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INDECOMPOSABILITY RESULTS FOR AMALGAMATED FREE PRODUCTS OF VON NEUMANN ALGEBRAS

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A von Neumann algebra $\mathcal{M}$ is said to be *prime* if it cannot be written as a tensor product $\mathcal{P}_1 \otimes \mathcal{P}_2$ of diffuse von Neumann algebras. We first review some well-known results on primality for finite von Neumann algebras. Using free probability theory, Ge in [5] proved that free group factors are prime. Subsequently, in [19] Stefan generalized Ge's results to subfactors of finite index of the interpolated free group factors. A major breakthrough came when, using C*-algebraic methods, Ozawa discovered [8] a class $\mathcal{S}$ of groups Γ such that the von Neumann algebra $L(\Gamma)$ is *solid*, i.e. the relative commutant of any diffuse von Neumann subalgebra is amenable. In particular, this entails that if $\Gamma \in \mathcal{S}$ is an infinite conjugacy class (ICC) group then every non-amenable subfactor of $L(\Gamma)$ is prime. In [8] it was shown that the class $\mathcal{S}$ contains the hyperbolic groups, the Lie groups of rank one, etc. An interesting result obtained recently [10] by Ozawa is that $\mathbf{Z}^2 \rtimes \mathrm{SL}(2, \mathbf{Z}) \in \mathcal{S}$. Ozawa was also the first one to prove in [9], an analogue of the Kurosh theorem for type II_1 factors. He showed that any free product of weakly exact type II_1 factors, i.e. factors that contain a weakly dense exact C*-algebra, is prime. From a completely different prospective, using L^2-derivations techniques, Peterson was able to generalize in [11] some of Ozawa's results. For instance, he showed that the free product of diffuse finite von Neumann algebras is prime as well as the group von Neumann algebra $L(\Gamma)$ of a countable discrete group Γ with a non-vanishing first L^2-Betti number ($\beta_1^{(2)}(\Gamma) > 0$). Finally, using the *deformation/spectral gap* rigidity principle, Popa proved in [13] that for any non-amenable group Γ acting by Bernoulli shift on the AFD II_1 factor R, the crossed product von Neumann algebra $(\bigotimes_\Gamma R) \rtimes \Gamma$ is prime. This is precisely the result which inspired the authors for the present work.

In the type III setting, there are fewer results. Using Stefan's results, Shlyakhtenko showed in [17] that the unique free Araki-Woods factor of type III_λ (see [18]) is prime. Vaes & Vergnioux [24] constructed type III factors associated with discrete quantum groups for which they proved generalized solidity, and thus primality. Gao & Junge, proved in [4] that any free product of amenable von Neumann algebras w.r.t. faithful normal states is generalized solid and thus prime.

In this joint work with Ionut Chifan, we provide new indecomposability results for amalgamated free products of (not necessarily tracial) von Neumann algebras $\mathcal{M} = \mathcal{M}_1 *_B \mathcal{M}_2$ over a common amenable von Neumann algebra B. Proofs are based on Popa's *deformation/rigidity* argument. We refer to [6, 7, 11, 13, 14, 15, 16, 25] for some applications of this theory.

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For the purpose of this work, we will extend the intertwining techniques from [16] as well as some results of [7] in the context of *semifinite* von Neumann algebras. Given a type III factor $\mathcal{M}$, these techniques allow us to work with its *core* $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$ (which is of type II_∞) rather than $\mathcal{M}$ itself. The main technical result roughly says that in a semifinite amalgamated free product, one can locate the position of the relative commutant of a non-amenable subfactor. The strategy to prove this result follows the *deformation/spectral gap* rigidity principle developed by Popa in [13, 14]. We briefly remind below the two concepts that we will play against each other to prove our main result:

(1) The first ingredient we will use is the "malleable deformation" by automorphisms (α_t, β) defined on $N *_B (B \otimes L(\mathbf{F}_2))$. This deformation was introduced for the first time in [7] and it represented one of the key tools that lead to the computation of the symmetry groups of amalgamated free products of weakly rigid factors. It was shown in [13] that this deformation automatically features a certain "transversality property" (see Lemma 2.1 in [13]) which will be of essential use in our proof.

(2) The second ingredient we will use is the following property proved by Popa in [14], for free products of finite von Neumann algebras. Roughly, with B a finite amenable von Neumann algebra, any von Neumann subalgebra $Q \subset N$ with no amenable direct summand has "spectral gap" with respect to the orthogonal complement of N in $N *_B (B \otimes L(\mathbf{F}_2))$. In other words, there exists a finite subset $F \subset \mathcal{U}(Q)$ such that if $x \in N *_B (B \otimes L(\mathbf{F}_2))$ almost commutes with all the unitaries $u \in F$, then x is almost contained in N.

We obtain the following theorem that generalizes many previous results on primality, and moreover gives new examples of prime factors (of type II_1 and III):

Theorem 1. *For $i = 1, 2$, let $\mathcal{M}_i$ be a von Neumann algebra. Let $B \subset \mathcal{M}_i$ be a common von Neumann subalgebra, with $B \neq \mathcal{M}_i$, such that there exists a faithful normal conditional expectation $E_i : \mathcal{M}_i \to B$. Assume that B is a finite von Neumann algebra of type* I, *e.g. B is finite dimensional or B is abelian. Denote by $\mathcal{M} = \mathcal{M}_1 *_B \mathcal{M}_2$ the amalgamated free product. If $\mathcal{M}$ is a non-amenable factor, then $\mathcal{M}$ is prime.*

Using some of Ueda's results on factoriality and non-amenability of plain free products and of amalgamated free products over a common Cartan subalgebra (see [20, 21, 22, 23]), we obtain the following corollaries:

Corollary 2. *For $i = 1, 2$, let $(\mathcal{M}_i, \varphi_i)$ be any von Neumann algebra endowed with a f.n. state. Assume that the centralizer $\mathcal{M}_1^{\varphi_1}$ is diffuse and $\mathcal{M}_2 \neq \mathbf{C}$. Then the free product $(\mathcal{M}, \varphi) = (\mathcal{M}_1, \varphi_1) * (\mathcal{M}_2, \varphi_2)$ is a prime factor.*

Corollary 3. *For $i = 1, 2$, let $\mathcal{M}_i$ be a non-type* I *factor, and $B \subset \mathcal{M}_i$ be a common Cartan subalgebra. Then the amalgamated free product $\mathcal{M} = \mathcal{M}_1 *_B \mathcal{M}_2$ is a prime factor.*

In particular, let $\Gamma = \Gamma_1 * \Gamma_2$ be a free product of countable infinite groups. Let $\sigma : \Gamma \curvearrowright (X, \mu)$ be a free action such that the measure μ is quasi-invariant under σ, and such that the restricted action $\sigma_{|\Gamma_i}$ is ergodic and non-transitive for $i = 1, 2$. Then the crossed product $L^\infty(X, \mu) \rtimes \Gamma$ is a prime factor. In the type II_1 case, we get a more general result:

Theorem 4. *For $i = 1, 2$, let M_i be a II_1 factor and $B \subset M_i$ be a common abelian von Neumann subalgebra such that $\tau_{1|B} = \tau_{2|B}$. Then the amalgamated free product $M = M_1 *_B M_2$ is a non-amenable II_1 factor. Thus, M is prime.*

We moreover obtain Bass-Serre type rigidity results for free products of equivalence relations. Let (X, μ) be the standard Borel non-atomic probability space. Let $\mathcal{R}$ be a countable Borel measure-preserving equivalence relation on (X, μ). Denote by $[\mathcal{R}]$, the *full group* of all Borel m.p. isomorphisms $\phi : X \to X$ such that $(x, \phi(x)) \in \mathcal{R}$ for almost every $x \in X$. Denote by $[[\mathcal{R}]]$, the set of all partial Borel m.p. isomorphisms $\phi : \mathrm{dom}(\phi) \to \mathrm{rng}(\phi)$, such that $(x, \phi(x)) \in \mathcal{R}$ for almost every $x \in \mathrm{dom}(\phi)$. A partial Borel isomorphism $\phi \in [[\mathcal{R}]]$ is said to be *properly outer* if $\phi(x) \neq x$, for almost any $x \in \mathrm{dom}(\phi)$. Remind the following notion of *freeness* for equivalence relations due to Gaboriau.

Definition 5 (Gaboriau, [3]). *Let $(\mathcal{R}_k)_{k \in \mathbf{N}}$ be a sequence of m.p. equivalence relations on the probability space (X, μ). The sequence $(\mathcal{R}_k)$ is said to be free if for any $n \geq 1$, for any $i_1 \neq \cdots \neq i_n \in \mathbf{N}$, for any $\phi_j \in [[\mathcal{R}_{i_j}]]$, whenever ϕ_j is properly outer, the product $\phi_1 \cdots \phi_n$ is still properly outer.*

In order to state the main result, we first introduce a few notations. Fix integers $m, n \geq 1$. For each $i \in \{1, \ldots, m\}$, and $j \in \{1, \ldots, n\}$ let

$$\begin{aligned} \Gamma_i &= G_i \times H_i \\ \Lambda_j &= G'_j \times H'_j \end{aligned}$$

be ICC (infinite conjugacy class) groups, such that G_i, G'_j are not amenable and H_i, H'_j are infinite. Note that Γ_i and Λ_j have a vanishing first L^2-Betti number (see [2, 12]). Denote $\Gamma = \Gamma_1 * \cdots * \Gamma_m$ and $\Lambda = \Lambda_1 * \cdots * \Lambda_n$.

Let $\sigma : \Gamma \curvearrowright (X, \mu)$ be a free ergodic m.p. action of Γ on the probability space (X, μ) such that $\sigma_i := \sigma_{|\Gamma_i}$ is still ergodic (see Appendix in [7]). Write $A = L^\infty(X, \mu)$, $M_i = A \rtimes \Gamma_i$, $M = A \rtimes \Gamma$, and $\mathcal{R}_{\sigma_i, \Gamma_i}, \mathcal{R}_{\sigma, \Gamma}$ the associated equivalence relations.

Likewise, denote by $\rho : \Lambda \curvearrowright (Y, \nu)$ a free ergodic m.p. action of Λ on the probability space (Y, ν) such that $\rho_j := \rho_{|\Lambda_j}$ is still ergodic. Write $B = L^\infty(Y, \nu)$, $N_j = B \rtimes \Lambda_j$, $N = B \rtimes \Lambda$, and $\mathcal{R}_{\rho_j, \Lambda_j}, \mathcal{R}_{\rho, \Lambda}$ the associated equivalence relations. Then we have

$$\begin{aligned} \mathcal{R}_{\sigma, \Gamma} &\simeq \mathcal{R}_{\sigma_1, \Gamma_1} * \cdots * \mathcal{R}_{\sigma_m, \Gamma_m} \\ \mathcal{R}_{\rho, \Lambda} &\simeq \mathcal{R}_{\rho_1, \Lambda_1} * \cdots * \mathcal{R}_{\rho_n, \Lambda_n}. \end{aligned}$$

We obtain the following analogues of Theorem 7.7 and Corollary 7.8 of [7]. The proofs are exactly the same. These results can be viewed as Bass-Serre type rigidity results.

Theorem 6. *If $\theta : M \to N^t$ is a $*$-isomorphism, then $m = n$, $t = 1$, and after permutation of indices there exist unitaries $u_j \in N$ such that for all j*

$$\begin{aligned} \mathrm{Ad}(u_j)\theta(M_j) &= N_j \\ \mathrm{Ad}(u_j)\theta(A) &= B. \end{aligned}$$

In particular $\mathcal{R}_{\sigma, \Gamma} \simeq \mathcal{R}_{\rho, \Lambda}$ and $\mathcal{R}_{\sigma_j, \Gamma_j} \simeq \mathcal{R}_{\rho_j, \Lambda_j}$, for any j.

Corollary 7. *If $\mathcal{R}_{\sigma, \Gamma} \simeq \mathcal{R}^t_{\rho, \Lambda}$, then $m = n$, $t = 1$, and after permutation of indices, we have $\mathcal{R}_{\sigma_j, \Gamma_j} \simeq \mathcal{R}_{\rho_j, \Lambda_j}$, for any j.*

Corollary 7 has been recently generalized by Alvarez & Gaboriau [1] to all non-amenable countable groups Γ_i, Λ_j with a vanishing first L^2-Betti number.

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