Uniform Sets and Complexity

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An element $\omega \in \{0, 1\}^N$ is called an infinite 0-1-word which is a mapping from $\mathbb{N}$ to $\{0, 1\}$, while it is also considered as an infinite sequence $\omega(0)\omega(1)\omega(2)\cdots$ of 0 and 1. On the other hand, an element $u$ in $\{0, 1\}^* := \bigcup_{k=0}^{\infty} \{0, 1\}^k$ is called a finite 0-1-word and represented as a finite sequence $u_1u_2\cdots u_k$ of 0 and 1, where $k = |u|$ is called the length of $u$ and is denoted by $|u|$. We also denote $\{0, 1\}^+ = \bigcup_{k=1}^{\infty} \{0, 1\}^k$.

The concatenation $uv$ or $u\omega$ of $u \in \{0, 1\}^*$ with $v \in \{0, 1\}^*$ or $\omega \in \{0, 1\}^N$ is defined as the word $u_1u_2\cdots u_kv_1v_2\cdots v_l$ or $u_1u_2\cdots u_k\omega(0)\omega(1)\omega(2)\cdots$, where $k = |u|$ and $l = |v|$, respectively. In this case, $u$ is called a prefix of $uv$ or $u\omega$, or equivalently, $uv$ or $u\omega$ is called an extension of $u$.

For $u \in \{0, 1\}^*$, the cylinder set $[u]$ determined by $u$ is defined by $[u] = \{ \omega \in \{0, 1\}^N; u$ is a prefix of $\omega\}$.

The prefix tree $G(\Omega) = (V, E)$ of a nonempty closed set $\Omega \subset \{0, 1\}^N$ is defined to be a directed graph such that the set $V$ of vertices is the set of cylinder sets $[u]$ which meet $\Omega$, and the set $E$ of edges is the set of the ordered pairs $([u], [v]) \in V \times V$ such that $v$ is an immediate extension of $u$, that is, $u$ is the prefix of $v$ such that $|v| = |u| + 1$.

Two nonempty closed sets $\Omega$, $\Lambda \subset \{0, 1\}^N$ are said to be isomorphic to each other if their prefix trees are isomorphic to each other. The class of all closed subsets of $\{0, 1\}^N$ isomorphic to $\Omega$ is denoted by $[\Omega]$ and is called the language structure of (or determined by) $\Omega$.

Define

$$\Theta_0 := \{0^\infty\}, \quad \Theta_1 := \{1^\infty\},$$

$$\Theta_\delta := \{\omega \in \{0, 1\}^N; \sum_{n\in\mathbb{N}} \omega(n) \leq 1\},$$

$$\Theta_{1-\delta} := \{\omega \in \{0, 1\}^N; \sum_{n\in\mathbb{N}} (1 - \omega(n)) \leq 1\},$$

$$\Theta_+ := \{\omega \in \{0, 1\}^N; \omega \text{ is increasing}\},$$

$$\Theta_- := \{\omega \in \{0, 1\}^N; \omega \text{ is decreasing}\},$$

$$\Theta_t := \{\omega \in \{0, 1\}^N; \omega \text{ is periodic}\}.$$
where $a^\infty = a a a \cdots$ for $a \in \{0, 1\}$ and $\omega \in \{0, 1\}^N$ is called increasing (decreasing) if $\omega(n) \leq \omega(m)$ ($\omega(n) \geq \omega(m)$, respectively) for any $n < m$.

All of $\Theta_\delta$, $\Theta_{1-\delta}$, $\Theta_+$, $\Theta_-$ are isomorphic to each other since for example, $G(\Theta_\delta)$ and $G(\Theta_+)$ are isomorphic (Figure 1). It also holds that $\Theta_\delta \cup \Theta_-$ and $\Theta_+ \cup \Theta_-$ are isomorphic, while $\Theta_\delta \cup \Theta_+$ is not isomorphic to $\Theta_\delta \cup \Theta_-$ (Figure 2).

**Definition 1.** For a nonempty closed set $\Omega \subset \{0, 1\}^\Sigma$, define the complexity function $p_\Omega(S) := \#\pi_S \Omega$, which is a function of finite sets $S \subset \Sigma$, where $\#$ denotes the number of elements in a set and $\pi_S : \{0, 1\}^\Sigma \to \{0, 1\}^S$ is the projection. We call $\Omega$ a uniform set if $p_\Omega(S)$ depends only on $\#S$. In this case, the function $p_\Omega(k) := p_\Omega(S)$ of $k = 1, 2, \cdots$, where $\#S = k$, is called the uniform complexity function of $\Omega$. We also define the maximal pattern complexity function of $\Omega$ as $p_\Omega^*(k) := \sup S : \#S = k p_\Omega(S)$ ($k = 1, 2, \cdots$). Note that $p_\Omega(k) = p_\Omega^*(k)$ ($k = 1, 2, \cdots$) if $\Omega$ is a uniform set.

Let $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\}$ be an infinite subset of $\mathbb{N}$. For $\omega \in \{0, 1\}^\Sigma$.
\(\{0,1\}^N\) and \(\Omega \subset \{0,1\}^N\), define \(\omega[N] \in \{0,1\}^N\) and \(\Omega[N] \subset \{0,1\}^N\) by
\[
\omega[N](n) := \omega(N_n) \quad (n \in N)
\]
\[
\Omega[N] := \{\omega[N] \in \{0,1\}^N; \omega \in \Omega\}.
\]
We sometimes identify the infinite subset \(N \subset N\) with an increasing injection \(n \mapsto N_n\) from \(N\) into itself. We use the same notation \(\Omega[S]\) for a finite set \(S = \{s_1 < s_2 < \cdots < s_k\} \subset N\) in the sense that
\[
\Omega[S] = \{\omega(s_1)\omega(s_2)\cdots\omega(s_k) \in \{0,1\}^k; \omega \in \Omega\}.
\]
For \(\Omega \subset \{0,1\}^\Sigma\), where \(\Sigma\) is a countably infinite set, and an injection \(\psi : N \to \Sigma\), denote
\[
\Omega \circ \psi := \{\omega \circ \psi \in \{0,1\}^N; \omega \in \Omega\}.
\]
Note that if \(\Omega\) is a uniform set, then \(\Omega \circ \psi\) is also a uniform set with the same complexity function.

**Definition 2.** A nonempty closed set \(\Omega \subset \{0,1\}^N\) is called a super-stationary set if \(\Omega[N] = \Omega\) holds for any infinite subset \(N\) of \(N\). Note that a super-stationary set is a uniform set and all of \(\Theta_0, \Theta_1, \Theta_{\delta}, \Theta_{1-\delta}, \Theta_{+}, \Theta_{-}\) are super-stationary sets.

**Definition 3.** A nonempty closed set \(\Omega \subset \{0,1\}^\Sigma\) is said to have a primitive factor \([\Omega \circ \phi]\) if \(\Omega \circ \phi\) is a super-stationary set, where \(\phi : N \to \Sigma\) is an injection and \([\Omega \circ \phi]\) is the language structure determined by \(\Omega \circ \phi\).

**Definition 4.** Let \(\Omega \subset \{0,1\}^N\) be a nonempty closed set. For \(\omega \in \Omega\) and \(k \in N\), we denote \(\omega|_k = \omega(0)\omega(1)\cdots\omega(k-1) \in \{0,1\}^k\). Let \(\Omega'\) be the set of accumulating points of \(\Omega\), that is,
\[
\Omega' = \{\omega \in \Omega; \#([\omega|_k] \cap \Omega) = \infty \text{ for any } k \in N\}.
\]
We call \(\Omega'\) the derived set of \(\Omega\). Clearly, \(\Omega'\) is a closed set (possibly, the empty set). We denote \(\Omega^{(0)} = \Omega\) and \(\Omega^{(i)} = (\Omega^{(i-1)})'\) for \(i = 1,2,\cdots\). The degree of \(\Omega\) is defined to be \(\delta = 0,1,2,\cdots\) such that \(\Omega^{(\delta)} \neq \emptyset\) and \(\Omega^{(\delta+1)} = \emptyset\), if such \(\delta\) exists, otherwise, \(\infty\). The degree of \(\Omega\) is denoted by \(\deg \Omega\). For completeness, we define \(\emptyset' = \emptyset\) and \(\deg \emptyset = -1\).

We have the following results.

**Theorem 5.** (Kamae [1]) Let \(\Omega\) be a nonempty closed subset of \(\{0,1\}^\Sigma\), where \(\Sigma\) is a countably infinite set.
(1) If there exists an injection \(\psi : N \to \Sigma\) such that \(\deg(\Omega \circ \rho) < \infty\), then there exists an increasing injection \(\phi : N \to N\) such that \(\Omega \circ \phi\) is a super-stationary set.
(2) If \(\deg(\Omega \circ \rho) = \infty\) for any injection \(\rho : N \to \Sigma\), then \(p_{\Omega}(k) = 2^k\) \((k = 1,2,\cdots)\).

Hence, any uniform set has a primitive factor and any uniform complexity function is realized by a super-stationary set.
Remark 6. (1) of the Main Theorem can be generalized easily to the case of general finite alphabet.

For $\xi = \xi_1 \xi_2 \cdots \xi_k \in \{0, 1\}^k$ and $\eta = \eta_1 \eta_2 \cdots \eta_l \in \{0, 1\}^l$ with $k \leq l$, we say that $\xi$ is a super-subword of $\eta$, if $\xi = \eta_{s_1} \eta_{s_2} \cdots \eta_{s_k}$ holds for some $1 \leq s_1 < s_2 < \cdots < s_k \leq l$. For this $\xi$ and $\omega \in \{0, 1\}^N$, we say that $\xi$ is a super-subword of $\omega$, if $\xi = \omega(s_1) \omega(s_2) \cdots \omega(s_k)$ holds for some $0 \leq s_1 < s_2 < \cdots < s_k < \infty$. In these cases, we denote $\xi \ll \eta$ or $\xi \ll \omega$.

For $\xi \in \{0, 1\}^*$, denote

$$\mathcal{P}(\xi) := \{\omega \in \{0, 1\}^N; \xi \ll \omega \text{ does not hold}\},$$

that is, $\mathcal{P}(\xi)$ is the set of infinite 0-1-words with the prohibited word $\xi$ as its super-subword. Denote for $\Xi \subset \{0, 1\}^*$,

$$\mathcal{Q}(\Xi) := \bigcup_{\xi \in \Xi} \mathcal{P}(\xi)$$

and

$$\mathcal{P}(\Xi) := \bigcap_{\xi \in \Xi} \mathcal{P}(\xi).$$

We call $\eta \in \{0, 1\}^* \cup \{0, 1\}^N$ a cover of $\Xi$ if $\xi \ll \eta$ holds for any $\xi \in \Xi$. It is called a minimal cover if in addition, any $\zeta \neq \eta$ is not a cover of $\Xi$. Let $L(\Xi)$ be the set of minimal covers of $\Xi$.

Theorem 7. (T. Kamae, H. Rao, B. Tan and Y-M. Xue [2]) (1) The class of super-stationary sets other than $\{0, 1\}^N$ coincides with the class of sets $\mathcal{Q}(\Xi)$ with nonempty finite sets $\Xi \subset \{0, 1\}^+$. It also coincides with the class of sets $\mathcal{P}(L(\Xi))$ with nonempty finite sets $\Xi \subset \{0, 1\}^+$.

(2) The complexity function $p_\Omega(k)$ of a super-stationary set $\Omega$ other than $\{0, 1\}^N$ is a polynomial function of $k$ for large $k$.

The following Corollary follows from above 2 theorems.

Corollary 8. The complexity function $p_\Omega(k)$ of a uniform set $\Omega$ is either $2^k$ ($k = 1, 2, \cdots$) or a polynomial function of $k$ for large $k$.

Let $X$ be a metrizable space with a continuous group or semi-group action $G$. For a family of subsets $A_1, A_2, \cdots, A_k$ of $X$, let $P(\{A_i; i = 1, 2, \cdots, k\})$ denote the partition of $X$ generated by these subsets, that is, the family of nonempty sets of the form

$$A_1^{i_1} \cap A_2^{i_2} \cap \cdots \cap A_k^{i_k} (i_1, i_2, \cdots, i_k \in \{0, 1\}),$$

where for a set $A \subset X$, we denote $A^1 = A$ and $A^0 = X \setminus A$.

Let $D$ be a nonempty subset of $X$. Define the maximal pattern complexity function $p_{X,G,D}^*(k)$ of the triple $(X, G, D)$ by

$$p_{X,G,D}^*(k) = \sup_{\tau \subset G, \#\tau = k} \#P(\sigma^{-1}D; \sigma \in \tau) \quad (k = 1, 2, \cdots). \quad (1)$$
Definition 9. For a set $U$ and $k \in \mathbb{N}$, $\mathcal{F}_k(U)$ denotes the family of sets $S \subset U$ with $\#S = k$. A countably infinite subset $\Sigma$ of $G$ is called an optimal position of the triple $(X, G, D)$ if

$$\#P(\{\sigma^{-1}D; \sigma \in \tau\}) = p^*_X,G,D(k),$$

holds for any $k = 1, 2, \cdots$ and $\tau \in \mathcal{F}_k(\Sigma)$.

Let $\Sigma \subset G$ be a countably infinite set. We call $\omega \in \{0, 1\}^\Sigma$ a name of the partition $P(\{\sigma^{-1}D; \sigma \in \Sigma\})$ if there exists $x \in X$ such that

$$\omega(\sigma) = \begin{cases} 1 & x \in \sigma^{-1}D \\ 0 & x \notin \sigma^{-1}D. \end{cases}$$

(3)

The closure of the set of names of the partition $P(\{\sigma^{-1}D; \sigma \in \Sigma\})$ is called the name set of $\Sigma$ with respect to the triple $(X, G, D)$.

The following theorem is clear from the definitions.

Theorem 10. The name set of any optimal position $\Sigma$ of a triple $(X, G, D)$ is a uniform set with the uniform complexity function $p^*_X,G,D$.

Example 11. Let $X = G = \mathbb{R}/\mathbb{Z}$. The action of $g \in G$ maps $x \in X$ to $x + g \in X$. Let $D$ be an interval $[a, b)$ in $X$ such that $a < b < a + 1$. Then, we have $p^*_{D, G}(k) = 2k (k = 1, 2, \cdots)$. In this case, a countably infinite subset $\Sigma$ of $G$ is an optimal position of $(X, G, D)$ if and only if for any $\sigma, \sigma' \in \Sigma$ with $\sigma \neq \sigma'$, $D - \sigma$ and $D - \sigma'$ intersect as well as their complements. This is also equivalent to the $\#P(\{\sigma^{-1}D, \sigma'^{-1}D\}) = 4$ for any $\{\sigma, \sigma'\} \in \mathcal{F}_2(\Sigma)$.

Let $\Omega$ be the name set of an optimal position $\Sigma$. Then, $\Omega$ is known to have the unique primitive factor $[\Theta_5 \cup \Theta_\perp] = [Q(11, 01)] ([?])$.

Example 12. Let $X = \mathbb{R}^2$ and $G = (\mathbb{R}/2\pi \mathbb{Z}) \times \mathbb{R}^2$. The action of $(\theta, (u, v))$ in $G$ maps $(x, y) \in X$ to the following $(x', y') \in X$:

$$\begin{cases} x' = x \cos \theta - y \sin \theta + u \\ y' = x \sin \theta + y \cos \theta + v. \end{cases}$$

Let $D$ be a line in $X$. Then, $g^{-1}D$ is also a line for any $g \in G$ and we have $p^*_{X, G, D}(k) = (1/2)k^2 + (1/2)k + 1 (k = 1, 2, \cdots)$. In this case, $\Sigma$ is an optimal position if and only if $\Sigma$ is a countably infinite subset of $G$ such that

(1) for any $\sigma, \sigma' \in \Sigma$ with $\sigma \neq \sigma'$, $\sigma^{-1}D \cap \sigma'^{-1}D \neq \emptyset$, and

(2) for any $\sigma, \sigma', \sigma'' \in \Sigma$ which are different each other,

$$\sigma^{-1}D \cap \sigma'^{-1}D \cap \sigma''^{-1}D = \emptyset.$$

Let $\Omega$ be the name set of an optimal position $\Sigma$. Then,

$$\Omega = \{\omega \in \{0, 1\}^\Sigma; \sum_{\sigma \in \Sigma} \omega(\sigma) \leq 2\}.$$

Hence, $\Omega$ has the unique primitive factor $[Q(111)]$. 

Example 13. (Y-M. Xue [5]) Let $X = G = \mathbb{R}^2$. The action of $g = (g_1, g_2) \in G$ maps $(x, y) \in X$ to $(x + g_1, y + g_2) \in X$. Let $D := \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$ be the unit disk. Then, we have $p_{X,G,D}^{*}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$. In this case, a countably infinite subset $\Sigma$ of $G$ is an optimal position if and only if $\#\mathcal{P}((\sigma^{-1}D; \sigma \in \tau)) = p_{X,G,D}^{*}(k) = 8$ for any $\tau \in \mathcal{F}_3(\Sigma)$. Moreover, $\Sigma$ satisfies this condition if $\Sigma \subset \{g \in G; g_1^2 + g_2^2 = r^2\}$ with $0 < r < 1$. Moreover, the name set $\Omega$ has a unique primitive factor $[\mathcal{Q}(101,010)]$.

All the examples so far admit a finitely determined optimal position. The following example does not admit an optimal position.

Example 14. Let $X = T_1 \cup T_2$ and $G = T_1 \times T_2$, where $T_i \cong \mathbb{R}/\mathbb{Z}$ $(i = 1, 2)$ and $T_1, T_2$ are disjoint each other. The action of $g = (g_1, g_2) \in G$ maps $x \in T_i$ to $x + g_i \in T_i$ for $i = 1, 2$. Let $D = [a_1, b_1] \cup [a_2, b_2]$, where $[a_i, b_i] \subset T_i$ and $a_i < b_i < a_{i+1}$ for $i = 1, 2$.

Then, we have $p_{X,G,D}^{*}(k) = 4k - 4$ $(k = 2, 3, \cdots)$. In this case, there is no optimal position since for any infinite subset $\Sigma$ of $G$, there exists a sequence $g_n = (g_{n,1}, g_{n,2}) \in \Sigma$ for $n = 1, 2, \cdots$ such that $g_{n,i}$ converges monotonously to, say $c_i \in T_i$, for $i = 1, 2$. Then, for any sufficiently large $n_0$, $\#\mathcal{P}((g_n^{-1}D; n = n_0 + 1,n_0 + 2,n_0 + 3)) = 6$ but not 8.

Definition 15. A nonempty closed set $\Omega \subset \{0,1\}^N$ is called a stationary set if $T\Omega = \Omega$, where $T : \{0,1\}^N \to \{0,1\}^N$ is the shift. Note that a super-stationary set is always stationary since $T\Omega = \Omega[\{1,2,\cdots\}]$. We call $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$ an optimal window of $\Omega$ if $p_{\Omega}(S) = p_{\Omega}^{*}(k)$ for any $k = 1, 2, \cdots$ and $S \subset \mathcal{N}$ with $\#S = k$.

Take a stationary set $\Omega \subset \{0,1\}^N$ as $X$ and the additive semi-group $\mathbb{N}$ as $G$. Let the action of $n \in \mathbb{N}$ to $\omega \in \Omega$ be $T^n\omega$. Let $D = \{\omega \in \Omega; \omega(0) = 1\}$. In this case, it is easy to see that $\Omega$.

Theorem 16. For an infinite subset $\mathcal{N}$ of $\mathbb{N}$, $\mathcal{N}$ is an optimal position of $(\Omega, \mathbb{N}, D)$ if and only if $\mathcal{N}$ is an optimal window of $\Omega$.

Hence, the following theorem follows from Theorem 4.1 of T. Kamae, H. Rao, B. Tan, Y-M. Xue [3].

Theorem 17. Let $\alpha \in \{0,1\}^\mathbb{N}$ be a recurrent pattern Sturmian word. Let $X = \overline{O}(\alpha)$, $G = \{T^n; n \in \mathbb{N}\}$ and $D = \{\omega \in \Omega; \omega(0) = 1\}$. Then, an optimal position of the triple $(X, G, D)$ exists.

Example 18. Let $\Omega = \overline{O}(\alpha)$ with the non-simple Toeplitz word $\alpha \in \{0,1\}^\mathbb{N}$ defined in Example 3 in N. Gjini, T. Kamae, B. Tan, and Y-M. Xue [4]. Then, $p_{\Omega}^{*}(k) = 2^k$ $(k = 1, 2, \cdots)$ holds. In this case, an optimal window does not exist. Take an arbitrary $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$. For any $k \in \mathbb{N}$, there exists $K \in \mathbb{N}$ with $K \geq k$ and $\xi \in \{0,1\}^K$ such that $\alpha = (\xi a_0)(\xi a_1)(\xi a_2)\cdots$ holds with $a_0, a_1, a_2, \cdots \in \{0,1\}$. There exists such
with the property that there exist 3 elements in $\mathcal{N}$, say $N_u < N_v < N_w$ with $N_u \not\equiv N_v \equiv N_w$ modulo $K + 1$. Then, either 001 or 101 is not in $\Omega[[N_u, N_v, N_w]]$. Hence, $\mathcal{N}$ is not an optimal window.

The following is the list of the language structures and the complexity functions with degree $\leq 1$.

(1) $[\Theta_0] = [Q(1)], p_\Omega(k) = 1,$
(2) $[\Theta_0 \cup \Theta_1] = [Q(0, 1)], p_\Omega(k) = 2,$
(3) $[\Theta_\delta] = [Q(11)], p_\Omega(k) = k + 1,$
(4) $[\Theta_\delta \cup \Theta_1] = [Q(11, 0)], p_\Omega(k) = k + 2 - 1_{k=1},$
(5) $[\Theta_\delta \cup \Theta_+] = [Q(11, 10)], p_\Omega(k) = 2k,$
(6) $[\Theta_\delta \cup \Theta_-] = [Q(11, 01)], p_\Omega(k) = 2k,$
(7) $[\Theta_\delta \cup \Theta_{1-\delta}] = [Q(11, 00)], p_\Omega(k) = 2k + 2 - 2 \cdot 1_{k=1},$
(8) $[\Theta_\delta \cup \Theta_+ \cup \Theta_-] = [Q(11, 10, 01)], p_\Omega(k) = 3k - 2 + 1_{k=1},$
(9) $[\Theta_\delta \cup \Theta_{1-\delta} \cup \Theta_+] = [Q(11, 00, 10)], p_\Omega(k) = 3k - 1 - 1_{k=2},$
(10) $[\Theta_\delta \cup \Theta_{1-\delta} \cup \Theta_+ \cup \Theta_-] = [Q(11, 00, 10, 01)], p_\Omega(k) = 4k - 4 + 2 \cdot 1_{k=1}.$

References


