}\} \cap \operatorname{supp}(\Phi)$ is a compact subset of G_m and Φ and f are locally constant on it.

We can now define the formal sum (*the zeta function*):

$$\mathrm{Z}(\Phi, T, f) = \sum_{N \in \mathbf{Z}} \left(\int_{\mathrm{G}_m, \nu(x)=q_{\mathrm{F}}^{-N}} \Phi(x) f(x) \, d\mu^{\times}(x) \right) T^N.$$

As Φ is compactly supported, for N small enough, we have :

$$\int_{\mathrm{G}_m, \nu(x)=q_{\mathrm{F}}^{-N}} \Phi(x) f(x) \, d\mu^{\times}(x) = 0.$$

Hence, $\mathrm{Z}(\Phi, T, f) \in \mathrm{R}((T))$.

2.2. In [**Mi3**] it is proved the following theorem:

Theorem 2.1. — *Let π be an irreducible* R*-representation of* G_m*. Then :*

(1) *There exists $P_0(\pi, T) \in \mathrm{R}[T]$ such that, for every coefficient f of π and every $\Phi \in S_{\mathrm{R}}(\mathscr{M}_m(\mathrm{D}))$, we have*

$$\mathrm{Z}(\Phi, T, f) \, P_0(\pi, T) \in \mathrm{R}\left[T, T^{-1}\right].$$

(2) *There exists a gamma factor $\gamma(T, \pi, \psi) \in \mathrm{R}(T)$ such that, for every coefficient f of π and every $\Phi \in S_{\mathrm{R}}(\mathscr{M}_m(\mathrm{D}))$, we have*

$$\mathrm{Z}\left(\widehat{\Phi}, q^{-\frac{1}{2}(n+1)} T^{-1}, \check{f}\right) = \gamma(T, \pi, \psi) \, \mathrm{Z}\left(\Phi, q^{-\frac{1}{2}(n-1)} T, f\right). \tag{2.1}$$

(3) *Set $\mathscr{Z}(\pi)$ the sub-*R*-vector space of* $\mathrm{R}(T)$ *generated by the functions $\mathrm{Z}\left(\Phi, Tq^{\frac{1-n}{2}}, f\right)$ with f coefficient of π and $\Phi \in S_{\mathrm{R}}(\mathscr{M}_m(\mathrm{D}))$. Then $\mathscr{Z}(\pi)$ is a fractional ideal* $\mathrm{R}[T, T^{-1}]$ *containing the constants. It admits a generator of the form*

$$\mathrm{L}(T, \pi) = \frac{1}{P_0(\pi, T)}$$

with $P_0(\pi, T) \in \mathrm{R}[T]$ and $P_0(\pi, 0) = 1$.

Set

$$\gamma(T,\pi,\psi)=\varepsilon(T,\pi,\psi)\frac{\mathrm{L}(q^{-1}T^{-1},\widetilde{\pi})}{\mathrm{L}(T,\pi)}.$$

Then the functional equation (2.1) reads:

$$\frac{\mathrm{Z}\left(\widehat{\Phi},T^{-1}q^{\frac{-1-n}{2}},\check{f}\right)}{\mathrm{L}(q^{-1}T^{-1},\widetilde{\pi})}=\varepsilon(T,\pi,\psi)\frac{\mathrm{Z}\left(\Phi,Tq^{\frac{1-n}{2}},f\right)}{\mathrm{L}(T,\pi)}.$$

2.3. The zeta functions allow us to construct a non-trivial intertwining operator between $\sigma_{m,m}$ and $\pi\otimes\widetilde{\pi}$, for each irreducible R-representation π of G_m. It is defined by:

$$\begin{aligned} Z_\pi &: S_R(\mathscr{M}_m(\mathrm{D}))\to V\otimes\widetilde{V}\\ Z_\pi(\Phi)(f) &= \lim_{T\to 1}\frac{Z(\Phi,T,f)}{L(Tq^{-(n-1)/2},\pi)}, \end{aligned}$$

for every $\Phi\in S_R(\mathscr{M}_m(\mathrm{D}))$, $f\in V\otimes\widetilde{V}$ coefficient of π and where $\lim_{T\to 1}\frac{Z(\Phi,T,\pi)}{L\left(Tq^{-(n-1)/2},\pi\right)}$ is the evaluation of the polynomial $\frac{Z(\Phi,T,f)}{L\left(Tq^{-(n-1)/2},\pi\right)}$ at $T=1$.

A classical argument (*cf.* [**MVW**, 3.III.5] which is also valid for R-representations, see [**MiThe**, 5.7.3], for more details), shows now that, for all $m\geq n$, there exists an irreducible composition factor π' of the induced R-representation $\natural\!-\!\mathrm{Ind}_{\mathrm{P}'_{m-n,n}}^{\mathrm{G}'_m}(1_{m-n}\otimes\widetilde{\pi})$ such that

$$\mathrm{Hom}_{\mathrm{G}_n\times\mathrm{G}'_m}(\sigma_{n,m},\pi\otimes\pi')\neq 0. \tag{2.2}$$

Remark 2.2. — Hence, for any algebraically closed field R of characteristic $l\neq p$, $n\leq m$ and π irreducible R-representation of G_n there exists at least one R-representation π' of G'_m such that (2.2) is satisfied.

The problem is now to prove that, under some other assumptions, this irreducible R-representation is unique.

3. The boundary of the metaplectic representation

3.1. Let

$$0=S_{t+1}\subset S_t\subset\cdots\subset S_1\subset S_0=S_{\mathrm{R}}(\mathscr{M}_{n,m}),$$

be the filtration of $\sigma_{n,m}$ by support (*cf.* [**Mi1**, §2]), and set

$$\sigma_k=S_k/S_{k+1}\simeq\natural\!-\!\mathrm{Ind}_{\mathrm{P}_{n-k,k}\mathrm{P}'_{m-k,k}}^{\mathrm{G}_n\mathrm{G}'_m}(\mu_k),$$

where μ_k is the R-representation of $\overline{\mathrm{P}}_{n-k,k}\mathrm{P}'_{m-k,k}$ on $S_{\mathrm{R}}(\mathrm{G}_k)$ defined by:

$$\mu_k(p,p')\Phi(h) = \Phi\left(p_4^{-1}hp'_4\right) = \rho_k(p_4,p'_4)\Phi(h),$$

for all $\Phi \in S_{\mathrm{R}}(\mathrm{G}_k)$, $h \in \mathrm{G}_k$, $p = \begin{pmatrix} p_1 & 0 \\ p_3 & p_4 \end{pmatrix}$, $p' = \begin{pmatrix} p'_1 & p'_2 \\ 0 & p'_4 \end{pmatrix}$ and ρ_k the natural R-representation of $\mathrm{G}_k \times \mathrm{G}'_k$ on $S_{\mathrm{R}}(\mathrm{G}_k)$ defined by

$$\rho_k(p_4,p'_4)\Phi(h) = \Phi\left(p_4^{-1}hp'_4\right). \tag{3.1}$$

Definition 3.1. — We say that an irreducible R-representation $\pi \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}_n)$ occurs on the boundary of $\sigma_{n,m}$ if there exists $k < n$ such that $\mathrm{Hom}_{\mathrm{G}_n}(\sigma_k,\pi) \neq 0$.

3.2. In [**Mi1**, Corollaire 2.3] we prove the following lemma, which is valid for any R:

Lemma 3.2. — *Let $\pi \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}_n)$. The following conditions are equivalent:*

(1) *The* R*-representation π does not occur on the boundary of $\sigma_{n,m}$.*

(2) *For every integer $k < n$, there doesn't exist a* R*-representation $\tau \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}_k)$, such that*

$$\mathrm{Hom}_{\mathrm{G}_n}\left(\natural\text{-}\mathrm{Ind}_{\overline{\mathrm{P}}_{n-k,k}}^{\mathrm{G}_n}(1_{n-k}\otimes\tau),\pi\right) \neq 0.$$

Remark 3.3. — One can prove that, for banal R-representations (see Section 5), these conditions are equivalent to the following:

(2') The L-function $\mathrm{L}(\pi,T)$ does not have a pole at $T = q^{-\frac{n-1}{2}}$.

3.3. We deduce as in [**Mi1**, 2.4]

Theorem 3.4. — *Let n,m be some positive integers $n \leq m$. Let $\pi \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}_n)$ and $\pi' \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}'_m)$ such that*

$$\mathrm{Hom}_{\mathrm{G}_n\times\mathrm{G}'_m}(\sigma_{n,m},\pi\otimes\pi') \neq 0.$$

Suppose that π does not occur on the boundary of $\sigma_{n,m}$. Then π' is a quotient of the induced R*-representation $\natural\text{-}\mathrm{Ind}_{\mathrm{P}'_{m-n,n}}^{\mathrm{G}'_m}(1_{m-n}\otimes\widetilde{\pi})$. Moreover,*

$$\dim\left(\mathrm{Hom}_{\mathrm{G}_n\times\mathrm{G}'_m}(\sigma_{n,m},\pi\otimes\pi')\right) = 1.$$

Remark 3.5. — In particular, if the representation $\natural\text{-}\mathrm{Ind}_{\mathrm{P}'_{m-n,n}}^{\mathrm{G}'_m}(1_{m-n}\otimes\widetilde{\pi})$ has a unique irreducible quotient (for example if π is a cuspidal R-representation or, more generally see Section 5), then there exists a unique π' such that

$$\mathrm{Hom}_{\mathrm{G}_n\times\mathrm{G}'_m}(\sigma_{n,m},\pi\otimes\pi') \neq 0.$$

4. Kudla's filtration

4.1. The computations of [**Mi1**, §3] are valid for any algebraically closed field R of characteristic $l \neq p$. We have then:

Proposition 4.1. — *Let t be an integer $0 \leq t \leq n$. The Jacquet module $r^{\mathrm{G}_n}_{t,n-t}(\sigma_{n,m})$ has composition factors τ_i for $i = 0, \ldots, \min\{t, m\}$, where*

$$\tau_i \simeq \mathrm{Ind}^{\mathrm{M}_{(t,n-t)}\times \mathrm{G}'_m}_{\mathrm{P}_{t-i,i}\times \mathrm{G}_{n-t}\times \mathrm{P}'_{i,m-i}} \left(\xi_{t,i} \otimes \rho_i \otimes \sigma_{n-t,m-i}\right),$$

ρ_i is defined by (3.1) and $\xi_{t,i}$ is the R-character

$$\xi_{t,i} = \begin{cases} \nu^{\frac{2m-n+t-i}{2}} & \text{on } \mathrm{G}_{t-i} \\ \nu^{\frac{2m-n+2t-i}{2}} & \text{on } \mathrm{G}_i \\ \nu^{\frac{t}{2}} & \text{on } \mathrm{G}_{n-t} \\ \nu^{\frac{-m-2t+i}{2}} & \text{on } \mathrm{G}'_i \\ \nu^{\frac{-2t+i}{2}} & \text{on } \mathrm{G}'_{m-i}. \end{cases}$$

We have a similar proposition (see [**Mi1**, 3.3]) for the Jacquet functor acting on G'_m.

4.2. This computation is used to prove the following proposition:

Proposition 4.2. — *Let n, m, r be some positive integers and $\pi \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}_n)$, $\pi' \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}'_m)$ such that $\pi \otimes \pi'$ is a quotient of $\sigma_{n,m}$. Let χ be an irreducible cuspidal R-representation of G_r non isomorphic to the R-characters of $\mathrm{D}^\times$, $\nu^{\frac{n+1}{2}}$ and $\nu^{\frac{2m-n+1}{2}}$. Then $a = b$ where a and b are defined by the following conditions:*

(1) *There exists $\rho \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}_{n-ra})$ such that π is a subrepresentation of*

$$\underbrace{\chi \times \chi \times \cdots \times \chi}_{a \text{ times}} \times \rho,$$

where a is maximal.

(2) *There exists $\rho' \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}'_{m-rb})$ such that π' is a subrepresentation of*

$$\rho' \times \underbrace{\nu^{\frac{m-n}{2}}\widetilde{\chi} \times \cdots \times \nu^{\frac{m-n}{2}}\widetilde{\chi}}_{b \text{ times}},$$

where b is maximal.

Moreover we have

$$\mathrm{Hom}\left(\sigma_{n-ra;m-ra}, \nu^{\frac{-ra}{2}}\rho \otimes \nu^{\frac{ra}{2}}\rho'\right) \neq 0.$$

Proof. — The proof of Proposition 4.4 in **[Mi1]** is valid in this setting, we will give an idea of how we use Proposition 4.1 to prove it.

Let $\pi \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}_n)$, $\pi' \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}'_m)$ and χ a cuspidal R-representation of G_r as in the proposition and let a be a positive integer such that there exists $\rho \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}_{n-ra})$ with π a subrepresentation of

$$\underbrace{\chi \times \chi \times \cdots \times \chi}_{a \text{ times}} \times \rho.$$

We suppose a to be maximal satisfying to these conditions.

As the Jacquet functor is exact, we get a surjectif morphism from $r^{\mathrm{G}_n}_{ra,n-ra}(\sigma_{n,m})$ onto $r^{\mathrm{G}_n}_{ra,n-ra}(\pi) \otimes \pi'$ and hence by Frobenius reciprocity we get a non-trivial morphism from $r^{\mathrm{G}_n}_{ra,n-ra}(\sigma_{n,m})$ onto $\chi \times \chi \times \cdots \times \chi \otimes \rho \otimes \pi'$.

By Proposition 4.1, there exists $i \in \{0, \ldots, ra\}$ such that

$$\mathrm{Hom}\left(\tau_i, \chi \times \chi \times \cdots \times \chi \otimes \rho \otimes \pi'\right) \neq 0.$$

As we have supposed that $\chi \not\simeq \nu^{\frac{2m-n+1}{2}}$ it is easy to check that only τ_{ra} can have such a quotient so we get:

$$\mathrm{Hom}\left(\tau_{ra}, \chi \times \chi \times \cdots \times \chi \otimes \rho \otimes \pi'\right) \neq 0.$$

Then, by Proposition 4.1

$$\mathrm{Hom}\left(\mathrm{Ind}^{\mathrm{M}_{(ra,n-ra)} \times \mathrm{G}'_m}_{\mathrm{M}_{(ra,n-ra)} \times \mathrm{P}'_{ra,m-ra}}\left(\xi_{ra,ra} \otimes \rho_{ra} \otimes \sigma_{n-ra,m-ra}\right), \chi \times \cdots \times \chi \otimes \rho \otimes \pi'\right) \neq 0.$$

Using again Frobenius reciprocity, after some simplifications, we get

$$\mathrm{Hom}\left(\mathrm{Ind}^{\mathrm{G}'_m}_{\mathrm{P}'_{ra,m-ra}}\left(\nu^{\frac{m-n}{2}}\widetilde{\chi} \times \cdots \times \nu^{\frac{m-n}{2}}\widetilde{\chi} \otimes \nu^{\frac{ra}{2}}\sigma_{n-ra,m-ra}\nu'^{\frac{-ra}{2}}\right), \rho \otimes \pi'\right) \neq 0.$$

Let $b \geq 0$ be now a maximal integer such that there exists $\rho' \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}'_{m-rb})$ with π' a subrepresentation of

$$\rho' \times \underbrace{\nu^{\frac{m-n}{2}}\widetilde{\chi} \times \cdots \times \nu^{\frac{m-n}{2}}\widetilde{\chi}}_{b \text{ times}}.$$

By Frobenius reciprocity, after conjugation, we get a non-trivial morphism from $\overline{r}^{\mathrm{G}'_m}_{rb,m-rb}(\pi')$ onto $\nu^{\frac{m-n}{2}}\widetilde{\chi} \times \cdots \times \nu^{\frac{m-n}{2}}\widetilde{\chi} \otimes \rho'$.

Hence, as before, we get:

$$\mathrm{Hom}\Big(\overline{r}^{\mathrm{G}'_m}_{rb,m-rb} \circ \mathrm{Ind}^{\mathrm{G}'_m}_{\mathrm{P}'_{ra,m-ra}} \Big(\nu^{\frac{m-n}{2}}\widetilde{\chi}\times\ldots\times\nu^{\frac{m-n}{2}}\widetilde{\chi}\otimes\nu^{\frac{ra}{2}}\sigma_{n-ra,m-ra}\nu'^{\frac{-ra}{2}}\Big),$$
$$\rho\otimes\nu^{\frac{m-n}{2}}\widetilde{\chi}\times\cdots\times\nu^{\frac{m-n}{2}}\widetilde{\chi}\otimes\rho'\Big)\neq 0.$$

Now we use the maximality of b, the fact that χ is not isomorphic to the R-character $\nu^{\frac{n+1}{2}}$ and Proposition 3.3 of [**Mi1**] to see that $b = a$ and finish the proof. For all details see [**Mi1**, Proposition 4.4]. □

5. Banal representations: Zelevinsky parameters

In this section we will make a brief account of the results in [**MS**] and [**Mi2**]. We define the set of *banal* representations and then we classify it in terms of segments.

5.1. Let fix R an algebraically closed field of characteristic $l \neq p$. Let C be a field of characteristic 0 such that it is an algebraic closure of a local field and its residue field is isomorphic to R. For example, if R is of characteristic 0 we can choose C to be an algebraic closure of the field $\mathrm{R}((T))$ of formal series with coefficients in R; if the characteristic of R is positive, we can choose C to be an algebraic closure of the fraction field of the ring of Witt vectors of R. If l is a prime number different from p and if R is an algebraic closure $\overline{\mathbb{F}}_l$ of $\mathbb{F}_l$, it is enough to take C as the algebraic closure $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$.

5.2. Let r be a positive integer and ρ a cuspidal R-representation of G_r. In [**MS**] we prove that there exists a R-character ν_ρ of the form ν^{b_ρ}, where b_ρ is an integer, such that if r' is a positive integer and ρ' is a cuspidal R-representation of $\mathrm{G}_{r'}$, the parabolically induced R-representation

$$\rho\times\rho'$$

is irreducible if, and only if, ρ' is not isomorphic to $\rho\nu_\rho$ or $\rho\nu_\rho^{-1}$. For example, if $\mathrm{R} = \mathbb{C}$ and $\mathrm{D} = \mathrm{F}$, then, for any cuspidal R-representation ρ, we can take $b_\rho = 1$.

We denote by $\rho\mathbb{Z}$ the set of classes of cuspidal R-representations of the form $\rho\nu_\rho^k$ where k is an integer. We remark that if $l > 0$, $\rho\mathbb{Z}$ is a finite set.

5.3. We say that the group G_m is *banal* if l does not divide the cardinal of the finite group $\mathrm{GL}_m(k_{\mathrm{D}})$.

Let π be an irreducible R-representation. We denote by $\mathrm{supp}(\pi)$ its cuspidal support. We will see it as a set (with multiplicities) of cuspidal R-representations

$$\mathrm{supp}(\pi) = \{\rho_1, \rho_2, \ldots, \rho_k\}.$$

We say that π is a *banal R-representation* if

(1) For all $\rho \in \mathrm{supp}(\pi)$, ρ is a R-representation of a banal group.
(2) for all $1 \leq i \leq k$, $\rho_i\mathbb{Z} \not\subseteq \mathrm{supp}(\pi)$.

5.4. Let r be a positive integer and ρ a cuspidal R-representation of G_n. Suppose G_n is a banal group. We need to fix a choice of b_ρ. As G_n is a banal group, there exists an integral cuspidal C-representation $\rho^\dagger$ such that $\rho \simeq \boldsymbol{r}_{\mathrm{C}}(\rho^\dagger)$ (see 1.3). To fix b_ρ we choose $b_{\rho^\dagger} > 0$ and suppose that $b_{\rho^\dagger} = b_\rho$ in R.

5.5. Let ρ be a cuspidal R-representation of G_n, $a, b \in \mathbb{Z}$, $a \leq b$. We set

$$\Delta = \left\{\nu_\rho^a\rho, \nu_\rho^{a+1}\rho, \ldots, \nu_\rho^b\rho\right\}.$$

We say that Δ is a segment and we will denote it often by $\Delta = \{a, b\}_\rho$.

A segment $\Delta = \{a, b\}_\rho$ is said to be *banal* if ρ is a R-representation of a banal group and $\rho\mathbb{Z} \not\subseteq \Delta$.

We say that $\{a, b\}_\rho$, $\{a', b'\}_{\rho'}$ are linked if $\{a, b\}_\rho \cup \{a', b'\}_{\rho'}$ is still a segment and $\{a, b\}_\rho \not\subseteq \{a', b'\}_{\rho'}$ and $\{a', b'\}_{\rho'} \not\subseteq \{a, b\}_\rho$. We say that $\{a, b\}_\rho$ precedes $\{a', b'\}_{\rho'}$ if they are linked and there exists $\tau \in \{a, b\}_\rho$ such that $\rho'\nu_{\rho'}^{a-1} \simeq \tau$.

To each banal segment $\Delta = \{a, b\}_\rho$ it corresponds an irreducible R-representation, denoted by $\langle\Delta\rangle$, defined as the unique quotient of the R-representation

$$\nu_\rho^a\rho \times \nu_\rho^{a+1}\rho \times \cdots \times \nu_\rho^b\rho.$$

For example, if $\mathrm{R} = \mathbb{C}$, then, for every segment Δ, the R-representation $\langle\Delta\rangle$ is essentially square integrable and, in fact, all essentially square integrable representations are of this form.

If $\Delta = \{a, b\}_\rho$ is a banal segment we denote by $\widetilde{\Delta} = \widetilde{\{a, b\}_\rho}$ the segment $\{-b, -a\}_{\tilde{\rho}}$, so that we have

$$\left\langle\widetilde{\Delta}\right\rangle = \widetilde{\langle\Delta\rangle}.$$

5.6. A *multisegment* is a multi-set of segments as above. We will usually see a multisegment $\mathbf{m}$ as an indexed set (with multiplicities) $(\Delta_1, \dots, \Delta_N)$, where N is a positive integer.

We denote by $\mathrm{supp}(\mathbf{m})$ the support of the multisegment $\mathbf{m} = (\Delta_1, \dots, \Delta_N)$, that is, the multiset of cuspidal R-representations defined by:

$$\mathrm{supp}(\mathbf{m})\,(\rho) = \sum_{\rho \in \Delta} \mathbf{m}\,(\Delta)\,,$$

for all cuspidal R-representations ρ. We will usually see it as an indexed set (with multiplicities) $\{\rho_1, \dots, \rho_t\}$.

A multisegment $\mathbf{m}$ is *banal* if for all $\rho \in \mathrm{supp}(\mathbf{m})$, ρ is a cuspidal R-representation of a banal group and for each cuspidal R-representation ρ, we have $\rho\mathbb{Z} \not\subseteq \mathrm{supp}(\mathbf{m})$.

5.7. In **[MS]** it is proved the following theorem:

Theorem 5.1. — (1) *Let $(\Delta_1, \dots, \Delta_N)$ be a banal multisegment. Suppose that for each pair of indices i, j such that $i < j$, Δ_i does not precede Δ_j. Then the* R*-representation $\langle\Delta_1\rangle \times \dots \times \langle\Delta_N\rangle$ has a unique irreducible quotient. We denote it by $\langle\Delta_1, \dots, \Delta_N\rangle$. It is a banal* R*-representation.*

(2) *The* R*-representations $\langle\Delta_1, \dots, \Delta_N\rangle$ and $\langle\Delta'_1, \dots, \Delta'_{N'}\rangle$ are isomorphic if, and only if, $(\Delta_1, \dots, \Delta_N)$ and $(\Delta'_1, \dots, \Delta'_{N'})$ are equal up to a rearrangement.*

(3) *Any banal* R*-representation of* G_m *is isomorphic to some representation of the form $\langle\Delta_1, \dots, \Delta_N\rangle$.*

5.8. To prove the local theta correspondence for R-representations we will need some results of **[Mi2]** which are valid in this setting.

Theorem 5.2. — *Let $\pi = \langle\Delta_1, \dots, \Delta_N\rangle$ be a banal* R*-representation. Let $\rho = \langle\Delta_1, \dots, \Delta_r\rangle$ be a banal* R*-representation such that one the following properties is satisfied:*

(1) *The* R*-representation ρ is a* R*-character of a banal group, or*

(2) *the multisegment $(\Delta'_1, \dots, \Delta'_r)$ is banal and for each pair of indices i, j such that $i \neq j$, $\Delta'_i = \Delta'_j$ or $\Delta'_i \cap \Delta'_j = \emptyset$.*

Suppose moreover that the multisegment $(\Delta_1, \dots, \Delta_N, \Delta'_1, \dots, \Delta'_r)$ is banal.

Then the R*-representation* $\pi \times \rho$ *(resp.* $\rho \times \pi$*) has a unique irreducible quotient and a unique irreducible subrepresentation and they appear with multiplicity 1 in the parabolically induced* R*-representation* $\pi \times \rho$ *(resp.* $\rho \times \pi$*).*

5.9. We need a last lemma which is proved in [**Mi1**] using the results in the appendix of [**Mi2**] and it is valid for R-representations with some modifications. Let n, m be a pair of positive integers such that $m \geq n$ and let $\pi = \langle \Delta_1, \ldots, \Delta_N \rangle$ be a banal R-representation of G_n. We say that π is *m-banal* if the multisegment $\left(\left\{\nu^{-\frac{m-n-1}{2}}\right\}, \ldots, \left\{\nu^{\frac{m-n-1}{2}}\right\}, \widetilde{\Delta_1}, \ldots, \widetilde{\Delta_N}\right)$ is still banal. In this case we denote by $\theta_m^*(\pi)$ the banal R-representation of G_m:

$$\theta_m^*(\pi) = \left\langle \left\{\nu^{-\frac{m-n-1}{2}}\right\}, \ldots, \left\{\nu^{\frac{m-n-1}{2}}\right\}, \widetilde{\Delta_1}, \ldots, \widetilde{\Delta_N} \right\rangle.$$

In particular, if $m = n$ then $\theta_m^*(\pi) \simeq \widetilde{\pi}$.

The following result is proved in [**Mi1**, Corollaire 6.5]:

Lemma 5.3. — *Let* χ *be a banal cuspidal* R*-representation of* G_r*, non isomorphic to the* R*-characters of* $\mathrm{D}^\times$ $\nu^{\frac{n+1}{2}}$ *and* $\nu^{\frac{2m-n+1}{2}}$*. Let* a *be a positive integer and* $\rho = \langle \mathbf{m} \rangle$ *a banal* R*-representation of* $\mathrm{Irr}_{\mathrm{R}}(G_{n-ra})$ *such that* $\mathbf{m} + \{\chi\}$ *is still a banal multisegment. Denote by* π *the unique irreducible subrepresentation of*

$$\underbrace{\chi \times \cdots \times \chi}_{a \text{ times}} \times \rho.$$

Suppose π *is* m*-banal and let* π' *be the unique subrepresentation of*

$$\nu^{\frac{-ra}{2}} \theta_{m-ra}^*(\nu^{\frac{-ra}{2}} \rho) \times \underbrace{\nu^{\frac{m-n}{2}} \widetilde{\chi} \times \cdots \times \nu^{\frac{m-n}{2}} \widetilde{\chi}}_{a \text{ times}}.$$

Then

$$\pi' \simeq \theta_m^*(\pi).$$

6. The proof, part I: uniqueness of the quotient

We are now ready to prove the bijectivity of the local theta correspondence for l-modular representations. The goal of this section is to prove the following theorem:

Theorem 6.1. — *Let n, m be a pair of integers such that $n \leq m$. Let π be a m-banal irreducible* R*-representation of* G_n*. There exists a unique* R*-representation π' of* G'_m *such that*

$$\mathrm{Hom}_{\mathrm{G}_n \times \mathrm{G}'_m}(\sigma_{n,m}, \pi \otimes \pi') \neq 0.$$

Moreover, we have $\dim\left(\mathrm{Hom}_{\mathrm{G}_n \times \mathrm{G}'_m}(\sigma_{n,m}, \pi \otimes \pi')\right) = 1$

Proof. — The proof is the same as for Theorem 5.1 of [**Mi1**]. Let us sketch it.

By induction hypothesis we can suppose that the theorem is true for all dual pair $(\mathrm{G}_i, \mathrm{G}'_j)$, such that $ij < nm$. We prove it for the pair $(\mathrm{G}_n, \mathrm{G}'_m)$.

Let $\pi' \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}'_m)$ such that $\pi \otimes \pi'$ is a quotient of $\sigma_{n,m}$ (we know that there exists such a quotient by Remark 2.2). We will prove that π' is uniquely determined by π.

Case 1. Suppose that there exists a triple (a, χ, ρ) where $a > 0$ is an integer, χ is a cuspidal R-representation of G_r (r being a positive integer) non isomorphic to the R-characters of $\mathrm{D}^\times$ $\nu^{\frac{n+1}{2}}$ and $\nu^{\frac{2m-n+1}{2}}$ and $\rho \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}_{n-ra})$ such that π is a subrepresentation of

$$\underbrace{\chi \times \chi \times \cdots \times \chi}_{a \text{ times}} \times \rho.$$

We suppose a to be maximal satisfying to these conditions.

Then, by Proposition 4.2, there exists $\rho' \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}'_{m-ra})$ such that π' is a subrepresentation of

$$\rho' \times \underbrace{\nu^{\frac{m-n}{2}} \widetilde{\chi} \times \cdots \times \nu^{\frac{m-n}{2}} \widetilde{\chi}}_{a \text{ times}}.$$

Moreover, we have:

$$\mathrm{Hom}\left(\sigma_{n-ra,m-ra}, \nu^{\frac{-ra}{2}} \rho \otimes \nu^{\frac{ra}{2}} \rho'\right) \neq 0.$$

By induction hypothesis, ρ' is uniquely determined by ρ and, by Theorem 5.2, π' is then the unique irreducible subrepresentation of

$$\rho' \times \underbrace{\nu^{\frac{m-n}{2}} \widetilde{\chi} \times \cdots \times \nu^{\frac{m-n}{2}} \widetilde{\chi}}_{a \text{ times}}.$$

Case 2. If there doesn't exist such a triple, it is very easy to see, using Lemma 3.2 that π does not occur on the boundary of $\sigma_{n,m}$. Then by Theorem 3.4, π' is an irreducible quotient of $\sharp\text{--}\mathrm{Ind}_{\mathrm{P}'_{m-n,n}}^{\mathrm{G}'_m}(1_{m-n} \otimes \widetilde{\pi})$. But, by Theorem 5.2, such a representation have just one irreducible quotient. □

7. The proof, part II: explicit correspondence

Theorem 7.1. — (1) *Let n, m be a pair of integers such that $n \leq m$. Let π be a m-banal irreducible* R*-representation of* G_n. *Denote by $\theta_m(\pi)$ the unique* R*-representation of* G'_m *given by Theorem 6.1. Then $\theta_m(\pi) = \theta^*_m(\pi)$ (see 5.9).*

(2) *The mapping $\pi \mapsto \theta_m(\pi)$ is a bijection between the set of m-banal irreducible* R*-representations π of* G_n *such that* $\mathrm{Hom}_{\mathrm{G}_n}(\sigma_{n,m}, \pi) \neq 0$ *and the set of banal irreducible* R*-representations π' of* G'_m *such that* $\mathrm{Hom}_{\mathrm{G}'_m}(\sigma_{n,m}, \pi') \neq 0$.

Proof. — The second par of the theorem is a consequence of the first one and Theorem 5.1. The idea of the proof of the first part is the same as for Theorem 6.1 of [**Mi1**]. As in the previous theorem, by induction hypothesis, we can suppose that the theorem is true for all dual pairs $(\mathrm{G}_i, \mathrm{G}'_j)$, such that $ij < nm$. Let us prove it for the pair $(\mathrm{G}_n, \mathrm{G}'_m)$.

Let π be a m-banal irreducible R-representation of G_n. Let us see that $\theta_m(\pi) \simeq \theta^*_m(\pi)$. We have again two cases.

Case 1. Suppose that there exists a triple (a, χ, ρ) where $a > 0$ is an integer, χ is a cuspidal R-representation of G_r (r being a positive integer) non isomorphic to the R-characters of $\mathrm{D}^\times$ $\nu^{\frac{n+1}{2}}$ and $\nu^{\frac{2m-n+1}{2}}$ and $\rho \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}_{n-ra})$ such that π is a subrepresentation of

$$\underbrace{\chi \times \chi \times \cdots \times \chi}_{a \text{ times}} \times \rho.$$

We suppose a to be maximal satisfying to these conditions.

Then, by Proposition 4.2, there exists $\rho' \in \mathrm{Irr}_{\mathrm{R}}(\mathrm{G}'_{m-ra})$ such that

$$\pi' \hookrightarrow \rho' \times \underbrace{\nu^{\frac{m-n}{2}}\widetilde{\chi} \times \cdots \times \nu^{\frac{m-n}{2}}\widetilde{\chi}}_{a \text{ times}}. \tag{7.1}$$

Moreover, we have:

$$\mathrm{Hom}\left(\sigma_{n-ra,m-ra}, \nu^{\frac{-ra}{2}}\rho \otimes \nu^{\frac{ra}{2}}\rho'\right) \neq 0.$$

By induction hypothesis, we get

$$\rho' \simeq \nu^{\frac{-ra}{2}}\theta^*_{m-ra}\left(\nu^{\frac{-ra}{2}}\rho\right). \tag{7.2}$$

In this case, the theorem is now a consequence of equations (7.1), (7.2) and Lemma 5.3.

Case 2. If there doesn't exist such a triple, the proof is the same as [**Mi1**, §9]: such representations have very particular Jacquet modules; using carefully the properties of the classification, in terms of segments, of banal R-representations, we get the remaining part of the theorem. We omit the details. □

8. Some examples in the non-banal case

In this last section we study the local theta correspondence in the non-banal case and its behavior by reduction modulo l.

8.1. We use the notations of paragraph 5.1. Let fix R an algebraically closed field of characteristic $l \neq p$. Let C be a field of characteristic 0 such that it is an algebraic closure of a local field and its residue field is isomorphic to R. Denote by $\mathcal{O}_{\mathrm{C}}$ its ring of integers.

8.2. First, let us give some counterexamples to the bijectivity of the local theta correspondence in the non-banal case. The theta correspondence may fail in two ways:

(1) For π an irreducible R-representation of G_n, there might exist π' an irreducible R-representation of G_m such that

$$\dim\left(\mathrm{Hom}_{\mathrm{G}_n \times \mathrm{G}_m}(\sigma_{n,m}, \pi \otimes \pi')\right) > 1.$$

The easiest example appears already when $n = m = 1$, π and π' are the trivial R-characters of $\mathrm{F}^\times$ and $q_{\mathrm{F}} \equiv 1 \mod l$. In this case we find two non-proportional intertwining operators between $\sigma_{1,1}$ and $\pi \otimes \pi'$ defined by:

$$\begin{aligned} \Phi &\mapsto \Phi(0), \\ \Phi &\mapsto Z(\Phi, 1, 1). \end{aligned}$$

See that this implies that $\mathcal{S}'_{\mathrm{R}}(\mathrm{F})$, the R-vector space of $\mathrm{F}^\times$-equivariant distributions on $\mathcal{S}_{\mathrm{R}}(\mathrm{F})$, is, when $q_{\mathrm{F}} \equiv 1 \mod l$, of dimension 2.

(2) For π an irreducible R-representation of G_n, there might exist several π' irreducible R-representations of G_m such that

$$\mathrm{Hom}_{\mathrm{G}_n \times \mathrm{G}_m}(\sigma_{n,m}, \pi \otimes \pi') \neq 0.$$

For π an irreducible R-representation of G_n, denote by $\mu_m(\pi)$ the number of irreducible R-representation π' of G_m (with multiplicities) such that $\pi \otimes \pi'$ is a quotient of $\sigma_{n,m}$.

Let us study in detail the theta correspondence for the dual pair $(\mathrm{GL}_1(\mathrm{F}), \mathrm{GL}_m(\mathrm{F}))$.

Theorem 8.1. — *Let ξ be a C-character of $\mathrm{GL}_1(\mathrm{F})$ with values in $\mathcal{O}_{\mathrm{C}}$. Denote by $\overline{\xi}$ its reduction. Then $\mu_m\left(\overline{\xi}\right) = 1$ but when*

$$\mathrm{L}(\xi, -m) \notin \mathcal{O}_{\mathrm{C}} \quad ;$$

in this case $\mu_m\left(\overline{\xi}\right) = 2$.

Proof. — The proof is similar to [**MiThe**, §4.5]. We omit the details. □

Remark 8.2. — We dispose of similar results for the dual pair $(\mathrm{GL}_2(\mathrm{F}), \mathrm{GL}_m(\mathrm{F}))$. It would be interesting to have a formula relating the multiplicities appearing in the theta correspondence to the integrality of some special values of the L-functions of Godement-Jacquet.

8.3. Let n, m be a pair of integers such that $n \leq m$. Let π be an integral irreducible C-representation of G_n. Suppose, just for the sake of simplicity, that it is R-irreducible. As the characteristic of C is 0, $\theta_m(\pi)$ is a well defined irreducible C-representation of G_m. It is an integral C-representation as, by Theorem 7.1, its cuspidal support is integral. It might not be R-irreducible.

Write now $\overline{\pi} = r_{\mathrm{C}}(\pi)$ and suppose first that $\overline{\pi}$ is m-banal (see 5.9). Then, by Theorem 6.1, $\theta_m(\overline{\pi})$ is a well defined irreducible R-representation of G_m and it appears as a composition factor of $r_{\mathrm{C}}(\theta_m(\pi))$. That is, we have a commutative diagram:

$$\begin{tikzcd} \pi \arrow[r, "\theta_m"] \arrow[d, "r_{\mathrm{C}}"'] & \theta_m(\pi) \arrow[d, "r_{\mathrm{C}}"'] & \text{C-representations} \arrow[d, "r_{\mathrm{C}}"'] \\ \overline{\pi} \arrow[r, bend right, "\theta_m"'] & \square \ \square \ \square & \text{R-representations} \end{tikzcd}$$

Suppose finally that we are in the non-banal case. Now $\theta_m(\overline{\pi})$ is not well defined. Still there is one irreducible R-representation $\overline{\pi}'$, appearing as a composition factor of $r_{\mathrm{C}}(\theta_m(\pi))$, such that $\overline{\pi} \otimes \overline{\pi}'$ is a quotient of the metaplectic R-representation $\sigma_{n,m}$. In the non-banal case it can appear some semi-simplification: for example Theorem 5.2 is no longer true. But it appears already at the level of $\Theta(\pi)$ (see introduction), that is why there might exist some irreducible R-representation $\overline{\pi}'_0$ which is not isomorphic to any composition factor of $r_{\mathrm{C}}(\theta_m(\pi))$ such that $\overline{\pi} \otimes \overline{\pi}'_0$ is a quotient of the metaplectic

R-representation $\sigma_{n,m}$. Now, the picture is:

$$
\begin{tikzcd}
 & & & & \Theta(\pi) \arrow[dll, dotted] \arrow[ddr, squiggly, "r_C"'] & \\
\pi \arrow[rr, "\theta_m"] \arrow[d, "r_C"'] & & \theta_m(\pi) \arrow[d, "r_C"'] & & & \\
\overline{\pi} \arrow[r, bend right] \arrow[rrrrr, bend right] & \square & \square & \square & & \square
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{C-representations} \arrow[d, "r_C"'] \\
\text{R-representations}
\end{tikzcd}
$$

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