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On a J_1-convergence theorem for stochastic processes on $D[0,\infty)$ having monotone sample paths and its applications

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Abstract

In this paper, we first show that Bingham's assertion on weak convergence under J_1-topology of stochastic processes with monotone sample paths is incorrect by giving a counter example and then remark that his assertion is valid if the limit process is continuous. Furthermore, we give a simple sufficient condition for weak convergence under J_1-topology of stochastic processes with monotone sample paths. We apply the result to the weak convergence under J_1-topology of hitting time processes of 1-dimensional generalized diffusion processes.

1 Introduction

Let $D[0,\infty)$ be the class of functions defined on $[0,\infty)$ to $\mathbb{R}$ which have left limits on $(0,\infty)$ and are right continuous on $[0,\infty)$. Bingham asserted in [2] that a sequence of random variables $\{X_n\}$ in $D[0,\infty)$, which has monotone sample paths, converges weakly to an $X_\infty \in D[0,\infty)$ in J_1-topology if X_∞ is continuous in probability and X_n converges to X_∞ in finite dimensional distributions sense.

In this paper, we remark that his proof contains an error and give an example which shows that his statement does not hold. We also remark that his conclusion is valid if we change the continuity in probability assumption for X_∞ to continuity of all sample functions.

We, then, give an additional condition so that the conclusion of Bingham's assertion holds and apply the result to the weak convergence in J_1-topology of additive processes and sums of independent random variables. The former result is then applied directly to weak convergences under J_1-topology of hitting time processes of 1-dimensional generalized diffusion processes to selfsimilar additive processes.

Define

$$\Delta(c,X,T)=\sup_{0\le t-c<t_1<t<t_2<t+c\le T}\min(|X(t_1)-X(t)|,|X(t_2)-X(t)|).$$

Stone [7] proved the following theorem.

Theorem 1.1 (Stone) *If $\{X_n : n=1,2,\dots\}$ are random variables in $D[0,\infty)$, X_n converges to a random variable X_∞ in $D[0,\infty)$ as $n\to\infty$ weakly under the J_1-topology on $D[0,\infty)$ if and only if*
(a) The finite-dimensional distributions of X_n converge to those of X_∞ as $n\to\infty$
(b) $\lim_{c\to 0}\limsup_{n\to\infty}P(\Delta(c,X_n,T)>\epsilon)=0$ for each $\epsilon>0$ and each $0<T<\infty$.

Bingham's assertion in next section reduces the proof to the above Stone's theorem.

2 On Bingham's assertion

Bingham asserted the following assertion as Theorem 3 in [2].
Bingham's assertion : Let $\{X_n\}_{n\geq 1}$ be a sequence of stochastic processes whose path-functions lie in $D[0,\infty)$. If
(i) the finite-dimensional distributions of X_n converge to those of X_∞,
(ii) the process X_∞ is continuous in probability and
(iii) the processes X_n have monotone path-functions,
then X_n converges to X_∞ weakly under the J_1-topology on $D[0,\infty)$

He defines the quantity

$$D(k,X,T) = \max\{|X(rT/k) - X((r-1)T/k)| : r = 1,2,\ldots,k\}.$$

He asserts that, by stochastic continuity of X_∞,

$$P(\Delta(Tk^{-1}, X_\infty, T) > \epsilon) \to 0 \text{ as } k \to \infty \tag{1}$$

for each $\epsilon > 0$ and $0 < T < \infty$ (for this convergence, the stochastic continuity assumption is not necessary if X_∞ is in $D([0,\infty))$, because $\Delta(Tk^{-1}, X, T) \to 0$ as $k \to \infty$ for $X \in D([0,\infty))$). Then he asserts that the inequalities

$$\frac{1}{2}\Delta(2Tk^{-1}, X, T) \leq D(k,X,T) \leq \Delta(Tk^{-1}, X, T) \tag{2}$$

holds and using these inequalities, he shows the validity of the condition (b) in Stone's theorem. But, the second inequality in (2) does not hold even for the following simple function:

$$x(t) = \begin{cases} 0, & 0 \leq t < a, \\ 1, & a \leq t \leq T, \end{cases}$$

where $0 < a < T$. For this function, $D(k,x,T) = 1$ while $\Delta(Tk^{-1}, x, T) = 0$ for large k. One may consider, instead of (1),

$$P(D(k, X_\infty, T) > \epsilon) \to 0 \text{ as } k \to \infty \tag{3}$$

for each $\epsilon > 0$ and $0 < T < \infty$. But this does not hold even for a simple stochastic process. Let $X(t)$ be a Poisson process with parameter λ and choose $\epsilon \in (0,1)$. Then,

$$P(D(k, X_\infty, T) \leq \epsilon) = \prod_{r=1}^{k} P(X(rT/k) - X((r-1)T/k) \leq \epsilon) = e^{-\lambda T}.$$

Moreover, the above assertion does not hold under the assumtions in the threorem. The following is a counter example : Let $T = 1$. Let

$$\begin{aligned} X_\infty(t,\omega) &= 2\mathbb{I}_{[\xi(\omega),1]}(t), \\ X_n(t,\omega) &= \mathbb{I}_{[\xi(\omega)-\frac{1}{n},1]}(t) + \mathbb{I}_{[\xi(\omega)+\frac{1}{n},1]}(t), \end{aligned}$$

where $\xi(\omega)$ is a random variable uniformly distributed on $[\frac{1}{3},\frac{2}{3}]$. Then $X_\infty(t)$ and $X_n(t)$ are non-decreasing and right continuous with left limits. Hence, X_∞ is stochastically continuous. $X_n(t)$ converges in finite dimensional distributions, but does not converge in J_1-topology. The last assertion is immediate from the fact that $\Delta(c, X_n, T) = 1$ for $\frac{1}{n} < c$. This example is stated as Problem 12.5 in [1].

Remark 2.1 *If we substitute the condition "continuous in probability" by "continuous with probability 1", then the above theorem is valid since in this case, $D(k, X_\infty(\omega), T) \to 0$ as $n \to \infty$ with probability 1 and hence* (3) *holds as $k \to \infty$ for each $\epsilon > 0$ and $0 < T < \infty$. By the assumption (i), $\limsup_{n\to\infty} P(D(k, X_n, T) > 2\epsilon) \le P(D(k, X_\infty, T) > \epsilon)$. Using this inequality, the convergence in* (3) *and the left inequality in* (2)*, we have the weak convergence under the J_1-topology.*

If the above Bingham's assertion were true, then the statement would be very useful. In fact, many papers use Bingham's "Theorem 3". For example, in Corollary 3.5 and Corollary 3.11 in [5] and Corollary 3.4 in [4]. Fortunately, sample paths of the limit processes in Corollary 3.11 in [5] and Corollary 3.4 in [4] are contiouous, the conclusion is valid by the above remark. In Corollary 3.5 in [5], limit processes are purely discontinuous. Hence the result in Remark 2.1 can not be applied.

In the following section, we investigate what additional condition is needed so that weak convergence under J_1-topology holds. The result is described as Theorem 3.1, in the following section. The conclusion of Corollary 3.5 in [5] is valid by Theorem 3.3, which is an application of Theorem 3.1.

3 Main results

Lemma 3.1 *Let $x \in D[0,1]$ with non-decreasing path. Let*

$$x_1(t) = \sum_{0 \le s \le t, x(s)-x(s-)>\epsilon} (x(s) - x(s-))$$

and let $x_2(t) = x(t) - x_1(s)$. Assume that $x_1(t)$ has at most one jump in a sub interval $(r, t]$ in $[0,1]$. Then, for $r < s < t$,

$$(x(s) - x(r)) \wedge (x(t) - x(s)) \le x_2(t) - x_2(r).$$

Proof First assume that there is no jump greater than ϵ in $(r, t]$. Then, for $r \le s \le t$,

$$(x(s) - x(r)) \wedge (x(t) - x(s)) \le (x(t) - x(r))/2 = (x_2(t) - x_2(r))/2.$$

Next, assume that $r < s_0 \le t$ be the jumping point of $x_1(t)$. We devide into three cases.
(1) $x(s_0-) < \frac{1}{2}(x(t) - x(r)) < x(s_0)$,
(2) $x(s_0) \le \frac{1}{2}(x(t) - x(r))$ and
(3) $\frac{1}{2}(x(t) - x(r)) \le x(s_0-)$.
If (1) holds, then

$$(x(s) - x(r)) \wedge (x(t) - x(s)) \le (x(s_0-) - x(r)) \vee (x(t) - x(s_0)) \le x_2(t) - x_2(r).$$

Now, assume that the case (2) or (3). Then

$$(x(s) - x(r)) \wedge (x(t) - x(s)) \le (x(t) - x(r))/2 \le x_2(t) - x_2(r).$$

□

Let X be a random variable in $D[0,\infty)$. For each $\epsilon > 0$,

$$X^{1,\epsilon}(t) = \sum_{0 < s \le t, X(s)-X(s-)>\epsilon} (X(s) - X(s-))$$

and $X^{2,\epsilon}(t) = X(t) - X^{1,\epsilon}(t)$. Let $\Omega_\delta^\epsilon(X, T)$ be a set of ω such that $X(t,\omega)$ has at most one jump greater than ϵ in any semi closed interval $(r, r+\delta]$ contained in $(0, T]$, where $\delta > 0$.

Theorem 3.1 *Suppose that $\{X_n\}$ is a sequence of stochastic processes in $D[0,\infty)$ with non-decreasing sample paths which converges in finite dimensional distributions sense to a stochastic process $X_\infty \in D[0,\infty)$ as $n \to \infty$. If*
(i) for each $\epsilon > 0$, $X_n^{2,\epsilon}$ converges to $X_\infty^{2,\epsilon}$ as $n \to \infty$ in finite dimensional distributions sense and
(ii) for each $\epsilon, T > 0$, $\lim_{\delta\downarrow 0}\limsup_{n\to\infty} P(\Omega_\delta^\epsilon(X_n,T)) = 1$,
then X_n converges to X_∞ weakly in J_1-topology on $D[0,\infty)$.

Proof Let X be a random variable in $D[0,\infty)$ and let $\omega \in \Omega_{T/k}^\epsilon(X,T)$. If $\frac{2\ell-1}{2k}T \le s \le \frac{2\ell+1}{2k}T$ and $s - \frac{T}{2k} < r < s < t < s + \frac{T}{2k}$ for $\ell = 1,\dots,k-1$, then, by Lemma 3.1,

$$(X(t) - X(s)) \wedge (X(s) - X(r) \le X^2(\frac{\ell+1}{k}T) - X^2(\frac{\ell-1}{k}T).$$

Hence

$$\Delta(\frac{1}{2k}T, X, T) \le \max_{1\le \ell \le k-1}\left\{X^2(\frac{\ell+1}{k}T) - X^2(\frac{\ell-1}{k}T)\right\}.$$

Denote the right hand side of the above inequality $E(k, X^2)$. We have

$$\limsup_{n\to\infty} P(E(k, X_n^{2,\epsilon}) > 2\epsilon) \le P(E(k, X_\infty^{2,\epsilon}) \ge 2\epsilon)$$

by (i). If k is sufficiently large, then $E(k, X_\infty^{2,\epsilon}(\omega)) < \frac{3}{2}\epsilon$. Hence

$$\lim_{k\to\infty} E(k, X_\infty^{2,\epsilon}(\omega)) \vee \frac{3}{2}\epsilon - \frac{3}{2}\epsilon = 0.$$

Then,

$$\begin{aligned} P(E(k, X_\infty^{2,\epsilon}) > 2\epsilon) &\le P(E(k, X_\infty^{2,\epsilon}) \vee \frac{3}{2}\epsilon > 2\epsilon) \\ &= P(E(k, X_\infty^{2,\epsilon}) \vee \frac{3}{2}\epsilon - \frac{3}{2}\epsilon > \frac{1}{2}\epsilon) \to 0 \text{ as } k \to \infty. \end{aligned} \tag{4}$$

This shows that

$$\begin{aligned} P(\Delta(\frac{1}{2k}, X_n, T) > 2\epsilon) &\le P(\Delta(\frac{1}{2k}, X_n, T) > 2\epsilon : \Omega_{T/k}^\epsilon(X_n,T)) + P((\Omega_{T/k}^\epsilon(X_n,T)^c) \\ &\le P(E(k, X_n^{2,\epsilon}) > 2\epsilon) + P((\Omega_{T/k}^\epsilon(X_n,T)^c) \end{aligned}$$

By assumption (ii) and (4), we have, for eqch $T > 0$ and $\epsilon > 0$,

$$\lim_{k\to\infty}\limsup_{n\to\infty} P(\Delta(\frac{1}{2k}, X_n, T) > 2\epsilon) = 0.$$

Appealing to Stone's theorem, we have the conclusion. □

We say that a stochastic process X is an additive process in law if it has independent increments, $X(0) = 0$ a.s. and it is stochastically continuous. Moreover, if the sample paths are contained in $D[0,\infty)$ with probability one, then we call X simply additive process. See [3] for detail.

Theorem 3.2 *Let $\{X_n\}$ be a sequence of additive processes having non-decreasing sample paths. If X_n converges to a stochastically continuous process X_∞ in finite dimensional distributions sense, then there is an additive process X in $D[0,\infty)$ with non-decreasing sample paths which is equivalent to X_∞ and X_n converges to X weakly under J_1-topology as $n \to \infty$.*

Proof If the finite dimensional distributions of X_n converge to the finite dimensional distributions of X_∞, then X_∞ is an additive process in law. It has a modification X whose sample paths are contained in $[0,\infty)$ and non-decreasing. For the convergence of X_n to X_∞ in finite dimensional distributions sense, it is necessary and sufficient that for every $t \ge 0$ the distribution of $X_n(t)$ converges to the distribution of $X_\infty(t)$. Let

$$Ee^{-zX_n(t)} = \exp\left(-\gamma_n(t) + \int_0^\infty (e^{-zx}-1)\nu_n(t,dx)\right)$$

and

$$Ee^{-zX_\infty(t)} = \exp\left(-\gamma_\infty(t) + \int_0^\infty (e^{-zx}-1)\nu\infty(t,dx)\right).$$

For the convergence of the distribution of $X_n(t)$ to the distribution of $X_\infty(t)$, it is necessary and sufficient that the Lévy measure $\nu_n(t,A)$ converges to the Lévy measure $\nu_\infty(t,A)$ for every continuity Borel set A of ν_∞ which is outside of a neighbourhood of 0 and $t \ge 0$ and $\gamma_n(t) \to \gamma(t)_\infty$ ([3]). This shows that for any $\epsilon > 0$, $X_n^{2,\epsilon}$ converges to $X_\infty^{2,\epsilon}$ as $n \to \infty$ in finite dimensional distributions sense since $\nu_\infty^{2,\epsilon}(t,\cdot) = \nu_n(t,\cdot\cap(0,\epsilon])$. Let $s_\ell = \frac{\ell}{k}T$ for $\ell = 0,1,\dots,k$ and $t_\ell = \frac{2\ell+1}{2k}T$ for $\ell = 0,1,\dots,k-1$. If the number of jumps of $X_n^{1,\epsilon}(t,\omega)$ contained in each interval $(s_\ell, s_{\ell+1}]$, $\ell = 0,1,\dots,k-1$ and $(t_\ell,t_{\ell+1}]$, $\ell = 0,1,\dots,k-2$ are atmost one, then $\omega \in \Omega^\epsilon_{T/k}(X_n,T)$. Probability that the number of jumps of $X_n^{1,\epsilon}(t,\omega)$ contained in each intervals $(s_\ell,s_{\ell+1}]$, $\ell = 0,1,\dots,k-1$ are atmost one is

$$\begin{aligned}&\prod_{\ell=0}^{k-1} e^{-\nu_n((s_\ell,s_{\ell+1}],(\epsilon,\infty))}(1+\nu_n((s_\ell,s_{\ell+1}],(\epsilon,\infty)))\\ = \;& e^{-\nu_n((0,T],(\epsilon,\infty))}\prod_{\ell=0}^{k-1}(1+\nu_n((s_\ell,s_{\ell+1}],(\epsilon,\infty))).\end{aligned}$$

By the convergence of X_n to X_∞ in finite dimensional distributions sense,

$$\begin{aligned}&\liminf_{n\to\infty} e^{-\nu_n((0,T],(\epsilon,\infty))}\prod_{\ell=0}^{k-1}(1+\nu_n((s_\ell,s_{\ell+1}],(\epsilon,\infty)))\\ \ge\;& e^{-\nu_\infty((0,T],(\epsilon,\infty))}\prod_{\ell=0}^{k-1}(1+\nu_\infty((s_\ell,s_{\ell+1}],(\epsilon,\infty))) \quad \text{as } n\to\infty.\end{aligned}$$

We have

$$\begin{aligned}&e^{-\nu_\infty((0,T],(\epsilon,\infty))}\prod_{\ell=0}^{k-1}(1+\nu_\infty((s_\ell,s_{\ell+1}],(\epsilon,\infty)))\\ \ge\;& \exp\left[-\nu_\infty((0,T],(\epsilon,\infty)) + \nu_\infty((0,T],(\epsilon,\infty))\left(1-\max_{0\le\ell\le k-1}\frac{1}{2}\nu_\infty((s_\ell,s_{\ell+1}],(\epsilon,\infty))\right)\right]\end{aligned}$$

by the inequality $\log(1+x) \ge x - \frac{1}{2}x^2$ for $x \ge 0$. Since X_∞ is stochastically continuous, $\nu_\infty(t,\cdot)$ is continuous in t and hence $\max_{0\le\ell\le k-1}\nu_\infty((\frac{\ell}{k}T,\frac{\ell+1}{k}T],(\epsilon,\infty)) \to 0$ as $k\to\infty$. The right hand side of the above inequality tends to 1 as $k \to \infty$. Similarly, we have that the probability that the number of jumps of $X_n^{1,\epsilon}(t,\omega)$ contained in each intervals $(t_\ell,t_{\ell+1}]$, $\ell = 0,1,\dots,k-2$ are atmost one tends to 1 as $n\to\infty$ and then $k\to\infty$. Hence

$$\lim_{k\to\infty}\limsup_{n\to\infty} P(\Omega^\epsilon_{T/k}(X_n,T)) = 1.$$

By appealing to Theorem 3.1 we have the weak converge under J_1-topology of $X_n \to X$. □

Skorohod [6] obtains the same result for general Lévy processes .

Theorem 3.3 *Let for each $n \geq 1$, $\{t_{n,k}\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive numbers such that $\lim_{k\to\infty} t_{n,k} = \infty$ and $\lim_{n\to\infty} \sup_{k\geq 1}(t_{n,k+1} - t_{n,k}) = 0$. Let for each $n \geq 1$, $\{Y_{n,k}\}_{k=1}^{\infty}$ be a sequence of independent nonnegative random variables such that for each $\epsilon > 0$,*

$$\lim_{n\to\infty} \max_{k\geq 1} P(Y_{n,k} > \epsilon) = 0.$$

If there is a measure $\nu(t)$ on $(0,\infty)$ continuous in $t > 0$ such that for each $t > 0$,

$$\lim_{n\to\infty} \sum_{t_{n,k}\leq t} P(Y_{n,k} > x) = \nu(t,(x,\infty)) \tag{5}$$

at every continuity point $x > 0$ of $\nu(t)$ and

$$\begin{aligned} &\lim_{\epsilon\to 0} \limsup_{n\to\infty} \sum_{t_{n,k}\leq t} \left(\int_{0<x<\epsilon} x^2 P(Y_{n,k} \in dx) - \left(\int_{0<x<\epsilon} xP(Y_{n,k} \in dx) \right)^2 \right) \\ = &\lim_{\epsilon\to 0} \liminf_{n\to\infty} \sum_{t_{n,k}\leq t} \left(\int_{0<x<\epsilon} x^2 P(Y_{n,k} \in dx) - \left(\int_{0<x<\epsilon} xP(Y_{n,k} \in dx) \right)^2 \right) = 0, \end{aligned}$$

$$\lim_{\epsilon\to 0} \lim_{n\to\infty} \sum_{t_{n,k}\leq t} \int_{0<x<\epsilon} xP(Y_{n,k} \in dx) = \gamma(t),$$

then $X_n(t) = \sum_{t_{n,k}\leq t} Y_{n,k}$ converges weakly under J_1-topology to an additive process $\{X(t)\}$ with nondecreasing sample paths such that

$$E(e^{-\theta X(t)}) = \exp\left(-\gamma(t) + \int_0^\infty (e^{-\theta x} - 1)\nu(t,dx) \right)$$

Proof Convergence of finite dimensional distributions of $X_n(t)$ to those of $X(t)$ is straightforward by the convergence of the distribution of triangular array of independent random variables to a infinitely divisible distribution. We show weak convergence under J_1-topology applying Theorem 3.1. Let $T > 0$ and let $s_\ell = \frac{\ell}{k}T$ for $\ell = 0,1,\ldots,k$ and $t_\ell = \frac{2\ell+1}{2k}T$ for $\ell = 0,1,\ldots,k-1$. If the number of jumps of $X_n^{1,\epsilon}(t,\omega)$ contained in each interval $(s_\ell, s_{\ell+1}]$, $\ell = 0,1,\ldots,k-1$ and $(t_\ell, t_{\ell+1}]$, $\ell = 0,1,\ldots,k-2$ are atmost one, then $\omega \in \Omega^{\epsilon}_{T/k}(X_n, T)$. The probability that there is no jump of $X_n(t)$ greater than ϵ in the interval $(s_\ell, s_{\ell+1}]$ is $\prod_{s_\ell<t_{n,j}\leq s_{\ell+1}} P_{n,j}([0,\epsilon])$ and the probability that there is one jump of $X_n(t)$ in $(s_\ell, s_{\ell+1}]$ is

$$\sum_{s_\ell<t_{n,j}\leq s_{\ell+1}} P_{n,j}((\epsilon,\infty)) \times \prod_{k\neq j, t_{n,k}\in(s_\ell,s_{\ell+1}]} P_{n,k}([0,\epsilon]).$$

The probability that there is atmost one jump of $X_n(t)$ in each $(s_\ell, s_{\ell+1}]$ is

$$\prod_{\ell=0}^{k-1} \left(\prod_{s_\ell<t_{n,j}\leq s_{\ell+1}} (1 - P_{n,j}((\epsilon,\infty))) \right) \left(1 + \sum_{s_\ell<t_{n,j}\leq s_{\ell+1}} P_{n,j}((\epsilon,\infty))/P_{n,j}([0,\epsilon]) \right). \tag{6}$$

Now, we use the inequalities $\log(1+x) \geq x - \frac{1}{2}x^2$ for $x > 0$ and $\log(1-x) \geq -x - x^2$ for $0 \leq x \leq \frac{1}{2}$. We have

(6)

$$\geq \exp\Big(\sum_{\ell=0}^{k-1}\Big[\sum_{s_\ell<t_{n,j}\leq s_{\ell+1}} \log(1-P_{n,j}((\epsilon,\infty))) + \log\Big(1+\sum_{s_\ell<t_{n,j}\leq s_{\ell+1}} P_{n,j}((\epsilon,\infty))/P_{n,j}([0,\epsilon])\Big)\Big]\Big)$$

$$\geq \exp\Big[\sum_{\ell=0}^{k-1}\Big\{\sum_{s_\ell<t_{n,j}\leq s_{\ell+1}}(-P_{n,j}((\epsilon,\infty))-\{P_{n,j}((\epsilon,\infty))\}^2) + \sum_{s_\ell<t_{n,j}\leq s_{\ell+1}} P_{n,j}((\epsilon,\infty))/P_{n,j}([0,\epsilon]) - \frac{1}{2}\Big(\sum_{s_\ell<t_{n,j}\leq s_{\ell+1}} P_{n,j}((\epsilon,\infty))/P_{n,j}([0,\epsilon])\Big)^2\Big\}\Big]$$

$$\geq \exp\Big[\sum_{\ell=0}^{k-1}\Big\{-\sum_{s_\ell<t_{n,j}\leq s_{\ell+1}} P_{n,j}((\epsilon,\infty))(1+\max_i P_{n,i}((\epsilon,\infty))) + \frac{1}{\max_i P_{n,j}([0,\epsilon])}\sum_{s_\ell<t_{n,j}\leq s_{\ell+1}} P_{n,j}((\epsilon,\infty))(1-\frac{1}{2\min_i P_{n,i}([0,\epsilon])}\sum_{s_\ell<t_{n,i}\leq s_{\ell+1}} P_{n,i}((\epsilon,\infty)))\Big\}\Big].$$

Since $\lim_{n\to\infty}\max_i P_{n,i}((\epsilon,\infty)) = 0$ and $\lim_{n\to\infty}\min_i P_{n,i}([0,\epsilon]) = 1$ by the assumption, the assumption (5) yields that the inferior limit as $n\to\infty$ of the last term of the above inequalities is greater than or equal to

$$\exp\Big[\sum_{\ell=0}^{k-1}\Big\{-\nu((s_\ell,s_{\ell+1}],((\epsilon,\infty)) + \nu((s_\ell,s_{\ell+1}],(\epsilon,\infty))\Big(1-\frac{1}{2}\nu((s_\ell,s_{\ell+1}],(\epsilon,\infty))\Big)\Big\}\Big].$$

Here $\nu([(s,t],\cdot) = \nu(t,\cdot)-\nu(s,\cdot)$. By the continuity of $\nu(t,\cdot)$ in t, the above quantity tends to 1 as $k\to\infty$. In the same way we have that the probability that $X_n(t)$ has atmost one jump greater than ϵ in each semiclosed interval $(t_\ell,t_{\ell+1}]$, $\ell=0,\ldots,k-2$, tends to 1 as $n\to\infty$ and then $k\to\infty$. Hence $\lim_{\delta\to 0}\limsup_{n\to\infty} P(\Omega_\delta^\epsilon(X_n,T))$. Since $X_n^{2,\epsilon}(t) = \sum_{t_j\leq t} Y_{n,j}1_{[0,\epsilon]}(Y_{n,j})$ and $P(Y_{n,j}1_{[0,\epsilon]}(Y_{n,j})\in B) = P_{n,j}((\epsilon,\infty))\delta_0(B) + P_{n,j}(B\cap[0,\epsilon])$, convergence of $X_n^{2,\epsilon}$ to $X^{2,\epsilon}$ for eqach $\epsilon>0$ in finite dimensional distributions sense is automatic from the convergence in finite distributions sense of X_n to X.

The above theorem guarantees the validity of Corollary 3.5 in [5].

Theorem 3.4 *Let for each $n\geq 1$, $\{Y_{n,i}\}_{i\geq 1}$ is a sequence of independent nonnegative random variables and let*

$$Y_n(t) = k \quad \text{if } \textstyle\sum_{j<k} Y_{n,j} < t \leq \sum_{j\leq k} Y_{n,j}.$$

If $\{X_n(t) = c(n)Y_n(nt)\}$ converges to $\{X(t)\}$ in finite dimensional distributions sense with $c(n)$ satisfying $\lim_{n\to\infty} c(n) = 0$, then X_n converges to X weakly under J_1-topology as $n\to\infty$.

Proof Since $X_n^{2,\epsilon}(t) = X_n(t)$ for $\epsilon > c(n)$, the processes $\{X_n\}$ satisfies the condition (i) of Theorem 3.1 and hence there is no jump greater than ϵ. This means that the condition (ii) is also satisfied. Hence X_n converges weakly under J_1-topology.

4 Limit theorems for hitting times of generalized diffusions

Let m be a non-decreasing, right continuous function defined on $[-\infty,\infty]$ to $[-\infty,\infty]$, which has left limit on $(0,\infty)$, satisfying $m(-\infty)=-\infty$, $m(\infty)=\infty$, $m(0-)=0$. We set

$$\begin{aligned}\ell_1 &= \sup\{x<0: m(x)=-\infty\}\\ \ell_2 &= \inf\{x>0: m(x)=\infty\}\end{aligned}$$

We denote by E_m the support of the measure induced by m restricted to (ℓ_1,ℓ_2). There naturally corresponds a strong Markov process $\{X_t\}$, called 1-dimensional generalized diffusion process on E_m, whose formal infinitesimal generator is $\frac{d}{dm}\frac{d}{dx}$, to m by changing the time of the Brownian motion (see [8]). The measure $m(dx)$ is called the speed measure of $\{X_t\}$. Denote the hitting time of x for $\{X_t\}$ by τ_x. The hitting time τ_x can be regarded as a stochastic process with time parameter $x\in E_m$ with independent increments. Moreover, this process can be extended to $x\in(\ell_1,\ell_2)$. This extended process is called generalized hitting time process in [8]. In [8], limit distribution of τ_x when the process starts at the origin and x tends to ℓ_2. Some of results in [8] are concerned with convergence of finite dimensional distributions. Since generalized hitting time process has non-decreasing sample paths, by Theorem 3.2, finite dimensional distributions leads to J_1-convergence.

Following [8], we introduce assumptions for asymptotics of m.
(C_γ) $(0\le\gamma<1)$: $\ell_2=\infty$ and $m(x)\sim x^{\gamma/(1-\gamma)}L(x)/(1-\gamma)$ as $x\to\infty$ with a function L slowly varying at ∞.
(C_1) $\ell_2=\infty$ and there is a function $s(x)$ regularly varying at ∞ and differentiable for large x such that the derivative $s'(s)$ is positive and monotone and

$$m(s(x))\sim e^x L(e^x) \text{ as } x\to\infty$$

with a function L slowly varying at ∞.
(C_γ) $(1<\gamma<\infty)$: $\ell_2<\infty$ and $m(x)\sim(\ell_2-x)^{\gamma/(1-\gamma)}K(\ell_2-x)/(\gamma-1)$ as $x\to\ell_2$ with a function K slowly varying at 0.
(C_∞) : $\ell_2<\infty$ and $m(x)\sim(\ell_2-x)^{-1}K(\ell_2-x)^{-1}$ as $x\to\ell_2$ with a function K slowly varying at 0.

Define $a(x)=\int_0^x m(y)dy$.

Theorem 4.1 *Assume that $\ell_1=-\infty$, (C_γ) holds for some $0\le\gamma<1$ and*

$$m(-x)/m(x)\to c\in[0,\infty) \text{ as } x\to\infty.$$

Then, the normalized generalized hitting time process $\{\tau_{tx}/a(x)\}$ with any starting point converges weakly under J_1-topology on $D[0,\infty)$ to a selfsimilar additive process $\{\tau_t^0\}$ with exponent $(1-\gamma)^{-1}$ as $x\to\infty$. The process $\{\tau_t^0\}$ is the hitting time process of the generalized diffusion process with speed measure

$$m^0(x)=\begin{cases} x^{\gamma/(1-\gamma)}/(1-\gamma), & x\ge 0,\\ -c|x|^{\gamma/(1-\gamma)}/(1-\gamma), & x<0.\end{cases}$$

starting at the origin

Theorem 4.2 *Assume that* $\ell_1 = -\infty$. *Let* $\widetilde{m}(x) = -m((-x)+) + m(0)$ *and* $\widetilde{a}(x) = \int_0^x |m(-y)|dy$. *Assume that* $\widetilde{m}$ *satisfies* (C_γ) *for some* $0 \le \gamma < 1$ *and*

$$a(x)/\widetilde{a}(x) \to 0 \text{ as } x \to \infty.$$

Then, the normalized generalized hitting time process $\{\tau_{tx}/\widetilde{a}(x)\}$ *with any starting point converges weakly under* J_1*-topology on* $D[0,\infty)$ *to a selfsimilar additive process* $\{\tau_t^0\}$ *with exponent* $(1-\gamma)^{-1}$ *as* $n \to \infty$. *The process* $\{\tau_t^0\}$ *is the hitting time process of the generalized diffusion process with speed measure*

$$m^0(x) = \begin{cases} 0, & x \ge 0, \\ -c|x|^{\gamma/(1-\gamma)}/(1-\gamma), & x < 0, \end{cases}$$

starting at the origin

Theorem 4.3 *Assume that* $\ell_1 = -\infty$, (C_1) *holds and*

$$x|m(-tx)|/a(x) \to 0 \text{ as } x \to \infty.$$

Then, the normalized generalized hitting time process $\{\tau_{s(t+x)}/a(s(x))\}$ *with any starting point converges weakly under* J_1*-topology on* $D[0,\infty)$ *to a process* $\{\tau_t^0\}$ *as* $x \to \infty$. *The process* $\{\tau_t^0\}$ *is the hitting time process of the generalized diffusion process with speed measure* $m^0(t) = e^t$, $t \in \mathbb{R}$ *and starting point* $-\infty$. $\{\tau^0_{\log t}\}_{t\ge 0}$ *is a selfsimilar additive process with exponent 1.*

Theorem 4.4 *Assume that* $\ell_1 = -\infty$, (C_γ) *holds for some* $1 < \gamma < \infty$. *Then, the normalized generalized hitting time process* $\{\tau_{\ell_2 - y|t|}/a(\ell_2 - y)\}_{t<0}$ *with any starting point converges weakly under* J_1*-topology on* $D[0,\infty)$ *to a process* $\{\tau_t^0\}$ *as* $y \downarrow 0$. *The process* $\{\tau_t^0\}$ *is the hitting time process of the generalized diffusion process with speed measure*

$$m^0(x) = \begin{cases} |x|^{\gamma/(1-\gamma)}/(\gamma-1), & x < 0, \\ \infty, & x > 0 \end{cases}$$

and starting point $-\infty$. $\{\tau^0_{t^{-1}}\}_{t\ge 0}$ *is a selfsimilar additive process with exponent* $(\gamma-1)^{-1}$.

Theorem 4.1 corresponds to Theorem 2, Theorem 4.2 corresponds to Theorem 3, Theorem 4.3 corresponds to Theorem 4 and Theorem 4.4 corresponds to Theorem 6 in [8], respectively.

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