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On a sufficient condition for starlikeness of meromorphic functions

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Abstract

A function $f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}$ is called convex of order $\alpha,\ 0<\alpha<1$ if $f(z)$ is analytic in $\mathbb{E}=\{z\,|\,|z|<1\}$ and satisfies the following inequality

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}>\alpha \quad \text{in } \mathbb{E}.$$

Then we denote by $f(z)\in\mathcal{K}(\alpha)$.
The family of starlike functions of order $\alpha,\ 0<\alpha<1$ shall be denoted by $S^{*}(\alpha)$ and is defined by the conditions that $f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}$ satisfies

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>\alpha \quad \text{in } \mathbb{E}.$$

Then it is well known that $f(z)\in\mathcal{K}(\alpha)$ implies $f(z)\in S^{*}(\beta)$ where

$$\beta=\begin{cases}\dfrac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})} & \text{if } \alpha\neq\dfrac{1}{2}\\[2ex] \dfrac{1}{2\log 2} & \text{if } \alpha=\dfrac{1}{2}.\end{cases}$$

The above result was completed by 3 papers by Jack [1], MacGregor [2] and Wilken and Feng [5].
In this paper, we will obtain the order of starlikeness of the function $F(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_{n}z^{n}$ which is meromorphic in $\mathbb{E}$ and satisfies the inequality

$$-\left(1+\mathrm{Re}\frac{zF''(z)}{F'(z)}\right)<\frac{3}{2}\alpha,\quad \frac{2}{3}<\alpha<1,\quad \text{in } \mathbb{E}.$$

1 Introduction

Let Σ denote the class of normalized functions $F(z)$ which are meromorphic in $\mathbb{E}$ and defined by

$$F(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_{n}z^{n}$$

with a simple pole at the origin.
Also let $\Sigma^{*}(\alpha),\ 0<\alpha<1$, denote the subclass of Σ consisting of the functions $F(z)$ which are univalent and starlike with respect to the origin in $\mathbb{E}$ or

$$-\mathrm{Re}\frac{zF'(z)}{F(z)}>\alpha \quad \text{in } \mathbb{E},$$

and let $\Sigma_k(\alpha)$, $0 < \alpha < 1$, denote the subclass of Σ consisting of the functions $F(z)$ which are univalent and convex in $\mathbb{E}$ or

$$-\left(1 + \mathrm{Re}\frac{zF''(z)}{F'(z)}\right) > \alpha \quad \text{in } \mathbb{E}.$$

2 Lemma

Lemma 1 *Let $p(z)$ be analytic in $\mathbb{E}$, $p(0) = 1$ and suppose that there exists a point $z_0 \in \mathbb{E}$ such that*

$$\mathrm{Re}\, p(z) > 0 \quad \textit{for} \quad |z| < |z_0|$$
$$\mathrm{Re}\, p(z_0) = 0 \quad \textit{and} \quad p(z_0) \neq 0.$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = iK$$

where K is real and

$$K \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \quad p(z_0) = ia \ \textit{and} \ 0 < a,$$

and

$$K \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \quad p(z_0) = -ia \ \textit{and} \ 0 < a.$$

A proof can be found in [3].

3 Theorem

Theorem 1 *Let $p(z)$ be analytic in $\mathbb{E}$, $p(0) = 1$ and suppose that*

$$\mathrm{Re}\left(p(z) - \frac{zp'(z)}{p(z)}\right) < \frac{3}{2}\alpha, \quad \frac{2}{3} < \alpha < 1 \quad \textit{in} \ \mathbb{E},$$

and suppose that for arbitrary r, $0 < r < 1$

$$\min_{|z| \leqq r} \mathrm{Re}\, p(z) = \mathrm{Re}\, p(z_0) \neq p(z_0), \quad |z_0| = r$$

or $\mathrm{Re}\, p(z)$ *on any circle $|z| = r$, $0 < r < 1$ does not take its minimum value on the real axis. Then we have*

$$\mathrm{Re}\, p(z) > \alpha \quad \textit{in} \ \mathbb{E}.$$

Proof. If there exists a point $z_0 \in \mathbb{E}$ such that

$$\mathrm{Re}\, p(z) > \alpha \quad \text{for} \quad |z| < |z_0|$$

and

$$\mathrm{Re}\, p(z_0) = \alpha \neq p(z_0),$$

putting

$$q(z) = \frac{p(z) - \alpha}{1 - \alpha}, \quad q(0) = 1$$

it follows that

$$\operatorname{Re} q(z) > 0 \quad \text{for} \quad |z| < |z_0|.$$

From the hypothesis of Theorem 1 and Lemma 1, we have

$$\operatorname{Re} q(z_0) = 0 \quad \text{and} \quad q(z_0) \neq 0$$

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{z_0 p'(z_0)}{p(z_0) - \alpha} = iK$$

where

$$K \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad p(z_0) - \alpha = ia \text{ and } 0 < a,$$

and

$$K \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad p(z_0) - \alpha = -ia \text{ and } 0 < a.$$

For the case, $p(z_0) - \alpha = ia$, $0 < a$, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{z_0 p'(z_0)}{p(z_0) - \alpha} \cdot \frac{p(z_0) - \alpha}{p(z_0)} = iK \frac{ia}{\alpha + ia}$$

$$= iK \frac{ia(\alpha - ia)}{\alpha^2 + a^2} = -\frac{\alpha a K - ia^2 K}{\alpha^2 + a^2}.$$

Putting

$$q(a) = \frac{1 + a^2}{\alpha^2 + a^2}, \quad 0 < a \quad \text{and} \quad 0 < \alpha < 1$$

then it follows that

$$q'(a) = \frac{2a(\alpha^2 - 1)}{(\alpha^2 + a^2)^2} < 0$$

and

$$\lim_{a \to \infty} q(a) = 1.$$

Therefore we have

$$\operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} = -\frac{\alpha}{\alpha^2 + a^2} a K$$

$$\leqq -\frac{\alpha}{2} \frac{1 + a^2}{\alpha^2 + a^2} \leqq -\frac{\alpha}{2}.$$

Then it follows that

$$\operatorname{Re}\left(p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) \geqq \alpha + \frac{\alpha}{2} = \frac{3}{2}\alpha.$$

This is a contradiction and for the case $p(z_0) - \alpha = -ia$, $0 < a$, applying the same method as the above, we also have a contradiction. It completes the proof. □

From Theorem 1, we have the following corollary.

Corollary 1 *Suppose that*

$$-\left(1 + \operatorname{Re} \frac{zF''(z)}{F'(z)}\right) < \frac{3}{2}\alpha, \quad \frac{2}{3} < \alpha < 1 \quad \textit{in} \;\; \mathbb{E}$$

and for arbitrary r, $0 < r < 1$

$$\min_{|z| \leqq r} \operatorname{Re}\left(-\frac{zF'(z)}{F(z)}\right) = \operatorname{Re}\left(-\frac{z_0 F'(z_0)}{F(z_0)}\right) \neq -\frac{z_0 F'(z_0)}{F(z_0)}$$

where $|z_0| = r$.
Then we have

$$F(z) \in \Sigma^*(\alpha).$$

In [4, Theorem 3 and 4], Robertson obtained the following result.

Robertson's result *Let* $F(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ *be meromorphic in* $\mathbb{E}$ *and suppose that*

$$F(z) \in \Sigma^*(0).$$

Then it follows that

$$-\left(1 + \mathrm{Re}\frac{zF''(z)}{F'(z)}\right) \geqq 0 \quad in \quad |z| \leqq \frac{1}{\sqrt{3}}.$$

References

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