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The reproducing property on parabolic Bergman and Bloch spaces

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1. Introduction

Let H be the upper half-space of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}(n \geq 1)$, that is, $H=\{(x,t) \in \mathbb{R}^{n+1}; x \in \mathbb{R}^{n}, t>0\}$. For $0<\alpha \leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)}=\partial_{t}+(-\Delta_{x})^{\alpha},$$

where $\partial_{t}=\partial/\partial t$ and Δ_{x} is the Laplacian with respect to x. A continuous function u on H is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u=0$ in the sense of distributions (for details, see section 2). For $1 \leq p<\infty$ and $\lambda>-1$, the parabolic Bergman space $\boldsymbol{b}_{\alpha}^{p}(\lambda)$ is the set of all $L^{(\alpha)}$-harmonic functions u on H which satisfy

$$\|u\|_{L^{p}(\lambda)}:=\left(\int_{H}|u(x,t)|^{p}t^{\lambda}dV(x,t)\right)^{1/p}<\infty,$$

where dV is the Lebesgue volume measure on H. Moreover, $\boldsymbol{b}_{\alpha}^{\infty}$ is the set of all $L^{(\alpha)}$-harmonic functions u on H which satisfy

$$\|u\|_{L^{\infty}}:=\operatorname*{ess\,sup}_{(x,t)\in H}|u(x,t)|<\infty.$$

We remark that $\boldsymbol{b}_{1/2}^{p}(\lambda)$ coincide with the usual harmonic Bergman spaces of Koo, Nam, and Yi [4]. The parabolic Bloch space $\mathcal{B}_{\alpha}$ is the set of all $L^{(\alpha)}$-harmonic and C^{1} class functions u on H which satisfy

$$\|u\|_{\mathcal{B}_{\alpha}}:=\sup_{(x,t)\in H}\{t^{\frac{1}{2\alpha}}|\nabla_{x}u(x,t)|+t|\partial_{t}u(x,t)|\}<\infty,$$

where $\partial_{k}=\partial/\partial x_{k}$, $\nabla_{x}=(\partial_{1},\cdots,\partial_{n})$. Moreover, let $\widetilde{\mathcal{B}}_{\alpha}:=\{u \in \mathcal{B}_{\alpha}; u(0,1)=0\}$. It is also known that $\widetilde{\mathcal{B}}_{\alpha}$ is a Banach space with the norm $\|\cdot\|_{\mathcal{B}_{\alpha}}$. And we remark that $\widetilde{\mathcal{B}}_{1/2}$ coincides with the harmonic Bloch space of [6].

Our aim of this paper is the study of reproducing property with fractional orders on parabolic Bergman and Bloch spaces. Ramey and Yi [6] study the reproducing property on harmonic Bergman and Bloch spaces. Furthermore, Nishio, Shimomura, and Suzuki [5] study the reproducing property on parabolic Bergman and Bloch spaces. In this paper, we introduce fractional derivatives and study the reproducing property with fractional orders on parabolic Bergman and Bloch spaces.

To state our main results, we give some definitions. We denote by $W^{(\alpha)}$ the fundamental solution of the parabolic operator $L^{(\alpha)}$. For a real number κ, a fractional differential operator $\mathcal{D}_{t}^{\kappa}$ is defined by $\mathcal{D}_{t}^{\kappa}=(-\partial_{t})^{\kappa}$ (for the explicit definitions of $W^{(\alpha)}$ and $\mathcal{D}_{t}$, see section 2). A function ω_{α}^{κ} on $H\times H$ is defined by

$$\omega_{\alpha}^{\kappa}(x,t;y,s)=\mathcal{D}_{t}^{\kappa}W^{(\alpha)}(x-y,t+s)-\mathcal{D}_{t}^{\kappa}W^{(\alpha)}(-y,1+s)$$

for all $(x,t),(y,s) \in H$. In Theorems A and B, we present results of Koo, Nam, and Yi [4] concerning with the reproducing property on harmonic Bergman and Bloch spaces.

THEOREM A ([4]). *$1 \leq p < \infty$ and $\lambda > -1$. And let $\kappa > \frac{\lambda+1}{p}$ be a real number. Then, the reproducing property*

$$u(x,t) = C_\kappa \int_H u(y,s) \mathcal{D}_t^\kappa W^{(\frac{1}{2})}(x-y,t+s) s^{\kappa-1} dV(y,s) \tag{1.1}$$

holds for all $u \in b^p_{1/2}(\lambda)$ and $(x,t) \in H$, where $\Gamma(\cdot)$ is the Gamma function, and $C_\kappa = 2^\kappa/\Gamma(\kappa)$. Moreover, (1.1) also holds whenever $p = 1$ and $\kappa = \lambda + 1$.

THEOREM B ([4]). *Let $\kappa > 0$ be a real number. Then, the reproducing property*

$$u(x,t) = C_\kappa \int_H u(y,s) \omega^\kappa_{1/2}(x,t;y,s) s^{\kappa-1} dV(y,s)$$

holds for all $u \in \widetilde{\mathcal{B}}_{1/2}$ and $(x,t) \in H$, where C_κ is the constant defined in Theorem A.

The following theorems are our main results. Theorem 1 gives the reproducing property on parabolic Bergman spaces, and Theorem 2 gives the reproducing property on the parabolic Bloch space. We remark that the condition for κ of Theorem 2 is the limiting case of Theorem 1 as $p \to \infty$.

THEOREM 1. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$. And let $\kappa > \frac{\lambda+1}{p}$ be a real number. Then, the reproducing property*

$$u(x,t) = C_\kappa \int_H u(y,s) \mathcal{D}_t^\kappa W^{(\alpha)}(x-y,t+s) s^{\kappa-1} dV(y,s) \tag{1.2}$$

holds for all $u \in b^p_\alpha(\lambda)$ and $(x,t) \in H$, where C_κ is the constant defined in Theorem A. Moreover, (1.2) also holds whenever $p = 1$ and $\kappa = \lambda + 1$.

THEOREM 2. *Let $0 < \alpha \leq 1$. And let $\kappa > 0$ be a real number. Then, the reproducing property*

$$u(x,t) = C_\kappa \int_H u(y,s) \omega^\kappa_\alpha(x,t;y,s) s^{\kappa-1} dV(y,s)$$

holds for all $u \in \widetilde{\mathcal{B}}_\alpha$ and $(x,t) \in H$, where C_κ is the constant defined in Theorem A.

2. Preliminaries

First, we recall the definition of $L^{(\alpha)}$-harmonic functions. We describe about the operator $(-\Delta_x)^\alpha$. Since the case $\alpha = 1$ is trivial, we only describe the case

$0 < \alpha < 1$. Let $C_c^\infty(H)$ be the set of all infinitely differentiable functions on H with compact support. For $0 < \alpha < 1$, $(-\Delta_x)^\alpha$ is the convolution operator defined by

$$(-\Delta_x)^\alpha \psi(x,t) = -c_{n,\alpha} \lim_{\delta \to 0+} \int_{|y-x|>\delta} (\psi(y,t) - \psi(x,t))|y-x|^{-n-2\alpha} dy \qquad (2.1)$$

for all $\psi \in C_c^\infty(H)$ and $(x,t) \in H$, where $c_{n,\alpha} = -4^\alpha \pi^{-n/2} \Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$. A continuous function u on H is said to be $L^{(\alpha)}$-harmonic on H if u satisfies the following condition: for every $\psi \in C_c^\infty(H)$,

$$\int_H |u \cdot \widetilde{L}^{(\alpha)} \psi| dV < \infty \quad \text{and} \quad \int_H u \cdot \widetilde{L}^{(\alpha)} \psi dV = 0, \qquad (2.2)$$

where $\widetilde{L}^{(\alpha)} = -\partial_t + (-\Delta_x)^\alpha$ is the adjoint operator of $L^{(\alpha)}$. By (2.1) and the compactness of $\mathrm{supp}(\psi)$ (the support of ψ), there exist $0 < t_1 < t_2 < \infty$ and a constant $C > 0$ such that $\mathrm{supp}(\widetilde{L}^{(\alpha)}\psi) \subset S = \mathbb{R}^n \times [t_1, t_2]$ and $|\widetilde{L}^{(\alpha)}\psi(x,t)| \le C(1+|x|)^{-n-2\alpha}$ for all $(x,t) \in S$. Thus, the integrability condition $\int_H |u \cdot \widetilde{L}^{(\alpha)}\psi| dV < \infty$ is equivalent to the following: for any $0 < t_1 < t_2 < \infty$,

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |u(x,t)|(1+|x|)^{-n-2\alpha} dV(x,t) < \infty. \qquad (2.3)$$

We introduce the fundamental solution of $L^{(\alpha)}$. For $x \in \mathbb{R}^n$, the fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ is defined by

$$W^{(\alpha)}(x,t) = \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{R}^n} \exp(-t|\xi|^{2\alpha} + \sqrt{-1}\, x \cdot \xi)\, d\xi & t > 0 \\ 0 & t \le 0, \end{cases}$$

where $x \cdot \xi$ denotes the inner product on $\mathbb{R}^n$. It is known that $W^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H and $W^{(\alpha)} \in C^\infty(H)$, where $C^\infty(H)$ is the set of all infinitely differentiable functions on H.

Next, we present definitions of fractional integral and differential operators. Let $C(\mathbb{R}_+)$ be the set of all continuous functions on $\mathbb{R}_+ = (0,\infty)$. For a positive real number κ, let $\mathcal{F}C^{-\kappa}$ be the set of all functions $\varphi \in C(\mathbb{R}_+)$ such that there exist constants ε, $C > 0$ with $|\varphi(t)| \le Ct^{-\kappa-\varepsilon}$ for all $t \in \mathbb{R}_+$. We remark that $\mathcal{F}C^{-\nu} \subset \mathcal{F}C^{-\kappa}$ if $0 < \kappa \le \nu$. For $\varphi \in \mathcal{F}C^{-\kappa}$, we can define the fractional integral of φ with order κ by

$$\mathcal{D}_t^{-\kappa}\varphi(t) = \frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1} \varphi(t+\tau) d\tau = \frac{1}{\Gamma(\kappa)} \int_t^\infty (\tau - t)^{\kappa-1} \varphi(\tau) d\tau, \quad t \in \mathbb{R}_+. \quad (2.4)$$

Furthermore, let $\mathcal{F}C^\kappa$ be the set of all functions $\varphi \in C(\mathbb{R}_+)$ such that $d_t^{\lceil\kappa\rceil}\varphi \in \mathcal{F}C^{-(\lceil\kappa\rceil - \kappa)}$, where $d_t = d/dt$ and $\lceil\kappa\rceil$ is the smallest integer greater than or equal to κ. In particular, we will write $\mathcal{F}C^0 = C(\mathbb{R}_+)$. For $\varphi \in \mathcal{F}C^\kappa$, we can also define the fractional derivative of φ with order κ by

$$\mathcal{D}_t^\kappa \varphi(t) = \mathcal{D}_t^{-(\lceil\kappa\rceil-\kappa)} (-d_t)^{\lceil\kappa\rceil} \varphi(t), \quad t \in \mathbb{R}_+. \qquad (2.5)$$

Also, we define $\mathcal{D}_t^0\varphi = \varphi$. We may often call both (2.4) and (2.5) *the fractional derivative of φ with order κ*. Moreover, we call $\mathcal{D}_t^\kappa$ *the fractional differential operator with order κ*.The following proposition shows that fractional differential operators hold the commutative and exponential laws under some conditions.

PROPOSITION 2.1 ([2]). *Let κ and ν be positive real numbers. Then, the following statements hold.*

(1) *If $\varphi \in \mathcal{F}C^{-\kappa}$, then $\mathcal{D}_t^{-\kappa}\varphi \in C(\mathbb{R}_+)$.*

(2) *If $\varphi \in \mathcal{F}C^{-\kappa-\nu}$, then $\mathcal{D}_t^{-\kappa}\mathcal{D}_t^{-\nu}\varphi = \mathcal{D}_t^{-\kappa-\nu}\varphi$.*

(3) *If $d_t^k\varphi \in \mathcal{F}C^{-\nu}$ for all integers $0 \le k \le \lceil\kappa\rceil - 1$ and $d_t^{\lceil\kappa\rceil}\varphi \in \mathcal{F}C^{-(\lceil\kappa\rceil-\kappa)-\nu}$, then $\mathcal{D}_t^{\kappa}\mathcal{D}_t^{-\nu}\varphi = \mathcal{D}_t^{-\nu}\mathcal{D}_t^{\kappa}\varphi = \mathcal{D}_t^{\kappa-\nu}\varphi$.*

(4) *If $d_t^{k+\lceil\nu\rceil}\varphi \in \mathcal{F}C^{-(\lceil\nu\rceil-\nu)}$ for all integers $0 \le k \le \lceil\kappa\rceil - 1$, $d_t^{\lceil\kappa\rceil+\ell}\varphi \in \mathcal{F}C^{-(\lceil\kappa\rceil-\kappa)}$ for all integers $0 \le \ell \le \lceil\nu\rceil - 1$, and $d_t^{\lceil\kappa\rceil+\lceil\nu\rceil}\varphi \in \mathcal{F}C^{-(\lceil\kappa\rceil-\kappa)-(\lceil\nu\rceil-\nu)}$, then $\mathcal{D}_t^{\kappa}\mathcal{D}_t^{\nu}\varphi = \mathcal{D}_t^{\kappa+\nu}\varphi$.*

Here, we give examples of the fractional derivatives of elementary functions.

EXAMPLE 2.2 ([2]). *Let $\kappa > 0$ and ν be real numbers. Then, we have the following.*

(1) $\mathcal{D}_t^{\nu}e^{-\kappa t} = \kappa^{\nu}e^{-\kappa t}$.

(2) *Moreover, if $-\kappa < \nu$, then* $\mathcal{D}_t^{\nu}t^{-\kappa} = \dfrac{\Gamma(\kappa+\nu)}{\Gamma(\kappa)}t^{-\kappa-\nu}$.

3. Fractional calculus on parabolic Bergman and Bloch spaces

In this section, we give basic properties of fractional derivaitves of the fundamental solution $W^{(\alpha)}$, and parabolic Bergman and Bloch functions. First, we give basic properties of fractional derivatives of the fundamental solution $W^{(\alpha)}$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a multi-index $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}_0^n$, let $\partial_x^\beta = \partial^{|\beta|}/\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}$.

PROPOSITION 3.1 ([2]). *Let $0 < \alpha \le 1$, $\beta \in \mathbb{N}_0^n$, and $\kappa > -\frac{n}{2\alpha}$ be a real number. Then the following statements hold.*

(1) *The derivative $\partial_x^\beta\mathcal{D}_t^\kappa W^{(\alpha)} = \mathcal{D}_t^\kappa\partial_x^\beta W^{(\alpha)}$ is well-defined. Moreover, there exists a constant $C > 0$ such that*

$$|\partial_x^\beta\mathcal{D}_t^\kappa W^{(\alpha)}(x,t)| \le C(t+|x|^{2\alpha})^{-\frac{n+|\beta|}{2\alpha}-\kappa}$$

for all $(x,t) \in H$.

(2) *If $0 < q < \infty$ and $\theta > -1$ satisfy the condition $\frac{n}{2\alpha} + \theta + 1 - (\frac{n+|\beta|}{2\alpha} + \kappa)q < 0$, then there exists a constant $C > 0$ such that*

$$\int_H |\partial_x^\beta\mathcal{D}_t^\kappa W^{(\alpha)}(x-y,t+s)|^q s^\theta dV(y,s) \le Ct^{\frac{n}{2\alpha}+\theta+1-(\frac{n+|\beta|}{2\alpha}+\kappa)q}$$

for all $(x,t) \in H$.

(3) *Let* ν *be a real number such that* $\kappa + \nu > -\frac{n}{2\alpha}$. *Then,*

$$\mathcal{D}_t^{\nu}\partial_x^{\beta}\mathcal{D}_t^{\kappa}W^{(\alpha)}(x,t) = \partial_x^{\beta}\mathcal{D}_t^{\kappa+\nu}W^{(\alpha)}(x,t)$$

for all $(x,t) \in H$.

(4) $\partial_x^{\beta}\mathcal{D}_t^{\kappa}W^{(\alpha)}$ *is* $L^{(\alpha)}$*-harmonic on* H.

We define a function $\omega_\alpha^{\beta,\kappa}$. Let $\beta \in \mathbb{N}_0^n$, and $\kappa > -\frac{n}{2\alpha}$ be a real number. The function $\omega_\alpha^{\beta,\kappa}$ on $H \times H$ is defined by

$$\omega_\alpha^{\beta,\kappa}(x,t;y,s) = \partial_x^{\beta}\mathcal{D}_t^{\kappa}W^{(\alpha)}(x-y,t+s) - \partial_x^{\beta}\mathcal{D}_t^{\kappa}W^{(\alpha)}(-y,1+s)$$

for all $(x,t),(y,s) \in H$. Here, we remark that $\omega_\alpha^{\kappa} = \omega_\alpha^{0,\kappa}$. In the following proposition, we give estimates of the function $\omega_\alpha^{\beta,\kappa}$.

PROPOSITION 3.2 ([3]). *Let* $0 < \alpha \le 1$, $\beta \in \mathbb{N}_0^n$, *and* $\kappa > -\frac{n}{2\alpha}$ *be a real number.*

(1) *For any compact set* $K \subset \mathbb{R}^n$ *and* $M > 1$, *there exist constants* $C_1, C_2 > 0$ *such that*

$$|\omega_\alpha^{\beta,\kappa}(x,t;y,s)| \le \frac{C_1|x|}{(1+s+|y|^{2\alpha})^{\frac{n+|\beta|+1}{2\alpha}+\kappa}} + \frac{C_2|t-1|}{(1+s+|y|^{2\alpha})^{\frac{n+|\beta|}{2\alpha}+\kappa+1}}$$

for all $(x,t) \in K \times [M^{-1}, M]$ *and* $(y,s) \in H$.

(2) *Let* $(x,t) \in H$ *be fixed. Then, there exists a constant* $C > 0$ *such that*

$$|\omega_\alpha^{\beta,\kappa}(x,t;y,s)| \le C(1+s+|y|^{2\alpha})^{-\frac{n+|\beta|}{2\alpha}-\kappa-\sigma}$$

for all $(y,s) \in H$, *where* $\sigma = \min\{1, \frac{1}{2\alpha}\}$.

(3) *Moreover, let* $\kappa > 0$ *be a real number. Then, there exists a constant* $C > 0$ *such that*

$$\int_H |\omega_\alpha^{\beta,\kappa}(x,t;y,s)| s^{\frac{|\beta|}{2\alpha}+\kappa-1} dV(y,s) \le C(1+\log(1+|x|)+|\log t|)$$

for all $(x,t) \in H$.

Next, we give basic properties of fractional derivatives of parabolic Bergman functions.

PROPOSITION 3.3 ([2]). *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda > -1$, $\beta \in \mathbb{N}_0^n$, *and* $\kappa > -(\frac{n}{2\alpha} + \lambda + 1)\frac{1}{p}$ *be a real number. If* $u \in \boldsymbol{b}_\alpha^p(\lambda)$, *then the following statements hold.*

(1) *The derivative* $\partial_x^\beta \mathcal{D}_t^\kappa u(x,t) = \mathcal{D}_t^\kappa \partial_x^\beta u(x,t)$ *is well-defined. Moreover, there exists a constant* $C > 0$ *such that*

$$|\partial_x^\beta \mathcal{D}_t^\kappa u(x,t)| \leq C t^{-\frac{|\beta|}{2\alpha} - \kappa - (\frac{n}{2\alpha} + \lambda + 1)\frac{1}{p}} \|u\|_{L^p(\lambda)}$$

for all $(x,t) \in H$.

(2) *Let* ν *be a real number such that* $\kappa + \nu > -(\frac{n}{2\alpha} + \lambda + 1)\frac{1}{p}$. *Then,*

$$\mathcal{D}_t^\nu \partial_x^\beta \mathcal{D}_t^\kappa u(x,t) = \partial_x^\beta \mathcal{D}_t^{\kappa+\nu} u(x,t)$$

for all $(x,t) \in H$.

(3) $\partial_x^\beta \mathcal{D}_t^\kappa u$ *is* $L^{(\alpha)}$*-harmonic on* H.

Finally, we give basic properties of fractional derivatives of parabolic Bloch functions.

PROPOSITION 3.4 ([3]). *Let* $0 < \alpha \leq 1$, $\beta \in \mathbb{N}_0^n$, *and* $\kappa \geq 0$ *be a real number. If* $u \in \mathcal{B}_\alpha$, *then the following statements hold.*

(1) *The derivative* $\partial_x^\beta \mathcal{D}_t^\kappa u(x,t) = \mathcal{D}_t^\kappa \partial_x^\beta u(x,t)$ *is well-defined. Moreover, for* $(\beta, \kappa) \neq (0,0)$, *there exists a constant* $C > 0$ *such that*

$$|\partial_x^\beta \mathcal{D}_t^\kappa u(x,t)| \leq C t^{-\frac{|\beta|}{2\alpha} - \kappa} \|u\|_{\mathcal{B}_\alpha}$$

for all $(x,t) \in H$.

(2) *Let* $\nu \geq 0$ *be a real number. Then,*

$$\mathcal{D}_t^\nu \partial_x^\beta \mathcal{D}_t^\kappa u(x,t) = \partial_x^\beta \mathcal{D}_t^{\kappa+\nu} u(x,t) \tag{3.1}$$

for all $(x,t) \in H$. *If* $\nu < 0$ *is a real number such that* $\kappa+\nu \geq 0$ *and* $(\beta, \kappa+\nu) \neq (0,0)$, *then* (3.1) *also holds.*

(3) $\partial_x^\beta \mathcal{D}_t^\kappa u$ *is* $L^{(\alpha)}$*-harmonic on* H.

We present more estimates of fractional derivatives of parabolic Bloch functions, which is the important tool for the proof of the reproducing property on $\widetilde{\mathcal{B}}_\alpha$.

PROPOSITION 3.5 ([3]). *Let* $0 < \alpha \leq 1$, $\beta \in \mathbb{N}_0^n$, *and* $\kappa \geq 0$ *be a real number.*

(1) *For any* $M > 1$, *there exists a constant* $C > 0$ *such that*

$$|\partial_x^\beta \mathcal{D}_t^\kappa u(x,t+s) - \partial_x^\beta \mathcal{D}_t^\kappa u(0,1+s)| \leq C\|u\|_{\mathcal{B}_\alpha} \left\{ \frac{|x|}{(1+s)^{\frac{|\beta|+1}{2\alpha}+\kappa}} + \frac{|t-1|}{(1+s)^{\frac{|\beta|}{2\alpha}+\kappa+1}} \right\}$$

for all $u \in \mathcal{B}_\alpha$, $(x,t) \in \mathbb{R}^n \times [M^{-1}, M]$, *and* $s \geq 0$.

(2) *Let* $(x,t) \in H$ *be fixed. Then there exists a constant* $C > 0$ *such that*

$$|\partial_x^\beta \mathcal{D}_t^\kappa u(x,t+s) - \partial_x^\beta \mathcal{D}_t^\kappa u(0,1+s)| \leq C(1+s)^{-\frac{|\beta|}{2\alpha} - \kappa - \sigma},$$

for all $u \in \mathcal{B}_\alpha$ *and* $s \geq 0$*, where* $\sigma = \min\{1, \frac{1}{2\alpha}\}$.

4. The reproducing property on parabolic Bergman and Bloch spaces

In this section, we give the reproducing property on parabolic Bergman and Bloch spaces. First, we present the Huygens property, which plays an important role for the proof of the reproducing property.

LEMMA 4.1 ([8]). *Let* $0 < \alpha \leq 1$*,* $1 \leq p < \infty$*, and* $\lambda > -1$*. If* $u \in \boldsymbol{b}^p_\alpha(\lambda)$*, then* u *satisfies the Huygens property, that is,*

$$u(x,t) = \int_{\mathbb{R}^n} u(x-y, t-s) W^{(\alpha)}(y,s)dy$$

holds for all $x \in \mathbb{R}^n$ *and* $0 < s < t < \infty$.

LEMMA 4.2 ([5]). *Let* $0 < \alpha \leq 1$*. If* $u \in \mathcal{B}_\alpha$*, then* u *satisfies the Huygens property, that is,*

$$u(x,t) = \int_{\mathbb{R}^n} u(x-y, t-s) W^{(\alpha)}(y,s)dy$$

holds for all $x \in \mathbb{R}^n$ *and* $0 < s < t < \infty$.

For $\delta > 0$ and a function u on H, we define an auxiliary function u_δ of u by $u_\delta(x,t) = u(x, t+\delta)$. We present the reproducing property for fractional derivatives of u_δ in Propositions 4.3 and 4.4, which play an important role for the proof of the reproducing property on $\boldsymbol{b}^p_\alpha(\lambda)$ and $\widetilde{\mathcal{B}}_\alpha$, respectively.

PROPOSITION 4.3 ([2]). *Let* $0 < \alpha \leq 1$*,* $1 \leq p < \infty$*,* $\lambda > -1$*, and* $\delta > 0$*. And let* $\nu > -(\frac{n}{2\alpha} + \lambda + 1)\frac{1}{p}$ *and* $\kappa \geq 0$ *be real numbers with* $\nu + \kappa > 0$*. Then,*

$$u_\delta(x,t) = C_{\nu+\kappa} \int_H \mathcal{D}^\nu_t u_\delta(y,s) \mathcal{D}^\kappa_t W^{(\alpha)}(x-y, t+s) s^{\nu+\kappa-1} dV(y,s)$$

holds for all $u \in \boldsymbol{b}^p_\alpha(\lambda)$ *and* $(x,t) \in H$.

PROPOSITION 4.4 ([3]). *Let* $0 < \alpha \leq 1$ *and* $\delta > 0$*. And let* $\kappa, \nu \geq 0$ *be real numbers with* $\kappa + \nu > 0$*. Then,*

$$u_\delta(x,t) - u_\delta(0,1) = C_{\nu+\kappa} \int_H \mathcal{D}^\nu_t u_\delta(y,s) \omega^\kappa_\alpha(x,t;y,s) s^{\nu+\kappa-1} dV(y,s)$$

holds for all $u \in \mathcal{B}_\alpha$ *and* $(x,t) \in H$.

Now, we give the reproducing property on parabolic Bergman and Bloch spaces.

THEOREM 4.5 ([2]). *Let* $0 < \alpha \leq 1$, $1 \leq p < \infty$, *and* $\lambda > -1$. *And let* $\nu > -\frac{\lambda+1}{p}$ *and* $\kappa > \frac{\lambda+1}{p}$ *be real numbers. Then, the reproducing property*

$$u(x,t) = C_{\nu+\kappa} \int_H \mathcal{D}_t^{\nu} u(y,s) \mathcal{D}_t^{\kappa} W^{(\alpha)}(x-y, t+s) s^{\nu+\kappa-1} dV(y,s) \tag{4.1}$$

holds for all $u \in \boldsymbol{b}_\alpha^p(\lambda)$ *and* $(x,t) \in H$. *Moreover,* (4.1) *also holds whenever* $p = 1$ *and* $\kappa = \lambda + 1$.

THEOREM 4.6 ([3]). *Let* $0 < \alpha \leq 1$. *And let* $\nu \geq 0$ *and* $\kappa > 0$ *be real numbers. Then, the reproducing property*

$$u(x,t) = C_{\nu+\kappa} \int_H \mathcal{D}_t^{\nu} u(y,s) \omega_\alpha^{\kappa}(x,t;y,s) s^{\nu+\kappa-1} dV(y,s)$$

holds for all $u \in \widetilde{\mathcal{B}}_\alpha$ *and* $(x,t) \in H$.

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