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Specker phenomenon in the countable case

Jun Nakamura

Waseda University

abstract

We show that Non-commutative Specker-Eda number $\mathfrak{se}_{\mathfrak{nc}}$ is equal to Specker-Eda number $\mathfrak{se}$. $\mathfrak{se}_{\mathfrak{nc}}$ is the smallest cardinalty of subgroups of $\times_{n<\omega}\mathbb{Z}_n$ which exhibit non-commutative Specker phenomenon. And $\mathfrak{se}$ is the smallest cardinaty of subgroups of $\mathbb{Z}^\omega$ which exhibit specker phenomenon.

1 Introduction

In 1950, E.Specker [1] proved that any homomorphism from $\mathbb{Z}^\omega$ to $\mathbb{Z}$ factors through $\mathbb{Z}^n$ for some n. This theorem is reduced to the following: $h(e_n) = 0$ for all but finitely many n for any homomorphism to $\mathbb{Z}$ where e_n is the element of $\mathbb{Z}^\omega$ whose n-th component is 1 and whose other component are all zero. A.Blass [4] named this fact "Specker phenomenon" in 1994.

E.Specker also established subgroups of $\mathbb{Z}^\omega$ which exhibit Specker phenomenon. But these subgroup have the cardinalty of the continuum $2^{\aleph_0}$. So, the next quetion naturally arises whether the smallest cardinalty of subgroups which exhibit Specker phenomenon is $2^{\aleph_0}$. In 1983, it was turned out that this question is undecidable on ZFC by K.Eda [2]. And in 1986, S.Kamo [3] also considered a related question in Cohen extension . Then, A.Blass [4] studied the cardinalty and named it Specker-Eda number $\mathfrak{se}$. He pointed out Eda's proof established that $\mathfrak{p} \leq \mathfrak{se} \leq \mathfrak{d}$. In 1994, he proved that $\mathfrak{e}_{\mathfrak{f}} \leq \mathfrak{se} \leq \mathfrak{b}$. Finally, in 1996, J.Brendle and S.Shelah [5] proved that $\mathfrak{se} = \mathfrak{e}_{\mathfrak{f}}$ =min$\{\mathfrak{e}, \mathfrak{b}\}$. Now, we consider the non-commutative case.

2 Definition of the complete free product

Definition 2.1.

Let G_i $(i \in I)$ be groups s.t $G_i \cap G_j = \{e\}$ for any $i \neq j \in I$. we call elements of $\bigcup_{i \in I} G_i$ letters. A word W is a function

$W : \overline{W} \rightarrow \bigcup_{i \in I} G_i$ $\overline{W}$ is a linearly ordered set and $\{\alpha \in \overline{W} \mid W(\alpha) \in G_i\}$ is finite for any $i \in I$. The class of all words is denoted by $\mathcal{W}(G_i : i \in I)$ (abbreviated by $\mathcal{W}$).

Definition 2.2. The identification of words
U and V are isomorphic ($U \equiv V$) if there exists an order isomorphism $\varphi : \overline{U} \rightarrow \overline{V}$ s.t $\forall \alpha \in \overline{U}$ $(U(\alpha) = V(\varphi(\alpha)))$.

It is easily seen that $\mathcal{W}$ becomes a set under this identification.

Definition 2.3. The restricted word of W
For a subset $X \subseteq I$, the restricted word W_X of W is given by the function

$W_X : \overline{W_X} \rightarrow \bigcup_{i \in I} G_i$ where $\overline{W_X} = \{\alpha \in \overline{W} | W(\alpha) \in \bigcup_{i \in X} G_i\}$ and $W_X(\alpha) = W(\alpha)$ for all $\alpha \in \overline{W_X}$. Hence $W_X \in \mathcal{W}$. If X is finite , then we can regard W_X as an element of the free product $*_{i \in X} G_i$.

Definition 2.4. The equivalence relation on words
U and V are equivalent ($U \sim V$) if $U_F = V_F$ for all $F \subset\subset I$ where we regard U_F and V_F as elements of the free product $*_{i \in F} G_i$.
So, "$U_F = V_F$" means that they are equivalent in the sense of the free product $*_{i \in F} G_i$.

Let $[W]$ be the equivalent class of a word W. The composition of two words and the inverse of a word are defined naturally. Thus $\mathcal{W}/\sim = \{[W] \mid W \in \mathcal{W}\}$ becomes a group.

Definition 2.5. The complete free product $\times_{i \in I} G_i$
$\times_{i \in I} G_i$ is the group $\mathcal{W}(G_i : i \in I)/\sim$. Clearly, if I is finite, then $\times_{i \in I} G_i$ is isomorphic to the free product $*_{i \in I} G_i$.

3 Non-commutative Specker-Eda number

Theorem 3.1. Non-commutative Specker's theorem (G.Higman [6], Theorem 1)
Let F be a free group. For any homomorphism h from $\times_{n<\omega}\mathbb{Z}_n$ to F, there exists a natural number n such that h factors through $*_{i<n}\mathbb{Z}_i$.

$$\begin{tikzcd}
\times_{n<\omega}\mathbb{Z}_n \arrow[rr, "h"] \arrow[d, "p_n"'] & & F \\
*_{i<n}\mathbb{Z}_i \arrow[urr, dashed, "\exists\overline{h}"'] & &
\end{tikzcd}$$

$h = \overline{h} \circ p_n$ $\quad p_n$: projection s.t $p_n(W) = W_n$

We can regard a word W as an element of the free product $*_{i<n}\mathbb{Z}_i * \times_{n\leq i<\omega}\mathbb{Z}_i$. So, this theorem means that there exists an n such that $h[\times_{n\leq i<\omega}\mathbb{Z}_i] = \{e\}$ where e is the identity of F. Similarly to the commutative case, this theorem is reduced the following: $h(\delta_n) = e$ for all but finitely n for any homomorphism h to F where δ_n is 1 of $\mathbb{Z}_n$. If $F = \mathbb{Z}$, we call this non-commutative Specker phenomenon(abbreviated by nc-Specker phenomenon).

Definition 3.1. Non-commutative Specker-Eda number
It is denoted by $\mathfrak{se}_{nc}$.

$\mathfrak{se}_{nc} = \min\{|G| \ : \ *_{n<\omega}\mathbb{Z}_n \leq G \leq \times_{n<\omega}\mathbb{Z}_n$ and G exhibits nc-Specker phenomenon.$\}$

Theorem 3.2. $\mathfrak{se}_{nc} = \mathfrak{se}$

Proof.
Firstly, we show that $\mathfrak{se} \leq \mathfrak{se}_{nc}$. Let $\sigma : \times_{n<\omega}\mathbb{Z}_n \to \mathbb{Z}^{\omega}$ be the canonical homomorphism such that $\sigma(W)(n) = W_{\{n\}}$ $(n < \omega)$ and G be a subgroup of $\times_{n<\omega}\mathbb{Z}_n$ which exhibits nc-Specker phenomenon and whose cardinality is $\mathfrak{se}_{nc}$. Then $\sigma[G]$ also exhibits Specker phenomenon. Because, let h :

$\sigma[G] \rightarrow \mathbb{Z}$ be a homomorphism. The composition of h and σ is a homomorphism from G to $\mathbb{Z}$. Therefore, $h(e_n) = h \circ \sigma(\delta_n) = 0$ for all but finitely many n. Then, we get $\mathfrak{se} \leq |\sigma[G]| \leq |G| = \mathfrak{se}_{\mathrm{nc}}$.

Next, we show that $\mathfrak{se}_{\mathrm{nc}} \leq \mathfrak{se}$. To show this, two lemmas are necessary.

Lemma 3.1. $x, a \in \mathbb{Z}$

$$\forall n < \omega \left(n! \mid x - \sum_{i=1}^{n-1} i!a \right) \Rightarrow x = 0 \textit{ and } a = 0$$

Proof.

we can prove by induction that $2 \leq n$ implies $\sum_{i=1}^{n-1} i! \leq 2(n-1)!$. Therefore, we can easily find a natural number n such that $|x - \sum_{i=1}^{n-1} i!a| < n!$ and $|x - \sum_{i=1}^{n} i!a| < (n+1)!$. It means that $x - \sum_{i=1}^{n-1} i!a = 0 = x - \sum_{i=1}^{n} i!a$ □

To show the second lemma, we consider the following words:U_∞, U_n. For $W \in \times_{n<\omega}\mathbb{Z}_n$, let $V_n = W_{\omega\backslash n}$. To define U_∞, U_n , we consider such a tree $T = \langle \bigcup_{n<\omega}(\prod_{1\leq m\leq n+1} m), \subseteq \rangle$ like the binary tree $< 2^{<\omega}, \subseteq >$. Then we order T lexicographically, i.e; If $x, y \in T$, define $x \lhd y$ iff $x(n) < y(n)$ where $n \in \mathrm{dom}(x) \cap \mathrm{dom}(y)$ is the least natural number such that $x(n) \neq y(n)$, or $\mathrm{dom}(x) < \mathrm{dom}(y)$. Consequently, T is linearly orded by this lexicographical order. Now, we define U_∞, U_n as follows.

$$\overline{U_\infty} = T,\ U_\infty(x) = V_n \quad (x \in \mathrm{Lev}_n(T) = \prod_{1\leq m\leq n+1} m)$$

$\overline{U_n} = \{y_n\} \cup \bigcup_{n+1\leq k<\omega} \mathrm{Lev}_k(T)$ where y_n is an arbitary element of $\mathrm{Lev}_n(T)$,

$$U_n(x) = V_k \quad (x \in \mathrm{Lev}_k(T))$$

These definition is not exact because V_n may not be a letter. But we can naturally regard them as the composition of infinitary many words, since V_n does not contain letters δ_i for $i < n$. And we find that they realy become a word.

$$
\begin{array}{cccccc}
 & & & V_3 & \cdots & \\
 & & & V_3 & \cdots & \\
 & & & V_3 & \cdots & \\
 & & V_2 & V_3 & \cdots & \\
 & V_1 & V_2 & \cdots & & \\
 & & V_2 & \ddots & & \\
U_\infty = V_0 & & & & & \\
 & & V_2 & \cdots & & \\
 & V_1 & V_2 & \cdots & & \\
 & & V_2 & V_3 & \cdots & \\
 & & & V_3 & \cdots & \\
 & & & V_3 & \cdots & \\
 & & & V_3 & \cdots & \cdots \\
 & & & & \ddots & \\
 & & & V_{n+2} & \cdots & \\
 & & & & \ddots & \\
 & & V_{n+1} & \vdots & & \\
 & & & V_{n+2} & \cdots & \\
 & & & & \ddots & \\
U_n = V_n & & \vdots & & & \\
 & & & V_{n+2} & \cdots & \\
 & & & & \ddots & \\
 & & V_{n+1} & \vdots & & \\
 & & & V_{n+2} & \cdots & \\
 & & & & \ddots &
\end{array}
$$

Now, we mention the second lemma.

Lemma 3.2. Let G be a subgroup of $\mathbf{x}_{n<\omega}\mathbb{Z}_n$ containing all δ_n.
If, for every $W \in \ker(\sigma)\cap G$, G contains U_∞ and all U_n coressponding to W,

then any homomorphism h from G to $\mathbb{Z}$ factors thorough $\sigma[G]$.

$$\begin{tikzcd} G \arrow[rr, "h"] \arrow[d, "\sigma"'] & & \mathbb{Z} \\ \sigma[G] \arrow[urr, dashed, "\exists\overline{h}"'] & & \end{tikzcd}$$

$h = \overline{h} \circ \sigma$ $\quad$ σ:canonical homomorphism

Proof.

It is sufficient to show that $\ker(\sigma) \cap G \subseteq \ker(h)$. Let G' be a commutator subgroup of G and $[W]$ be a element of G/G'. Let $W \in \ker(\sigma) \cap G$. Then $[W] = [V_n]$ for all n because G/G' is an abelian group. By the figure of U_∞, U_n, we have

$$\begin{aligned} [U_\infty] &= \sum_{i=1}^{n-1} i![V_{i-1}] + n![U_{n-1}] \\ &= \sum_{i=1}^{n-1} i![W] + n![U_{n-1}] \end{aligned}$$

And there exists a homomorphism $h_0 : G' \to \mathbb{Z}$ s.t $h(x) = h_0([x])$ for any $x \in G$ by the homomorphism theorem because $G' \subseteq \ker(h)$. Therefore, we get

$$n! \mid h_0([U_\infty]) - \sum_{i=1}^{n-1} i! h_0([W]) \quad \text{for all } n$$

So, we have $h(W) = h_0([W]) = 0$ by Lemma 3.1. $\square$

Now, we return to the proof of $\mathfrak{se}_{\mathrm{nc}} \leq \mathfrak{se}$. Our goal is getting a subgroup of $\mathbf{X}_{n<\omega}\mathbb{Z}_n$ whose cardinality is $\mathfrak{se}$ and which exhibits nc-Specker phenomenon. In the diagram of Lemma 3.2, if $\sigma[G]$ exhibits Specker phenomenon, then G also exhibits nc-Specker phenomenon because $h(\delta_n) = \overline{h}(e_n)$. So, we take a subgroup H of $\mathbb{Z}^\omega$ whose cardinality is $\mathfrak{se}$ and which exhibits Specker phenomenon. $\sigma^{-1}[H]$ also exhibits nc-Specker phenomenon, but, unfortunately, the cardinality of $\ker(\sigma)$ is $2^{\aleph_0}$. Let X be a set such that $\sigma[X] = H$ and $|X| = \mathfrak{se}$. Then let G be the smallest subgroup which contains X and satisfies the clause ofLemma 3.2. Obviously, the size of G is $\mathfrak{se}$. And $\sigma[G]$ contains H, so $\sigma[G]$ also exhibits Specker phenomenon. Therefore, G is the desired subgroup. $\square$

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