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PROBLEMS ON QUANTUM GROUPS

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1. QUANTUM GROUPS

Throughout this survey, we will mainly treat quantum groups of compact or discrete type. Our standard references are [2, 9, 19, 45]. We denote by $\otimes$ minimal tensor products or spatial tensor products for C^*-algebras or von Neumann algebras, respectively. The leg notations are frequently used. For example, let $T\in B(H\otimes H)$, where H is a Hilbert space. Set the transposition of the i-th and j-th tensor components, $\sigma_{ij}\in B(H\otimes H\otimes H)$ for $i,j=1,2,3$. Then $T_{12}=T\otimes 1$, $T_{13}=\sigma_{23}T_{12}\sigma_{23}$ and so on.

1.1. Compact quantum groups

The following definition of a compact quantum group has been introduced by S. L. Woronowicz [45]:

Definition 1.1 (Woronowicz). A *compact quantum group* (c.q.g.) $\mathbb{G}$ is a pair $(C(\mathbb{G}),\delta)$ that satisfies the following conditions:

(1) $C(\mathbb{G})$ is a separable unital C^*-algebra;
(2) (Coproduct) The map $\delta\colon C(\mathbb{G})\to C(\mathbb{G})\otimes C(\mathbb{G})$ is a coproduct, i.e. it is a faithful unital $*$-homomorphism satisfying the co-associativity condition,
$$(\delta\otimes\mathrm{id})\circ\delta=(\mathrm{id}\otimes\delta)\circ\delta;$$
(3) (Cancellation property) The vector spaces $\delta(C(\mathbb{G}))(\mathbb{C}\otimes C(\mathbb{G}))$ and $\delta(C(\mathbb{G}))(C(\mathbb{G})\otimes\mathbb{C})$ are dense in $C(\mathbb{G})\otimes C(\mathbb{G})$.

Example 1.2. A compact group $\mathbb{G}$ is regarded as a compact quantum group. Indeed, via the identification $C(\mathbb{G})\otimes C(\mathbb{G})=C(\mathbb{G}\times\mathbb{G})$, a coproduct δ is defined by
$$\delta(x)(r,s):=x(rs)\text{ for all }x\in C(\mathbb{G}),r,s\in\mathbb{G}.$$
The cancellation property means $rs=rt$ or $sr=tr$ imply $s=t$ for $r,s,t\in\mathbb{G}$. Note that a compact semigroup with cancellation property is a compact group.

As in a compact group theory, the following state called *Haar state* plays an important role.

Theorem 1.3 (Woronowicz). *There uniquely exists a state* $h\in C(\mathbb{G})^*$ *such that*
$$(\mathrm{id}\otimes h)(\delta(a))=h(a)1=(h\otimes\mathrm{id})(\delta(a))\ \textit{for all}\ a\in C(\mathbb{G}).$$

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1.2. Reduced quantum groups

We should note that h may not be faithful in general. For example, the full group C^*-algebra $C^*\mathbb{F}_2$ is a c.q.g. with a coproduct $\delta(r) = r \otimes r$ for $r \in \mathbb{F}_2$. The Haar state is given by $h(r) = 0$ if $r \neq e$. However, h is not faithful because $C^*\mathbb{F}_2 \ncong C^*_{\mathrm{red}}\mathbb{F}_2$.

Let $N_h := \{a \in C(\mathbb{G}) \mid h(a^*a) = 0\}$. Then it is known that N_h is in fact an ideal of $C(\mathbb{G})$, and we can consider the *reduced compact quantum group* $C(\mathbb{G}_{\mathrm{red}}) := C(\mathbb{G})/N_h$ with a natural coproduct. By definition, h is faithful on $C(\mathbb{G}_{\mathrm{red}})$.

Let $(L^2(\mathbb{G}), \Omega_h)$ be the GNS representation associated with the Haar state h, that is,

- $L^2(\mathbb{G})$ is a Hilbert space;
- $\Omega_h \in L^2(\mathbb{G})$ is the GNS cyclic vector, i.e. we have $L^2(\mathbb{G}) = \overline{C(\mathbb{G})\Omega_h}$ and $h(a) = (a\Omega_h, \Omega_h)$.

Note that N_h is precisely equal to the kernel of the GNS representation.

1.3. Multiplicative unitaries

From the bi-invariance of the state h, the following theorem holds:

Theorem 1.4. *There exist unitary operators $V, W \in B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ satisfying*

$$V(a\Omega_h \otimes \xi) = \delta(a)(\Omega_h \otimes \xi) \text{ for all } a \in C(\mathbb{G}), \xi \in L^2(\mathbb{G});$$

$$W^*(\xi \otimes a\Omega_h) = \delta(a)(\xi \otimes \Omega_h) \text{ for all } a \in C(\mathbb{G}), \xi \in L^2(\mathbb{G}).$$

Then V and W satisfy the following notable *pentagon equations*:

$$V_{12}V_{13}V_{23} = V_{23}V_{12}, \quad W_{12}W_{13}W_{23} = W_{23}W_{12}. \tag{1.1}$$

By definition, we have the following implementation formula:

$$V(a \otimes 1)V^* = \delta(a) = W^*(1 \otimes a)W \text{ for all } a \in C(\mathbb{G}_{\mathrm{red}}). \tag{1.2}$$

1.4. Von Neumann closures of quantum groups

We denote by $L^\infty(\mathbb{G})$ the weak closure of $C(\mathbb{G}_{\mathrm{red}})$ in $B(L^2(\mathbb{G}))$. The coproduct δ extends to the normal morphism from $L^\infty(\mathbb{G})$ into $L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$ through (1.2). Then the pair $(L^\infty(\mathbb{G}), \delta)$ is called the *von Neumann algebraic compact quantum group*. It is non-trivial that there exists a modular automorphism for h, and the Haar state $h(\cdot) = (\cdot\Omega_h, \Omega_h)$ is faithful on $L^\infty(\mathbb{G})$ [45].

1.5. Kac type quantum groups

Definition 1.5. A compact quantum group is said to be of *Kac type* when the Haar state is tracial, i.e. $h(ab) = h(ba)$ for all $a, b \in C(\mathbb{G})$.

A compact group or C^*-group algebra of a discrete group are typical examples of Kac type quantum groups, but they are not all. As an example, $SU_{-1}(n)$ is of Kac type, which is neither commutative nor co-commutative. Readers should note the first such example discovered by G. I. Kac and V. G. Paljutkin [17].

1.6. Representation theory

Definition 1.6. Let H be a Hilbert space. A unitary $v \in B(H) \otimes L^\infty(\mathbb{G})$ is called a *(right unitary) representation* if it satisfies

$$(\mathrm{id} \otimes \delta)(v) = v_{12}v_{13}. \tag{1.3}$$

Similarly we can define a left representation. The above equality is the translation of the equality $v(rs) = v(r)v(s)$, $r, s \in \mathbb{G}$ in terms of a Hopf algebra.

Example 1.7. The multiplicative unitaries $V \in B(L^2(\mathbb{G})) \otimes L^\infty(\mathbb{G})$ and $W \in L^\infty(\mathbb{G}) \otimes B(L^2(\mathbb{G}))$ are right and left representations, respectively. Indeed using the pentagon equation (1.1), we have

$$(\mathrm{id} \otimes \delta)(V) = V_{23}V_{12}V_{23}^* = (V_{12}V_{13}V_{23})V_{23}^* = V_{12}V_{13}.$$

Similarly we obtain $(\delta \otimes \mathrm{id})(W) = W_{13}W_{23}$.

There are the following three operations:

- **(direct sum)**
$$v_1 \oplus v_2 := \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \in B(H_1 \oplus H_2) \otimes L^\infty(\mathbb{G});$$
- **(tensor product)**
$$v_1 \otimes v_2 := (v_1)_{13}(v_2)_{23} \in B(H_1 \otimes H_2) \otimes L^\infty(\mathbb{G});$$
- **(conjugation)** Let $v = (v_{ij})_{i,j \in I}$ be a matrix form of a representation. Consider
$$v^c := (v_{ij}^*)_{i,j \in I}$$
which may not be a unitary, but this still satisfies (1.3). In fact, if the dimension is finite, then it is unitarizable, i.e. there exists a positive invertible operator Q such that
$$\overline{v} := (Q^{1/2} \otimes 1)v^c(Q^{-1/2} \otimes 1)$$
is a unitary matrix, where Q is usually canonically chosen (see §1.9).

The important point is that the Peter-Weyl theory holds even for the quantum setting. Let us introduce the intertwiner space between unitary representations $v_i \in B(H_i) \otimes L^\infty(\mathbb{G})$, $i = 1, 2$,

$$\mathrm{Mor}(v_1, v_2) := \{T \in B(H_1, H_2) \mid (T \otimes 1)v_1 = v_2(T \otimes 1)\}.$$

Definition 1.8. Let $v \in B(H) \otimes L^\infty(\mathbb{G})$ be a unitary representation.

- A unitary representation is said to be *irreducible* when $\mathrm{Mor}(v, v) = \mathbb{C}1_H$.
- Let $w \in B(K) \otimes L^\infty(\mathbb{G})$ be a unitary representation. We say that v and w are *equivalent* if $\mathrm{Mor}(v, w)$ contains a unitary.

Theorem 1.9 (Woronowicz). *The following hold:*

(1) An irreducible representation is finite dimensional;
(2) A finite dimensional representation is the direct sum of irreducibles;

(3) *Let* $v \in B(H) \otimes L^\infty(\mathbb{G})$ *be a finite dimensional representation. Then* $v \in B(H) \otimes C(\mathbb{G})$.

Define the following subspace of $C(\mathbb{G})$ called the *smooth part*:

$$A(\mathbb{G}) := \mathrm{span}\{v_{ij} \mid v \text{ finite dimensional representation}\}.$$

Theorem 1.10 (Woronowicz). *The following hold:*

(1) $A(\mathbb{G})$ *is a unital* $*$*-subalgebra that is dense in* $C(\mathbb{G})$*;*
(2) *The set* $\{v_{\pi_{ij}}\}_{ij \in I_\pi}$, $\pi \in \mathrm{Irr}(\mathbb{G})$ *is a linear basis of* $A(\mathbb{G})$*;*
(3) *The Haar state* h *is faithful on* $A(\mathbb{G})$.

We denote by $\mathrm{Irr}(\mathbb{G})$ the set of equivalent classes of irreducible representations of $\mathbb{G}$. For each $\pi \in \mathrm{Irr}(\mathbb{G})$, we choose a corresponding representation $v_\pi \in B(H_\pi) \otimes L^\infty(\mathbb{G})$. Note that $\dim(H_\pi) < \infty$ from the previous theorem. The trivial one dimensional representation is denoted by $\mathbf{1}$. For $\pi \in \mathrm{Irr}(\mathbb{G})$, the conjugation is denoted by $\overline{\pi}$, which is the unique element of $\mathrm{Irr}(\mathbb{G})$ such that $\mathrm{Mor}(\pi \otimes \overline{\pi}, \mathbf{1}) \neq 0$ or $\mathrm{Mor}(\overline{\pi} \otimes \pi, \mathbf{1}) \neq 0$.

Let us define the $\mathbb{Z}$-module,

$$\mathcal{R}_{\mathbb{G}} := \bigoplus_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathbb{Z}\pi.$$

Setting $N_{\pi\rho}^\sigma := \dim \mathrm{Mor}(\pi \otimes \rho, \sigma)$ for $\pi, \rho, \sigma \in \mathrm{Irr}(\mathbb{G})$, we define the following product structure on $\mathcal{R}_{\mathbb{G}}$:

$$\pi \cdot \rho = \sum_{\sigma \in \mathrm{Irr}(\mathbb{G})} N_{\pi\rho}^\sigma \sigma.$$

The $\mathbb{Z}$-ring $\mathcal{R}_{\mathbb{G}}$ is called the *representation ring* of $\mathbb{G}$.

Definition 1.11. We say that a compact quantum group $\mathbb{G}$ has *commutative fusion rules* if $\mathcal{R}_{\mathbb{G}}$ is commutative.

1.7. Hopf algebra structure

By Theorem 1.10, we can introduce the maps $\varepsilon \colon A(\mathbb{G}) \to \mathbb{C}$ and $\kappa \colon A(\mathbb{G}) \to A(\mathbb{G})$ defined by

$$\varepsilon(v_{\pi_{ij}}) = \delta_{ij} \quad \text{for } \pi \in \mathrm{Irr}(\mathbb{G}), i, j \in I_\pi,$$
$$\kappa(v_{\pi_{ij}}) = v_{\pi_{ji}}^* \quad \text{for } \pi \in \mathrm{Irr}(\mathbb{G}), i, j \in I_\pi.$$

Theorem 1.12 (Woronowicz). *The following hold:*

(1) ε *is a* $*$*-homomorphism satisfying*

$$(\varepsilon \otimes \mathrm{id}) \circ \delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \delta \quad \text{on } A(\mathbb{G});$$

(2) κ *is an anti-homomorphism satisfying* $\kappa(\kappa(a)^*)^* = a$ *for* $a \in A(\mathbb{G})$ *and*

$$m \circ (\kappa \otimes \mathrm{id}) \circ \delta = \varepsilon = m \circ (\mathrm{id} \otimes \kappa) \circ \delta \quad \text{on } A(\mathbb{G}),$$

where $m \colon A(\mathbb{G}) \otimes A(\mathbb{G}) \to A(\mathbb{G})$ *is the multiplication.*

So, $A(\mathbb{G})$ has a Hopf $*$-algebra structure.

1.8. Modular objects

We introduce the *Woronowicz characters* $\{f_z\}_{z\in\mathbb{C}}$ on $A(\mathbb{G})$ [45, Theorem2.4]. The multiplicative functional $f_z\colon A(\mathbb{G}) \to \mathbb{C}$ is uniquely determined by the following properties:

(1) $f_0 = \varepsilon$;
(2) For any $a \in A(\mathbb{G})$, the function $\mathbb{C} \ni z \mapsto f_z(a) \in \mathbb{C}$ is entirely holomorphic;
(3) $(f_{z_1} \otimes f_{z_2}) \circ \delta = f_{z_1+z_2}$ for any $z_1, z_2 \in \mathbb{C}$;
(4) For any $z \in \mathbb{C}$ and $a \in A(\mathbb{G})$, $f_z(\kappa(a)) = f_{-z}(a)$, $f_z(a^*) = \overline{f_{-\bar{z}}(a)}$;
(5) For any $a \in A(\mathbb{G})$, $\kappa^2(a) = (f_1 \otimes \mathrm{id} \otimes f_{-1})\big((\delta \otimes \mathrm{id})(\delta(a))\big)$;
(6) For any $a, b \in A(\mathbb{G})$, $h(ab) = h\big(b\,(f_1 \otimes \mathrm{id} \otimes f_1)\big((\delta \otimes \mathrm{id})(\delta(a))\big)\big)$.

The modular automorphism group $\{\sigma_t^h\}_{t\in\mathbb{R}}$ is given by

$$\sigma_t^h(x) = (f_{it} \otimes \mathrm{id} \otimes f_{it})\big((\delta \otimes \mathrm{id})(\delta(x))\big) \quad \text{for all } t \in \mathbb{R},\ x \in A(\mathbb{G}).$$

We define the following map $\tau_t\colon A(\mathbb{G}) \to A(\mathbb{G})$ by

$$\tau_t(x) = (f_{it} \otimes \mathrm{id} \otimes f_{-it})\big((\delta \otimes \mathrm{id})(\delta(x))\big) \quad \text{for all } t \in \mathbb{R},\ x \in A(\mathbb{G}).$$

Then $\{\tau_t\}_{t\in\mathbb{R}}$ is a one-parameter automorphism group on $A(\mathbb{G})$ and it is called the *scaling automorphism group.* Since the Haar state h is invariant under the $*$-preserving maps σ_t^h and τ_t, we can extend them to the maps on $C(\mathbb{G}_{\mathrm{red}})$, and on $L^\infty(\mathbb{G})$.

1.9. Quantum Peter-Weyl theorem

Let $v \in B(H) \otimes A(\mathbb{G})$ be a finite dimensional representation. We set

$$Q_v := (\mathrm{id} \otimes f_1)(v),$$

which is an invertible positive operator on H. For $\pi \in \mathrm{Irr}(\mathbb{G})$, we write Q_π instead of Q_{v_π}. When $\mathbb{G}$ is of Kac type, then $Q_v = 1$ for any representation v.

Definition 1.13. The value $\mathrm{Tr}_H(Q_v)$ is called the *quantum dimension* of v, and denoted by $\dim_q(v)$.

Since it can be shown that $\mathrm{Tr}(Q_\pi^{-1}) = \mathrm{Tr}(Q_\pi)$, we have $\dim_q(v) \geq \dim H$.

Theorem 1.14. *The Haar state h satisfies the following generalized orthogonality:*

$$(\mathrm{id} \otimes h)(v_\pi(\xi\eta^* \otimes 1)v_\rho^*) = \delta_{\pi,\rho} \dim_q(\pi)(Q_\pi\xi, \eta),$$
$$(\mathrm{id} \otimes h)(v_\pi^*(\xi\eta^* \otimes 1)v_\rho) = \delta_{\pi,\rho} \dim_q(\pi)(Q_\pi^{-1}\xi, \eta),$$

where $\pi, \rho \in \mathrm{Irr}(\mathbb{G})$, $\xi \in H_\pi$ and $\eta \in H_\rho$.

The key point of the proof of this result is to observe that $(\mathrm{id} \otimes h)(v_\pi(\xi\eta^* \otimes 1)v_\rho^*) \in \mathrm{Mor}(\rho, \pi)$. The matrix form is sometimes useful. Take an ONB $\{\varepsilon_{\pi_i}\}_{i\in I_\pi}$. Then we have

$$h(v_{\pi_{i,j}} v_{\rho_{k,\ell}}^*) = \delta_{\pi,\rho} \dim_q(\pi)^{-1} Q_{\pi_{\ell,j}} \delta_{i,k}, \quad h_{\mathbb{G}}(v_{\pi_{i,j}}^* v_{\rho_{k,\ell}}) = \delta_{\pi,\rho} \dim_q(\pi)^{-1} (F_\pi^{-1})_{k,i} \delta_{j,\ell}.$$

In particular, $h(v_{\pi_{ij}}) = \delta_{\pi,1}\delta_{ij}$.

We see that the matrix $(Q_{\overline{\pi}}^{1/2} \otimes 1)v_{\pi}^{c}(Q_{\overline{\pi}}^{-1/2} \otimes 1)$ is a unitary representation equivalent to $v_{\overline{\pi}}$.

Now note that $\dim \mathrm{Mor}(\pi \otimes \overline{\pi}, \mathbf{1}) = 1$, and take $t_\pi \in \mathrm{Mor}(\pi \otimes \overline{\pi}, \mathbf{1})$ such that $t_\pi^* t_\pi = \dim_q(\pi)$. The following theorem is due to Woronowicz, but our notation is slightly different because we do not use special ONB (see [8]).

Theorem 1.15 (Quantum Peter-Weyl theorem). *One has the following unitary isomorphism:*

$$L^2(\mathbb{G}) \to \bigoplus_{\pi \in \mathrm{Irr}(\mathbb{G})} H_{\overline{\pi}} \otimes H_\pi,$$

which maps $(\omega_{\xi,\eta} \otimes \mathrm{id})(v_\pi)\Omega_h$ *to* $(1 \otimes \eta^*) \circ t_{\overline{\pi}} \otimes \xi$. *intertwines the left and right* $\mathbb{G}$*-actions.*

1.10. **Non-trivial examples 1**

For a classical compact Lie group $\mathbb{G}$, we can construct the *q-deformation* $\mathbb{G}_q$ [18], where $q \in [-1, 1] \setminus \{0\}$. If $q = 1$, $\mathbb{G}_1$ is nothing but the original $\mathbb{G}$. The object corresponding to $q = 0$ is considered as a *quantum semigroup* which is not a quantum group because that does not have a Haar state.

Now we explain the simplest and the most important example $SU_q(2)$ [44]. The continuous function algebra $C(SU_q(2))$ is the universal C^*-algebra generated by four elements x, u, v and y with the following relations:

$$ux = qxu, \quad vx = qxv, \quad yu = quy, \quad yv = qvy, \quad uv = vu,$$
$$xy - q^{-1}uv = 1 = yx - quv,$$
$$x^* = y, \quad u^* = -q^{-1}v.$$

To introduce a coproduct δ, the following 2 by 2 unitary matrix is useful:

$$v(\pi_{1/2}) := \begin{pmatrix} x & u \\ v & y \end{pmatrix}.$$

Then the coproduct δ is given by

$$\begin{pmatrix} \delta(x) & \delta(u) \\ \delta(v) & \delta(y) \end{pmatrix} := \begin{pmatrix} x \otimes 1 & u \otimes 1 \\ v \otimes 1 & y \otimes 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \otimes x & 1 \otimes u \\ 1 \otimes v & 1 \otimes y \end{pmatrix},$$

This means $v(\pi_{1/2})$ is a representation, which is in fact irreducible.

It is known that $\mathrm{Irr}(SU_q(2))$ is naturally identified with the positive half integers $(1/2)\mathbb{Z}_{\geq 0}$. Each $\nu \in \mathrm{Irr}(SU_q(2))$ is called the *spin* and the dimension of $v(\pi_\nu)$ is $2\nu + 1$. The quantum dimension of π_ν is given by the *q-integer* $[2\nu + 1]_q := (q^{-2\nu-1} - q^{2\nu+1})/(q^{-1} - q)$ [20].

On tensor products, we have the same formula (Clebsh-Gordan rule) as that of $SU(2)$,

$$\pi_\mu \otimes \pi_\nu = \pi_{|\mu-\nu|} \oplus \pi_{|\mu-\nu|+1} \oplus \cdots \oplus \pi_{\mu+\nu-1} \oplus \pi_{\mu+\nu}.$$

In particular, $SU_q(2)$ has commutative fusion rules [20, 44].

This phenomena hold for every q-deformation of a classical compact Lie group, that is, the fusion rule is invariant under the q-deformation, and it is commutative.

1.11. **Non-trivial example 2**

Our second interesting example is a *universal quantum group*. There are a lot of variants, and we explain the only original examples here [35].

Definition 1.16. Let $F \in GL(n, \mathbb{C})$.

- (Universal orthogonal quantum group $A_o(F)$) Assume that $F\overline{F} = \pm 1$. The function algebra $C(A_o(F))$ is the universal C^*-algebra generated by u_{ij}, $i, j = 1, \ldots, n$, which satisfy
$$u = (F \otimes 1)u^c(F^{-1} \otimes 1),$$
where $u = (u_{ij})_{ij}$ and $u^c = (u_{ij}^*)_{ij}$.
- (Universal quantum group $A_u(F)$) The function algebra $C(A_u(F))$ is the universal C^*-algebra generated by u_{ij}, $i, j = 1, \ldots, n$ such that u and $(F \otimes 1)u^c(F^{-1} \otimes 1)$ are unitary.

The both coproducts are given by
$$\delta(u_{ij}) = \sum_{i,j=1}^{n} u_{ik} \otimes u_{kj}.$$

This means the matrix u is a representation, which is in fact irreducible.

By definition, we obtain the surjective morphism $r\colon C(A_u(F)) \to C(A_o(F))$ as quantum groups. Hence $A_o(F)$ is a quantum subgroup of $A_u(F)$.

The fusion rules of $A_o(F)$ and $A_u(F)$ are computed by T. Banica [3, 4]. The c.q.g. $A_o(F)$ has the same fusion rule as $SU(2)$, and in fact, it is monoidally equivalent to $SU_q(2)$ for some q [8] (see §3.4). We can regard $\mathrm{Irr}(A_u(F))$ as the free monoid $\mathbb{N} \star \mathbb{N}$ whose product is written like xy for $x, y \in \mathbb{N} \star \mathbb{N}$. Let α and β are the generators. We define the conjugation on $\mathbb{N} \star \mathbb{N}$ such that $\overline{\alpha} = \beta$. The representation ring is $\mathcal{R} = \mathbb{Z}[\mathbb{N} \star \mathbb{N}]$ as a $\mathbb{Z}$-module. The product structure $x \cdot y$ for $x, y \in \mathbb{N} \star \mathbb{N}$ is given by
$$x \cdot y := \sum_{\{a \in \mathbb{N}\star\mathbb{N} \mid x = x_0 a,\ y = \bar{a} y_0\}} x_0 y_0.$$

For example, $\alpha \cdot \alpha = \alpha^2$ and $\alpha \cdot \beta = \alpha\beta + 1$. So, the fusion rule does not depend on F. In particular, $\dim \mathrm{Mor}(x \otimes y, z) = 0$ or 1 for all $x, y, z \in \mathrm{Irr}(A_u(F))$.

Recall the definition of $A_u(F)$ where we have taken a unitary matrix $u \in B(\mathbb{C}^n) \otimes C(A_u(F))$. In fact, u and $\overline{u}$ are irreducible representations corresponding to α and β, respectively.

2. Discrete quantum groups

Let $\mathbb{G}$ be a c.q.g. In this section, we study basic properties of the dual $\widehat{\mathbb{G}}$.

2.1. Right and left group algebras

Recall the multiplicative unitaries V, W, which is right and left representations of $\mathbb{G}$ on $L^2(\mathbb{G})$. We introduce the following subspaces:

$$R(\mathbb{G}) := \overline{\operatorname{span}}^{\mathrm{w}}\{(\mathrm{id} \otimes \omega)(V) \mid \omega \in L^\infty(\mathbb{G})_*\},$$

$$L(\mathbb{G}) := \overline{\operatorname{span}}^{\mathrm{w}}\{(\omega \otimes \mathrm{id})(W) \mid \omega \in L^\infty(\mathbb{G})_*\}.$$

We call them *right, left group algebras*, respectively.

Define the maps $\beta, \beta^\ell \colon B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G})) \otimes B(L^2(\mathbb{G}))$,

$$\beta(x) := V^*(1 \otimes x)V \text{ for } x \in B(L^2(\mathbb{G})),$$

$$\beta^\ell(x) := W(x \otimes 1)W^* \text{ for } x \in B(L^2(\mathbb{G})).$$

Theorem 2.1. *The following hold:*

- *$R(\mathbb{G})$ and $L(\mathbb{G})$ are von Neumann algebras;*
- *The restrictions $\Delta := \beta|_{R(\mathbb{G})}$ and $\Delta^\ell := \beta^\ell|_{L(\mathbb{G})}$ define coproducts, i.e.*

$$\Delta(R(\mathbb{G})) \subset R(\mathbb{G}) \otimes R(\mathbb{G}), \quad (\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

and

$$\Delta(L(\mathbb{G})) \subset L(\mathbb{G}) \otimes L(\mathbb{G}), \quad (\Delta^\ell \otimes \mathrm{id}) \circ \Delta^\ell = (\mathrm{id} \otimes \Delta^\ell) \circ \Delta^\ell.$$

Proof. We prove the second statement. Take $(\mathrm{id} \otimes \omega)(V) \in R(\mathbb{G})$. Then using the pentagon equation, we have

$$\begin{aligned}\Delta((\mathrm{id} \otimes \omega)(V)) &= V^*(1 \otimes (\mathrm{id} \otimes \omega)(V))V \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega)(V_{12}^* V_{23} V_{12}) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega)(V_{13} V_{23}) \in R(\mathbb{G}) \otimes R(\mathbb{G}).\end{aligned}$$

□

This theorem implies that the pair $(R(\mathbb{G}), \Delta)$ is a bialgebra. In fact, it is known that there exist weights φ and ψ on $R(\mathbb{G})$ such that

- $\varphi((\omega \otimes \mathrm{id})(\Delta(x))) = \omega(1)\psi(x)$ for all $\omega \in R(\mathbb{G})_*^+$, $x \in R(\mathbb{G})_+$.
- $\psi((\mathrm{id} \otimes \omega)(\Delta(x))) = \omega(1)\varphi(x)$ for all $\omega \in R(\mathbb{G})_*^+$, $x \in R(\mathbb{G})_+$;

Therefore, $\widehat{\mathbb{G}} := (R(\mathbb{G}), \Delta)$ is a quantum group in the sense of [19].

Using Theorem 1.15, we obtain the isomorphism,

$$R(\mathbb{G}) \to \bigoplus_{\pi \in \mathrm{Irr}(\mathbb{G})} B(H_\pi).$$

Hence $\widehat{\mathbb{G}}$ is also called a *discrete quantum group.* Similarly $L(\mathbb{G})$ is a discrete quantum group, too. They are acting on $L^2(\mathbb{G})$ standardly, and

$$R(\mathbb{G})' = L(\mathbb{G}).$$

A discrete quantum group is also characterized by the existence of a normal counit. Indeed in our case, the normal counit $\hat{\varepsilon}$ is given by the evaluation of $x \in R(\mathbb{G})$ at $\pi = \mathbf{1}$, i.e. $\hat{\varepsilon}(x) = x_{\mathbf{1}} \in \mathbb{C}$.

When we study actions of $\widehat{\mathbb{G}}$, the following description of the coproduct Δ is quite useful:

$$\Delta(x_\pi) = \sum_{\rho,\sigma\in\mathrm{Irr}(\mathbb{G})} \sum_{S\in\mathrm{ONB}(\mathrm{Mor}(\pi,\rho\otimes\sigma))} Sx_\pi S^* \quad \text{for } x_\pi \in B(H_\pi) \subset R(\mathbb{G}),$$

where $\mathrm{ONB}(\mathrm{Mor}(\pi, \rho\otimes\sigma))$ is a set of orthonormal bases of $\mathrm{Mor}(\rho\otimes\sigma, \pi)$ with the inner product $(S, T) := T^*S$.

2.2. Actions of quantum groups

Definition 2.2. Let $\mathbb{G} = (L^\infty(\mathbb{G}), \delta)$ be a locally compact quantum group and M a von Neumann algebra. A map $\alpha\colon M \to M \otimes L^\infty(\mathbb{G})$ is called a *(right) action* when it satisfies the following:

- α is a unital faithful normal $*$-homomorphism;
- $(\mathrm{id}\otimes\delta)\circ\alpha = (\alpha\otimes\mathrm{id})\circ\alpha$.

A left action is similarly defined.

Example 2.3. The map $\beta\colon B(L^2(\mathbb{G})) \to R(\mathbb{G}) \otimes B(L^2(\mathbb{G}))$ is a left action of $\widehat{\mathbb{G}}$. Similarly $\alpha\colon B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G})) \otimes L^\infty(\mathbb{G})$ defined by $\alpha(x) = V(x\otimes 1)V^*$ is a right action of $\mathbb{G}$.

2.3. Quantum subgroups and left (right) coideals

There are several ways to define a quantum subgroup of a c.q.g. Here, we adopt the following definition.

Definition 2.4. Let $\mathbb{G}$ and $\mathbb{H}$ be compact quantum groups. We say that $\mathbb{H}$ is a *quantum subgroup* of $\mathbb{G}$ if there exists a unital $*$-homomorphism $r_\mathbb{H}\colon A(\mathbb{G}) \to A(\mathbb{H})$ such that

- $r_\mathbb{H}$ is surjective;
- $\delta_\mathbb{H}\circ r_\mathbb{H} = (r_\mathbb{H}\otimes r_\mathbb{H})\circ\delta_\mathbb{G}$.

This definition is weaker than the usual C^*-version, which requires $r_\mathbb{H}$ is a C^*-homomorphism from $C(\mathbb{G})$ onto $C(\mathbb{H})$.

Note that $\mathbb{H}$ acts on $A(\mathbb{G})$ from the both sides as $\mathbb{H} \overset{\gamma^\ell}{\curvearrowright} A(\mathbb{G}) \overset{\gamma^r}{\curvearrowleft} \mathbb{H}$ defined by

$$\gamma^\ell := (r_\mathbb{H}\otimes\mathrm{id})\circ\delta_\mathbb{G}, \quad \gamma^r := (\mathrm{id}\otimes r_\mathbb{H})\circ\delta_\mathbb{G}.$$

Then we define the *non-commutative quotient spaces* by the following fixed point algebras:

$$A(\mathbb{H}\backslash\mathbb{G}) := \{a\in A(\mathbb{G}) \mid \gamma^\ell(a) = 1\otimes a\},$$
$$A(\mathbb{G}/\mathbb{H}) := \{a\in A(\mathbb{G}) \mid \gamma^r(a) = a\otimes 1\}.$$

The weak closures in $B(L^2(\mathbb{G}))$ are denoted by $L^\infty(\mathbb{H}\backslash\mathbb{G})$ and $L^\infty(\mathbb{G}/\mathbb{H})$.

Note that the left $\mathbb{H}$-action γ^ℓ and the right $\mathbb{G}$-action $\delta_\mathbb{G}$ are commuting, i.e. $(\mathrm{id}\otimes\delta_\mathbb{G})\circ\gamma^\ell = (\gamma^\ell\otimes\mathrm{id})\circ\delta_\mathbb{G}$. Hence $\mathbb{G}$ is also acting $A(\mathbb{H}\backslash\mathbb{G})$ by δ. Similarly the coproduct δ defines a left action on $A(\mathbb{G}/\mathbb{H})$. Since these actions are preserving the Haar state, they extend to the quotient spaces $L^\infty(\mathbb{H}\backslash\mathbb{G})$ or $L^\infty(\mathbb{G}/\mathbb{H})$, respectively. They are typical examples of right or left coideals.

Definition 2.5. Let $B \subset L^\infty(\mathbb{G})$ be a von Neumann subalgebra. Then we say that

- B is a left coideal if $\delta(B) \subset L^\infty(\mathbb{G}) \otimes B$;
- B is a right coideal if $\delta(B) \subset B \otimes L^\infty(\mathbb{G})$;
- a left (right) coideal B is of *quotient type* if $B = L^\infty(\mathbb{G}/\mathbb{H})$ (resp. $L^\infty(\mathbb{H}\backslash\mathbb{G})$) for some quantum subgroup $\mathbb{H}$.

Thanks to Gelfand theorem, every left coideal is of quotient type when $\mathbb{G}$ is a compact group [1]. However, this is not true in general [26, 27, 30]. Indeed, we have the following characterization [31].

Theorem 2.6 (Tomatsu). *Let $B \subset L^\infty(\mathbb{G})$ be a right coideal. Then the following are equivalent:*

- *B is of quotient type;*
- *There exists an expectation $E_B\colon L^\infty(\mathbb{G}) \to B$ preserving the Haar state, and moreover $\widehat{\mathbb{G}}$ acts on B, i.e. $\beta(B) \subset R(\mathbb{G}) \otimes B$.*

This theorem has been proved for co-amenable quantum groups [31], but the same proof works because we have changed the definition of quantum subgroups.

2.4. Amenability and co-amenability

For details of the theory of amenability for quantum groups, readers are referred to [5, 6, 7, 29] and references therein.

Definition 2.7. We say that $\widehat{\mathbb{G}}$ is *amenable* when there exists an *invariant mean* m on $R(\mathbb{G})$, that is, $m \in R(\mathbb{G})^*$ is a state such that

$$m((\mathrm{id} \otimes \omega)(\Delta(x))) = \omega(1)m(x) = m((\omega \otimes \mathrm{id})(\Delta(x))).$$

In this case, $\mathbb{G}$ is said to be *co-amenable.*

Theorem 2.8 (Bedos-Murphy-Tuset, Tomatsu). *The following are equivalent:*

- *$\mathbb{G}$ is co-amenable;*
- *$C(\mathbb{G}_{\mathrm{red}})$ has a bounded counit ε, which is a $*$-homomorphism $\varepsilon\colon C(\mathbb{G}_{\mathrm{red}}) \to \mathbb{C}$ such that $(\varepsilon \otimes \mathrm{id}) \circ \delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \delta$.*

If $\mathbb{G}$ is co-amenable, then $C(\mathbb{G}_{\mathrm{red}})$ is the universal C^*-algebra of $A(\mathbb{G})$, which is proved by using Fell absorption technique.

It is easy to see that any quantum subgroup of a co-amenable c.q.g. is also co-amenable. Since $A_o(F)$ $(n \geq 3)$ or $A_u(F)$ has non-co-amenable quantum subgroups, they are not co-amenable. It is known that the q-deformation of a classical compact Lie group is co-amenable. In particular, $SU_q(2)$ is co-amenable.

3. Problems

In this section, some interesting open problems are listed.

3.1. Minimal actions

One of the most difficult problem in a quantum group theory is the existence of minimal actions on amenable factors. The definition of minimality is the following:

Definition 3.1. An action $M \overset{\alpha}{\curvearrowleft} \mathbb{G}$ is said to be *minimal* if it satisfies

- (trivial relative commutant) $(M^\alpha)' \cap M = \mathbb{C}$;
- (full spectrum) $L^\infty(\mathbb{G}) = \overline{\mathrm{span}}^{\mathrm{w}}\{(\omega \otimes \mathrm{id})(\alpha(M)) \mid \omega \in M_*\}$.

The first condition is concerned with the high non-commutativity of the fixed point algebra in the ambient algebra. In particular, M must be a factor. The second one is also called *faithfulness* of α. Indeed if $\mathbb{G}$ is a compact group, then that is equivalent to the injectivity of the group homomorphism $\alpha\colon \mathbb{G} \to \mathrm{Aut}(M)$, and the trivial action is excluded.

Example 3.2. Let $\mathbb{G} \subset U(n)$ be a closed subgroup. We present a typical construction of a minimal action of $\mathbb{G}$ on the amenable type II_1 factor $\mathcal{R}_0$.

Set the unitary representation $\pi\colon \mathbb{G} \to U(n+1)$,

$$\pi(g) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\mathbb{G}$ acts on $M_{n+1}(\mathbb{C})^{\otimes k}$ by adjoint action $\mathrm{Ad}\,\pi^{\otimes k}(g)$ for each $k \geq 0$. The action extends to the UHF $M_{n+1}(\mathbb{C})^{\otimes\infty}$, which preserves the unique trace. Hence we obtain an action of $\mathbb{G}$ on its weak closure, i.e. the amenable type II_1 factor $\mathcal{R}_0$. Indeed it is minimal [44]. It is known that any minimal action of a compact group on $\mathcal{R}_0$ is unique up to conjugacy [25, 22].

We have seen an example of a minimal action of a compact Lie group. The first example of a minimal action of a c.q.g. was constructed by Y. Ueda [33].

Let $\mathbb{G}$ be a c.q.g. Take a von Neumann algebra N such that $N' \cap (N \star L^\infty(\mathbb{G})) = \mathbb{C}$. Then consider the *Ueda action* $\alpha := \mathrm{id} \star \delta$ on a factor $M := N \star L^\infty(\mathbb{G})$. By definition, α is minimal. The point is that M is not amenable. So, our problem is the following:

Problem 3.3. *Let $\mathbb{G}$ be a compact quantum group of non-Kac type. Does there exist a minimal action on an amenable factor?*

If there would exist such an action, $\mathbb{G}$ would have to be co-amenable [36]. Although we have not known if it is true or not even for $SU_q(2)$, Izumi's observation seems to be relevant.

Let $v \in B(H) \otimes L^\infty(\mathbb{G})$ be a finite dimensional representation. Then as in the previous example, we can define the infinite tensor product action α on an amenable factor $\mathcal{R}$, which is a Powers factor. According to Izumi theory, we have the isomorphism between $\mathbb{G}$-algebras,

$$(\mathcal{R}^\alpha)' \cap \mathcal{R} \cong H^\infty(\widehat{\mathbb{G}}, P_\mu),$$

where $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ is the *Poisson boundary* associated with the random walk on $\widehat{\mathbb{G}}$ with a probability measure μ [13]. Therefore, α is never minimal when $\mathbb{G}$ is of non-Kac type.

Many operator algebraists have interest in this problem because it has an application to Jones' problem, that is, to determine the index values for irreducible subfactors of $\mathcal{R}_0$ [16]. If Proposition 3.3 would be affirmatively solved for $SU_q(2)$, then the set of possible values greater than 4 would become $(4, \infty)$ [34]. They are attained for Wassermann subfactors:

$$M^\alpha \subset (B(\mathbb{C}^2) \otimes M)^{\widetilde{\alpha}},$$

where $\widetilde{\alpha} = \mathrm{Ad}(v_{\pi_{1/2}})_{13} \circ (\mathrm{id} \otimes \alpha)$.

However, we might solve Jones' problem without proving the existence. In fact, if we had found an example of an action α on an amenable factor M such that

- $M = M^\alpha \oplus M_{\pi_{1/2}} \oplus 0 \oplus M_{\pi_{3/2}} \oplus \dots$ (spectral decomposition);
- $\{M, \alpha\}$ is conjugate to $\{B(\mathbb{C}^2) \otimes M, \widetilde{\alpha}\}$,

then the Wassermann inclusion would give an index $\dim_q(\pi_{1/2})^2 = (q^{-1} + q)^2$.

3.2. Centers of Poisson boundaries

We briefly recall the notion of the Poisson boundary for a discrete quantum group. We refer to [13] for definitions of terminology.

Let $\phi_\pi \in B(H_\pi)_*$ be the right $\mathbb{G}$-invariant state. Define a transition operator P_π on $R(\mathbb{G})$ by $P_\pi(x) = (\mathrm{id} \otimes \phi_\pi)(\Delta_R(x))$ for $x \in R(\mathbb{G})$. When $\widehat{\mathbb{G}}$ is a discrete group, P_g, $g \in \widehat{\mathbb{G}}$, is nothing but the right translation of functions by $g \in \widehat{\mathbb{G}}$, which is an automorphism. However, the map P_π is not an automorphism but a faithful normal u.c.p. map in general.

For a probability measure μ on $\mathrm{Irr}(\mathbb{G})$, we set a non-commutative Markov operator,

$$P_\mu := \sum_{\pi \in \mathrm{Irr}(\mathbb{G})} \mu(\pi) P_\pi.$$

We assume μ is generating, that is, $\mathrm{supp}(\mu)$ generates $\mathrm{Irr}(\mathbb{G})$ as a "semigroup", i.e. for any $\pi \in \mathrm{Irr}(\mathbb{G})$, there exist $\rho_1, \dots, \rho_n \in \mathrm{supp}(\mu)$ such that the representation π is contained in the tensor product representation $\rho_1 \otimes \cdots \otimes \rho_n$.

Then we define an operator system,

$$H^\infty(\widehat{\mathbb{G}}, P_\mu) := \{x \in R(\mathbb{G}) \mid P_\mu(x) = x\}.$$

We often regard $\mathrm{id} - P_\mu$ as a Laplace operator on $\widehat{\mathbb{G}}$, and we say that each element of $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ is *P_μ-harmonic.* That operator system has the von Neumann algebra structure defined by

$$x \cdot y = \lim_{n \to \infty} P_\mu^n(xy) \quad \text{for } x, y \in H^\infty(\widehat{\mathbb{G}}, P_\mu), \tag{3.1}$$

where the limit is taken in the strong topology [13, Theorem 3.6]. The von Neumann algebra $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ is called the (non-commutative) *Poisson boundary* of $\{R(\mathbb{G}), P_\mu\}$.

We know that $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ is isomorphic to $(\mathcal{R}^{\mathbb{G}})' \cap \mathcal{R}$, where $\mathcal{R} \curvearrowleft \mathbb{G}$ is an ITP action [13]. Thanks to Takesaki theorem on an expectation [28], we see that $(\mathcal{R}^{\mathbb{G}})' \cap \mathcal{R}$ is amenable, and so is $H^\infty(\widehat{\mathbb{G}}, P_\mu)$.

Now we recall the actions $\widehat{\mathbb{G}} \overset{\beta}{\curvearrowright} B(L^2(\mathbb{G})) \overset{\alpha}{\curvearrowleft} \mathbb{G}$ defined by

$$\beta(x) := V^*(1 \otimes x)V, \quad \alpha(x) = V(x \otimes 1)V^*.$$

Since we can prove P_μ and α or β are commuting on $R(\mathbb{G})$, the Poisson boundary is a $\widehat{\mathbb{G}}$-$\mathbb{G}$-von Neumann algebra. We should note that if $\widehat{\mathbb{G}}$ is a discrete group, then α is trivial. Hence a non-triviality of α on $R(\mathbb{G})$ is a purely quantum phenomenon.

In Poisson boundary theory, one of the most important problem is the following:

Problem 3.4 (Identification problem). *Realize $\widehat{\mathbb{G}}$-$\mathbb{G}$-von Neumann algebra $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ more concretely.*

For example, if $\mathbb{G}$ is the q-deformation of a classical compact Lie group, then the Poisson boundary is isomorphic to a quantum flag manifold [13, 15, 31]. Also for $A_o(F)$ or $A_u(F)$, the computation has been done [38, 39, 40].

Here, we propose a new problem on a Poisson boundary. Recall that P_μ and α are commuting, and P_μ acts on the fixed point algebra $R(\mathbb{G})^\alpha$, which is nothing but the center $Z(R(\mathbb{G})) = \ell_\infty(\mathrm{Irr}(\mathbb{G}))$. Hence we introduce the classical part of a Poisson boundary,

$$H^\infty(\widehat{\mathbb{G}}, P_\mu)_{\mathrm{class}} := H^\infty(\widehat{\mathbb{G}}, P_\mu) \cap Z(R(\mathbb{G})).$$

Let us denote the center of $H^\infty(\widehat{\mathbb{G}}, P_\mu)$ by $Z(H^\infty(\widehat{\mathbb{G}}, P_\mu))$. It is trivial by (3.1) that the classical part $H^\infty(\widehat{\mathbb{G}}, P_\mu)_{\mathrm{class}}$ is contained in $Z(H^\infty(\widehat{\mathbb{G}}, P_\mu))$. Now we present following our problem:

Conjecture 3.5. *Let $\mathbb{G}$ be a compact quantum group and μ a generating probability measure. Then the following equality holds:*

$$H^\infty(\widehat{\mathbb{G}}, P_\mu)_{\mathrm{class}} = Z(H^\infty(\widehat{\mathbb{G}}, P_\mu)).$$

When $\mathbb{G}$ has a commutative fusion, then the classical part is trivial [12, 13]. So, the conjecture means the factoriality of $H^\infty(\widehat{\mathbb{G}}, P_\mu)$.

There are some positive observations about this conjecture. For $SU_q(2)$ case, it is true because the quantum flag manifold (or a Podleś sphere) $L^\infty(\mathbb{T}\backslash SU_q(2))$ is a type I_∞ factor. However, that is unknown for other q-deformations. The conjecture holds even for non-amenable examples such as $A_o(F)$ and $A_u(F)$ [37, 39]. It has seemed to be affirmative so far.

3.3. 2-cocycles

The next problem is about 2-cocycle deformations of a locally compact quantum group. Let $\mathbb{G}$ be a locally compact quantum group with the coproduct δ.

Definition 3.6. An element $\Omega \in L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$ is called a *2-cocycle* if it satisfies the following:

- Ω is a unitary:
- $(\Omega \otimes 1)(\delta \otimes \mathrm{id})(\Omega) = (1 \otimes \Omega)(\mathrm{id} \otimes \delta)(\Omega)$.

Let us define the map $\delta_\Omega := \mathrm{Ad}\,\Omega \circ \delta$, which is a new coproduct on $L^\infty(\mathbb{G})$. Hence $(L^\infty(\mathbb{G}), \delta_\Omega)$ is a bi-algebra. In fact, K. De Commer recently proved that there exist the left and right invariant weights on it, that is, $\mathbb{G}_\Omega := (L^\infty(\mathbb{G}), \delta_\Omega)$ is a locally compact quantum group [10].

If $\mathbb{G}$ is discrete, $\mathbb{G}$ is again discrete (use the counit). If $\mathbb{G}$ is a compact Kac, then $\mathbb{G}_\Omega$ is also compact. However, the compactness is not invariant in general [10]. His example is involving free products of $SU_q(2)$. So, we propose the following problem.

Problem 3.7. *Find a co-amenable compact quantum group $\mathbb{G}$ which has a 2-cocycle Ω so that $\mathbb{G}_\Omega$ is not compact.*

3.4. Monoidal equivalences

Definition 3.8. Let $\mathbb{G}$ and $\mathbb{G}_1$ be compact quantum groups. We say that they are *monoidally equivalent* when there exist the following maps both denoted by φ:

- a bijection φ: $\mathrm{Irr}(\mathbb{G}) \to \mathrm{Irr}(\mathbb{G}_1)$;
- bijective linear maps φ: $\mathrm{Mor}(x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_n) \to \mathrm{Mor}(\varphi(x_1) \otimes \cdots \otimes \varphi(x_m), \varphi(y_1) \otimes \cdots \otimes \varphi(y_n))$ for $x_1, \ldots, x_m, y_1, \ldots, y_n \in \mathrm{Irr}(\mathbb{G})$,

such that

- $\varphi(1) = 1$;
- $\varphi(S^*) = \varphi(S)^*$;
- $\varphi(ST) = \varphi(S)\varphi(T)$;
- $\varphi(S \otimes T) = \varphi(S) \otimes \varphi(T)$.

This means the tensor category of $\mathbb{G}$ and $\mathbb{G}_1$ are isomorphic, and so are their fusion algebras, in particular. The dimensions of their representation spaces may not be equal, but their quantum dimensions are invariant, i.e. $\dim_q(x) = \dim_q(\varphi(x))$.

The following problem is proposed by T. Banica.

Problem 3.9. *Assume that $\mathbb{G}$ has commutative fusion rules. Then does there exist a co-amenable compact quantum group $\mathbb{G}_1$ such that $\mathbb{G} \sim \mathbb{G}_1$?*

The fusion algebra of $A_o(F)$ is exactly same as the one of $SU_q(2)$ [3]. So, the problem is true for $A_o(F)$.

3.5. Universal quantum groups

Denote $A_u(1_n)$ by $A_u(n)$. It is known that $L^\infty(A_u(n))$ is a type II_1 factor (the trace is the Haar state) [4, 40].

Problem 3.10 (Banica). *Is the factor $L^\infty(A_u(n))$ isomorphic to a free group factor $L(\mathbb{F}_r)$ for some $r \in \mathbb{N}$?*

If $n = 2$, it is proved by Banica [4] with $r = 2$.

Even if the statement of the previous problem does not hold, it is expectable that the factor $L^\infty(A_o(n))$ or $L^\infty(A_u(n))$ have similar properties to free group factors. Indeed, $L^\infty(A_o(n))$ has Akemann-Ostrand property, it is solid [23, 40].

Problem 3.11. *Show that the factor $L^\infty(A_o(n))$ has no Cartan subalgebras.*

3.6. Ergodic actions of $SU(n)$

We have considered several problems related with compact quantum groups, but there still remain some problems for compact groups.

Problem 3.12 (Jones). *Can $SU(n)$ ergodically act on $\mathcal{R}_0$?*

The first breakthrough was made by A. Wassermann [42, 43]. He classified all ergodic actions of $SU(2)$, and indeed there does not exist an ergodic action on $\mathcal{R}_0$. Hence probably $SU(n)$ can not act on $\mathcal{R}_0$ ergodically. One of the key point of this problem is to require the factoriality. If $n \geq 3$, then $\mathbb{T}^2$ embeds into $SU(n)$. Inducing an ergodic action from $\mathbb{T}^2 \curvearrowright \mathcal{R}_0$ (consider the irrational rotation algebra), we see the induced algebra, which is of type II_1, has a non-trivial center.

Even if we focus on very restricted cases, the answer has not been given. Let $\mathbb{G} = SU(n)$ and $\omega \in R(\mathbb{G}) \otimes R(\mathbb{G})$ a 2-cocycle. Then $\mathbb{G}$ ergodically acts on the twisted group algebra $R_\omega(\mathbb{G})$, whose type has not been computed yet.

There is a relating problem [41]. Let us introduce the 2-cohomology set $H^2(\widehat{\mathbb{G}})$ which consists of equivalence classes of 2-cocycles.

Conjecture 3.13 (A. Wassermann). *Each element of $H^2(\widehat{\mathbb{G}})$ is coming from $H^2(\widehat{\mathbb{T}_{\max}})$ via the embedding $R(\mathbb{T}_{\max}) \subset R(\mathbb{G})$, where $\mathbb{T}_{\max}$ is the maximal torus of $\mathbb{G} = SU(n)$. In particular, $H^2(\widehat{\mathbb{G}})$ is a finite set.*

Of course, the following problem is still open.

Problem 3.14. *Classify all ergodic actions of $SU_q(2)$.*

Readers are referred to [8, 24, 30] for examples or partial classification results.

3.7. Galois correspondence

In [14, 32], a Galois correspondence for a minimal action of a c.q.g. is proved. The next step is to generalize this result to locally compact quantum group actions.

Problem 3.15. *Does the Galois correspondence hold for any minimal actions of locally compact quantum groups?*

Readers are referred to [11, 14, 32].

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