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Toward asymptotic non-degeneracy results for the mean field equations on surfaces[1]

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Abstract

The purpose of this note is to collect some facts seems to necessary to get asymptotic non-degeneracy results for the mean field equations on surfaces. Some facts in this note would be new but some are presented by other researchers. Nevertheless we would like to present here and give some comments for them.

This note is prepared for the joint work with Dr. Tomohiko Sato of Gakushuin University and Prof. Takashi Suzuki of Osaka University.

1 Preliminaries

Let (M,g) be a two-dimensional compact orientable Riemannian manifold without boundary. In this note, we are concerned with the blow-up sequences of solutions for the mean field equation on (M,g):

$$-\Delta_g u = \rho\left(\frac{h(x)e^u}{\int_\Omega h(x)e^u} - \frac{1}{|M|}\right), \qquad \int_M u = 0, \tag{1}$$

where $h(x)$ is a non-negative smooth function and ρ is a positive parameter. Our main interest is the property so-called the *asymptotic non-degeneracy* of the linearized problems of (1) around the solutions of those sequences. The property seems to be useful to understand the global structure of the solution set $\{(u,\rho)\}$ for (1).

Motivated by several background of the problem (see, e.g., [6, 9] and references therein), the behaviour of the blow-up sequences of solutions for (1) is widely studied by several authors. Here we recall the followings:

Fact 1.1 ([6, Theorem 0.2]). *Let $\{h_k\} \subset W^{1,\infty}(M)$ satisfy*

$$\liminf_{k\longrightarrow\infty} \min_M h_k > 0, \quad \limsup_{k\longrightarrow\infty}\left(\max_M h_k + \|\nabla h_k\|_{L^\infty(M)}\right) < \infty$$

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and $\{\rho_k\} \subset \mathbf{R}$ *satisfy*

$$\rho_k \longrightarrow \rho \in \mathbf{R}.$$

Then for every sequence $\{u_k\}$ *of solutions for (1) with* $h = h_k$ *and* $\rho = \rho_k$ *satisfying*

$$\max_M |u_k| \longrightarrow \infty,$$

there exist distinct m *points* $\{p_1, \ldots, p_m\} \subset M$ *and a subsequence of* $\{u_k\}$, *still denoted by the same symbols, satisfying*

$$u_k \longrightarrow 8\pi \sum_{j=1}^{m} G(x, p_j) \quad \text{in } C^2_{\text{loc}}\left(M \backslash \{p_1, \ldots, p_m\}\right), \tag{2}$$

where $G(x, y)$ *denotes the Green function of* $(-\Delta_g)^{-1}$ *with respect to the condition* $\int_M \cdot = 0$.

We note that as a consequence of Fact 1.1 we have $\rho \in 8\pi\mathbf{N}$ and

$$\rho_k \frac{h_k(x) e^{u_k}}{\int_\Omega h_k(x) e^{u_k}} \longrightarrow 8\pi \sum_{j=1}^{m} \delta_{p_j} \quad \text{weakly} * \text{ in } \mathcal{M}(M). \tag{3}$$

The location of the blow-up points is also distinguished:

Fact 1.2 ([9, Theorem 2.2]). *Under the situation of Fact 1.1, suppose there exists* $h \in C^1(M)$ *and*

$$h_k \longrightarrow h \quad \text{in } W^{1,\infty}(M).$$

Then for every p_j *and every isothermal chart* (U, ψ) *around* p_j *satisfying*

$$\psi(p_j) = 0, \quad g = e^{\xi(X)}\left(dX_1^2 + dX_2^2\right), \tag{4}$$

we have

$$\nabla_X \left\{ 4\pi \tilde{G}(X, X) + \sum_{l \neq j} 8\pi G(X, p_l) + \log h(X) + \xi(X) \right\} \Bigg|_{X=0} = 0,$$

where $f(X) = f(\psi^{-1}(X))$ *for every function* $f(x)$ *on* M *and*

$$\tilde{G}(x, y) = G(x, y) + \frac{1}{2\pi} \log |\psi(x) - \psi(y)|.$$

The problem (1) is the Euler-Lagrange equation of the functional

$$J_\rho(u) := \frac{1}{2}\int_M |\nabla u|^2 - \rho \log\left(\frac{1}{|M|}\int_M h(x)e^u\right)$$

on $u \in E := \left\{u \in H^1(M) \mid \int_M u = 0\right\}$. Therefore we are able to recognise Fact 1.2 as a fact insists that the critical points of $J_{\rho_k}(u)$ are *asymptotically* controlled by the critical points of

$$4\pi\tilde{G}(X,X) + \sum_{l\neq j} 8\pi G(X,p_l) + \log h(X) + \xi(X). \tag{5}$$

We start our study with thinking that this observation might be true also on the level of the *linearized problems*, that is,

Conjecture 1.3. *Under the situation of Fact 1.2, suppose $h \in C^2(M)$,*

$$h_k \longrightarrow h \quad \text{in } C^2(M),$$

and $X = 0$ is a non-degenerate critical point of (5) for every $j \in \{1,\dots,m\}$. Then u_k is a non-degenerate critical point of J_{ρ_k} for every $k \gg 1$.

The proof of this conjecture would be based on the argument of Gladiali-Grossi [5, Thoerem 1] and subsequent Sato-Suzuki [11, Theorem 1.4]. They proved the similar phenomena for the Liouville-Gel'fand problems on a bounded smooth domain $\Omega \subset \mathbf{R}^2$:

$$-\Delta v = \lambda V(x)e^v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{6}$$

where λ is a positive constant and $V(x) \equiv 1$ for Gladiali-Grossi and $V(x)$ is a positive C^1 function on $\overline{\Omega}$ for Sato-Suzuki. The classification of the possible limits of the blow-up sequences of solutions for (6) as $\lambda \longrightarrow 0$ are established by [8] for $V \equiv 1$ and [7] for other cases, which are summarized as follow:

Fact 1.4. *Let $\{(v_k,\lambda_k)\}$ be a sequence of solutions of (6) satisfying*

$$\lambda_k \longrightarrow 0, \quad \|v_k\|_{L^\infty(\Omega)} \longrightarrow \infty, \quad \limsup_{k\longrightarrow\infty} \lambda_k \int_\Omega V(x)e^{v_k} < \infty.$$

Then there exist distinct m (interior) points $\{p_1,\dots,p_m\} \subset \Omega$ and a subsequence of $\{v_k\}$, still denoted by the same symbols, satisfying

$$v_k \longrightarrow 8\pi\sum_{j=1}^m G_\Omega(x,p_j) \quad \text{in } C^2_{\text{loc}}\left(\overline{\Omega}\backslash\{p_1,\dots,p_m\}\right),$$

where $G_\Omega(x,y)$ *denotes the Green function of* $(-\Delta)^{-1}$ *with the Dirichlet boundary condition* $\cdot|_{\partial\Omega}=0$. *Moreover, for every* p_j *we have*

$$\nabla\left\{4\pi\tilde{G}_\Omega(x,x)+\sum_{l\neq j}8\pi G_\Omega(x,p_l)+\log V(x)\right\}\Bigg|_{x=p_j}=0,$$

where

$$\tilde{G}_\Omega(x,y)=G_\Omega(x,y)+\frac{1}{2\pi}\log|x-y|.$$

In this situation, the result of Sato-Suzuki, which contains that of Gladiali-Grossi, is as follows:

Fact 1.5 ([11, Theorem 1.4]). *Under the situation of Fact 1.4, suppose* $m=1$, $V(x)$ *is* C^2 *near* $p:=p_1$, *and* p *is a non-degenerate critical point of*

$$4\pi\tilde{G}_\Omega(x,x)+\log V(x).$$

Then v_k *is non-degenerate for every* $k\gg 1$, *that is, the linearized operator* $-\Delta-\lambda_k V(x)e^{v_k}$ *in* Ω *with the Dirichlet boundary condition* $\cdot|_{\partial\Omega}=0$ *is invertible.*

We note that we intended to extend Fact 1.5 threefold. First we would like to extend it to the cases on manifolds. It need some localization of the arguments established by Gladiali-Grossi and Sato-Suzuki, which are in some sense global over Ω. Second we would like to extend it to the cases of the mean field equation (1) that have *non-local* nonlinear term, which causes the linearized operator to be more complicated. Third we would like to consider the cases of many blow-up points. Part of the results are already established in the doctoral dissertation of Dr. Sato [10, Theorem 5] (for the mean field equation in Ω with the Dirichlet boundary condition $\cdot|_{\partial\Omega}=0$ and $m=1$). Examining the argument in detail, we seems to be able to resolve the above threefold extensions by a unified manner, but unfortunately we have not finished it yet.

2 Observations

Before we start detailed calculations, we would like to observe several facts obtained by standard arguments.

2.1 On the weak limits of sequences of solutions for linearized problem

Following the argument of [5, 11], we would like to prove Conjecture 1.3 by contradiction. So suppose u_k be a *degenerate* critical point of J_{λ_k} for $k \gg 1$. Then we are able to take $w_k \in E$ satisfying the following properties:

$$\begin{aligned} -\Delta_g w_k &= \rho_k \frac{h_k e^{u_k} w_k \int_M h_k e^{u_k} dv_g - h_k e^{u_k} \int_M h_k e^{u_k} w_k dv_g}{\left(\int_M h_k e^{u_k} dv_g\right)^2}, \\ &= \rho_k \frac{h_k e^{u_k}}{\int_M h_k e^{u_k} dv_g} \left(w_k - \frac{\int_M h_k e^{u_k} w_k dv_g}{\int_M h_k e^{u_k} dv_g} \right), \\ &=: \rho_k \frac{h_k e^{u_k}}{\int_M h_k e^{u_k} dv_g} \left(w_k + c_k \right), \end{aligned} \tag{7}$$

$$\begin{aligned} \int_M w_k dv_g &= 0, \\ \|w_k\|_{L^\infty(M)} &= 1. \end{aligned} \tag{8}$$

From (8), we have

$$c_k = -\frac{\int_M h_k e^{u_k} w_k dv_g}{\int_M h_k e^{u_k} dv_g} = O(1) \in [-1, 1] \tag{9}$$

as $k \longrightarrow \infty$ and we may assume that there exist $c_\infty \in [-1, 1]$ such that

$$c_k \longrightarrow c_\infty \quad \text{as } k \longrightarrow \infty,$$

taking subsequence if necessary. Also from (8) and the well-know behaviours (2) and (3) of (sub-) sequence of solutions $\{u_k\}$, we are also able to assume that there exists $w_\infty \in L^\infty(M) \cap C^{2,\alpha}_{\mathrm{loc}}(M \backslash \{p_1, \cdots, p_m\})$ satisfying

$$w_k \longrightarrow w_\infty \quad \text{weakly} * \text{ in } L^\infty(M) \text{ and in } C^{2,\alpha}_{\mathrm{loc}}(M \backslash \{p_1, \cdots, p_m\}) \tag{10}$$

for each $0 < \alpha < 1$ without loss of generality. Especially we have

$$\int_M w_\infty dv_g = \lim_{k \longrightarrow \infty} \int_M w_k dv_g = 0. \tag{11}$$

Moreover, we have

$$-\Delta_g w_k \longrightarrow 0 \quad \text{in } L^\infty_{\mathrm{loc}}(M \backslash \{p_1, \ldots, p_m\})$$

and

$$-\Delta_g w_\infty = 0 \quad \text{in } M \backslash \{p_1, \ldots, p_m\}.$$

Then from the removable singularity theorem for bounded harmonic functions we are able to conclude that

$$w_\infty \equiv 0 \quad \text{in } M \tag{12}$$

since w_∞ is normalized (11). Therefore we have

$$w_k \longrightarrow w_\infty \equiv 0 \quad \text{weakly} * \text{ in } L^\infty(M) \text{ and in } C^{2,\alpha}_{\text{loc}}(M\backslash\{p_1,\cdots,p_m\}). \tag{13}$$

Consequently, we might be able to *localize* the proof on each neighbourhood of p_j.

Here we should be remarked that we are not able to determine the number c_∞ so far because we have not got the uniform convergence of w_k around the blow-up points. Nevertheless the following is obvious: if w_k uniformly converges to a function $w_{\infty,j}$ in a sufficiently small neighbourhood of p_j for each $j = 1,\cdots,m$, the function $w_{\infty,j} \equiv 0$ for each j. If these local convergences are established for every j, we get $c_\infty = 0$.

2.2 Localization

Taking isothermal chart (U,ψ) around p_j satisfying (4), we are able to get

$$-\Delta u_k = \rho_k e^\xi \left(\frac{h_k e^{u_k}}{\int_\Omega h_k(x)e^{u_k}} - \frac{1}{|M|} \right) \quad \text{in } \Omega = \psi(U) \subset \mathbf{R}^2$$

from (1) with $h = h_k$ and $u = u_k$. To simplify the presentation, we use x instead of X for the coordinate of Ω and also $f(x)$ instead of $f(X) = f(\psi^{-1}(X))$ for a function f on M.

Let $H_\xi(x)$ and B_k satisfy

$$-\Delta H_\xi = -e^\xi \frac{1}{|M|} \quad \text{in } \Omega, \quad H_\xi = 0 \quad \text{on } \partial\Omega$$

and

$$-\Delta B_k = 0 \quad \text{in } \Omega, \quad B_k = u_k \quad \text{on } \partial\Omega.$$

Then setting

$$u_k^L = u_k - \rho_k H_\xi - B_k, \quad v_k^L = w_k + c_k,$$

we get

$$-\Delta u_k^L = \lambda_k V_k(x) e^{u_k^L} \quad \text{in } \Omega, \quad u_k^L = 0 \quad \text{on } \partial\Omega \tag{14}$$

and

$$-\Delta v_k^L = \lambda_k V_k(x) e^{u_k^L} v_k^L \quad \text{in } \Omega, \quad v_k^L = w_k + c_k = O(1) \quad \text{on } \partial\Omega, \tag{15}$$

where

$$\lambda_k = \frac{\rho_k}{\int_M h_k e^{u_k}}, \quad V_k(x) = h_k e^{\xi + \rho_k H_\xi + B_k} = e^{\rho_k H_\xi + B_k + \log h_k + \xi}.$$

We note that

$$\lambda_k V_k(x) e^{u_k^L} \longrightarrow 8\pi\delta_0 \quad \text{weakly} * \text{ in } \mathcal{M}(\overline{\Omega})$$

from (3) and

$$\lambda_k \longrightarrow 0, \quad \|u_k^L\|_{L^\infty(\Omega)} \longrightarrow \infty, \quad \lim_{k\longrightarrow\infty} \lambda_k \int_\Omega V_k e^{u_k^L} \longrightarrow 8\pi$$

Moreover, we have

$$\|v_k^L\|_{L^\infty(\Omega)} = \|w_k + c_k\|_{L^\infty(U)} = O(1)$$

from (8) and (9). We also note that

$$v_k^L = w_k + c_k \longrightarrow c_\infty \quad \text{uniformly on } \partial\Omega \tag{16}$$

(or even more regularly) from (13). Therefore the localized problem seems to differ from the problems considered in [11] only in the *non-homogeneous* and *asymptotically constant* boundary condition of the linearized equation (15) (and varying $V = V_k$).

It also should be remarked that

$$\begin{aligned}\log V_k &= \rho_k H_\xi + B_k + \log h_k + \xi \\ &\longrightarrow 8\pi G(\cdot, p_j) + \sum_{l\neq j} 8\pi G(\cdot, p_l) - 8\pi G_\Omega(\cdot, 0) + \log h + \xi \\ &= 8\pi \tilde{G}(\cdot, p_j) + \sum_{l\neq j} 8\pi G(\cdot, p_l) - 8\pi \tilde{G}_\Omega(\cdot, 0) + \log h + \xi.\end{aligned}$$

Therefore assuming that the conclusion of Fact 1.5 holds for our situation, we get formally $v_k^L \longrightarrow 0$ uniformly in Ω if $0 \in \Omega$ is a non-degenerate critical point of

$$4\pi\tilde{G}_\Omega(\cdot,\cdot) + \log V(\cdot) \tag{17}$$

$$\begin{aligned}&= 4\pi G(\cdot,\cdot) + \sum_{l\neq j} 8\pi G(\cdot, p_l) + \log h + \xi \\ &\quad + \{4\pi\tilde{G}_\Omega(\cdot,\cdot) - 8\pi\tilde{G}_\Omega(\cdot, 0)\} - \{4\pi\tilde{G}(\cdot,\cdot) - 8\pi\tilde{G}(\cdot, p_j)\} \\ &=: (\mathrm{I}) + (\mathrm{II}).\end{aligned} \tag{18}$$

Here (I) is exactly (5) but we have extra terms (II). Nevertheless it should be remarked that $\nabla(\mathrm{II})|_{x=0}$ from the symmetries $G(x,y) = G(y,x)$ and $G_\Omega(x,y) = G_\Omega(y,x)$. Therefore it does not contradict Fact 1.2. To control (II) seems to be a point left for us to prove Conjecture 1.3. Here I would like to remark that it is more natural to consider the set of blow-up points $\{p_1, \cdots, p_m\}$ to be a critical point of

$$\sum_{j=1}^{m} 4\pi\tilde{G}(X_j, X_j) + \sum_{l\neq j} 4\pi G(X_j, X_l) + \sum_{j=1}^{m} (\log h(X_j) + \xi(X_j)) \tag{19}$$

instead of (5). We note that in (19) we assumed that all p_j is in one local isothermal chart. So it seems to be better that we define (19) by a *global* manner and consider the non-degeneracy of the global version of (19).

Despite some extra terms observed in (18), I expect that we would get $v_k^L = w_k + c_k \longrightarrow 0$, that is, $w_k \longrightarrow$ *constant* as $k \longrightarrow 0$ in a neighbourhood of each blow-up point. Consequently we would get $w_k \longrightarrow 0$ uniformly in M from (10) and (12).

3 Gladiali-Grossi's argument

3.1 Our settings

In this section, we would like to consider the localized problem (14) and (15) in the framework of Gladiali-Grossi[5]. To simplify the presentation we assume

$$V_k \equiv 1,$$

that is, we consider the following problem in a bounded smooth domain $\Omega \subset \mathbf{R}^2$ in the rest of this note: u_k and w_k are smooth functions on $\overline{\Omega}$ satisfying

$$-\Delta u_k = \lambda_k e^{u_k} \quad \text{in } \Omega, \quad u_k = 0 \quad \text{on } \partial\Omega, \tag{20}$$

$$-\Delta v_k = \lambda_k e^{u_k} v_k \quad \text{in } \Omega, \quad v_k \longrightarrow c_\infty \in \mathbf{R} \quad \text{uniformly in } \partial\Omega, \tag{21}$$

where

$$
\begin{aligned}
&\lambda_k \longrightarrow 0 \quad \text{as } k \longrightarrow \infty,\\
&\lambda_k \int_\Omega e^{u_k} \longrightarrow 8\pi,\\
&\lambda_k e^{u_k} \longrightarrow 8\pi\delta_0 \quad \text{weakly} * \text{ in } \mathcal{M}(\overline{\Omega}),\\
&\lambda_k e^{u_k} \longrightarrow 0 \quad \text{locally uniformly in } \overline{\Omega}\backslash\{0\},\\
&u_k \longrightarrow 8\pi G_\Omega(\cdot,0) \quad \text{in } C^2_{\mathrm{loc}}(\Omega\backslash\{0\}),\\
&\nabla R(0) = 0,\\
&\|v_k\|_{L^\infty(\Omega)} = O(1).
\end{aligned}
$$

Here we set

$$\tilde{G}_\Omega(x,x) =: R(x),$$

which is called the Robin function of Ω.

We note that Gladiali-Grossi's case [5] is the case $v_k \equiv 0$ on $\partial\Omega$ in (21) and they proved $v_k \longrightarrow 0$ uniformly in Ω assuming

$$(\partial_i\partial_j R(0))_{i,j=1,2} \tag{22}$$

is a invertible 2×2 matrix. We would like to consider if the same conclusion holds under above settings.

3.2 On the one point blow-up mean field Dirichlet case

It also should be noticed that the above situation also contains some special case $v_k|_{\partial\Omega}$ is constant for each k, which correspond to the one point blow-up sequence of the mean field equation in Ω with the Dirichlet boundary condition:

$$
\begin{aligned}
&-\Delta u_k = \rho_k \frac{e^{u_k}}{\int_M e^{u_k}} \quad \text{in } \Omega, \quad u_k = 0 \quad \text{on } \partial\Omega\\
&\rho_k \longrightarrow 8\pi, \quad \rho_k \frac{e^{u_k}}{\int_M e^{u_k}} \longrightarrow 8\pi\delta_0 \quad \text{weakly} * \text{ in } \mathcal{M}(\overline{\Omega}).
\end{aligned}
$$

In this case, corresponding (normalized) linearized problem becomes as follows:

$$
\begin{aligned}
-\Delta_g w_k &= \rho_k \frac{e^{u_k} w_k \int_\Omega e^{u_k} - e^{u_k}\int_\Omega e^{u_k} w_k}{\left(\int_\Omega e^{u_k}\right)^2},\\
&= \rho_k \frac{e^{u_k}}{\int_\Omega e^{u_k}}\left(w_k + c_k\right), \quad \text{in } \Omega, \qquad (23)\\
w_k &= 0, \quad \text{in } \partial\Omega,\\
\|w_k\|_{L^\infty(\Omega)} &= 1,
\end{aligned}
$$

where

$$c_k = -\frac{\int_\Omega e^{u_k} w_k}{\int_\Omega e^{u_k}}.$$

Therefore, setting $v_k = w_k + c_k$ we have (20) and (21) with

$$v_k \equiv c_k = O(1) \quad \text{on } \partial\Omega. \tag{24}$$

This *one point blow-up mean field Dirichlet case* is treated in [10] and we also see later in this note.

3.3 Rescaling

The proof of Gladiali-Grossi is based on the rescaling argument. Let $x_k \in \Omega$ be a point satisfying

$$\max_{x\in\Omega} u_k (= \|u_k\|_{L^\infty(\Omega)}) = u_k(x_k)$$

and δ_k be a number determined by

$$\delta_k^2 \lambda_k e^{\|u_k\|_{L^\infty(\Omega)}} = 1.$$

We note that

$$\|u_k\|_{L^\infty(\Omega)} = -2\log\lambda_k + 2\log 8 - 8\pi R(0) + o(1) \quad \text{as } k \longrightarrow \infty \tag{25}$$

is known [5, Thorem 7]. Therefore

$$\delta_k = \lambda_k^{-\frac{1}{2}} e^{-\frac{1}{2}\|u_k\|_{L^\infty(\Omega)}} = \lambda_k^{\frac{1}{2}} \times O(1) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{26}$$

Under these notations, we define

$$\begin{aligned} \tilde{u}_k(x) &= u_k(\delta_k x + x_k) - \|u_k\|_{L^\infty(\Omega)}, \\ \tilde{v}_k(x) &= v_k(\delta_k x + x_k). \end{aligned}$$

Then $\tilde{u}_k(x)$ and $\tilde{v}_k(x)$ satisfy the following on $\Omega_k = (\Omega - x_k)/\delta_k$:

$$\begin{aligned} &-\Delta\tilde{u}_k = e^{\tilde{u}_k}, \quad \tilde{u}_k(x) \leqq \tilde{u}_k(0) = 0, \quad \text{in } \Omega_k, \quad \int_{\Omega_k} e^{\tilde{u}_k} \longrightarrow 8\pi \\ &-\Delta\tilde{v}_k = e^{\tilde{u}_k}\tilde{v}_k \quad \text{in } \Omega_k, \quad \|\tilde{v}_k\|_{L^\infty(\Omega_k)} = O(1) \quad \text{as } k \longrightarrow \infty. \end{aligned}$$

Thanks to the Brezis-Merle's theory [1], we are able to see that $\{\tilde{u}_k\}$ is locally uniformly bounded. Then taking subsequence if necessary, we are able to get $\tilde{u}_k \longrightarrow \tilde{u}_\infty \in C^{2,\alpha}_{\mathrm{loc}}(\mathbf{R}^2)$ for every $0 < \alpha < 1$ such that

$$-\Delta\tilde{u}_\infty = e^{\tilde{u}_\infty}, \quad \tilde{u}_\infty(x) \leqq \tilde{u}_\infty(0) = 0, \quad \text{in } \mathbf{R}^2, \quad \int_{\mathbf{R}^2} e^{\tilde{u}_\infty} < \infty.$$

The solution of this equation is know to exist uniquely [3]:

$$\tilde{u}_\infty = \log \frac{1}{\left(1+\frac{|x|^2}{8}\right)^2}.$$

On the other hand, thanks to the convergence of $\{\tilde{u}_k\}$, we are able to get $\tilde{v}_k \longrightarrow \tilde{v}_\infty \in C^{2,\alpha}_{\mathrm{loc}}(\mathbf{R}^2)$ for every $0<\alpha<1$ taking subsequence if necessary. Here $\tilde{v}_\infty$ satisfies

$$-\Delta \tilde{v}_\infty = e^{\tilde{u}_\infty}\tilde{v}_\infty, \quad \|\tilde{v}_\infty\|_{L^\infty(\mathbf{R}^2)} < \infty.$$

The solution of this equation is also know to exist and classified [2]: there exists $\mathbf{a}=(a_1,a_2)\in\mathbf{R}^2$ and $b\in\mathbf{R}$ such that

$$\tilde{v}_\infty = \sum_{i=1}^{2} \frac{a_i x_i}{8+|x|^2} + b\frac{8-|x|^2}{8+|x|^2}$$

So far, we get same conclusions as Gladiali-Grossi.

The lest of the proof is divided into the following steps:

3.4 Step. 1: The asymptotic behaviour when $\mathbf{a}\neq\mathbf{0}$.

Assuming $\mathbf{a}\neq\mathbf{0}$, Gladiali-Grossi show the following asymptotic behaviour:

Fact 3.1 ([5, (3.13)]). *If $v_k \equiv 0$ on $\partial\Omega$ and $\mathbf{a}\neq\mathbf{0}$, we have*

$$\frac{v_k}{\delta_k} = 2\pi\,(\mathbf{a}\cdot\nabla_2)\,G_\Omega(\cdot,0) + o(1) \quad \textit{locally uniformly in } \overline{\Omega}\backslash\{0\}, \tag{27}$$

where ∇_2 denote the ∇ with respect to the second component of $G_\Omega(x,y)$.

The proof of this lemma is established by the representation

$$v_k(x) = \int_\Omega G_\Omega(x,y)\lambda_k e^{u_k(x)} v_k(x)dx$$

using the Dirichlet boundary value condition of v_k. Therefore we might imagine that the following behaviour hold for our *non-homogeneous* boundary condition of (21):

$$\frac{P_{H_0^1(\Omega)}v_k}{\delta_k} = 2\pi\,(\mathbf{a}\cdot\nabla_2)\,G_\Omega(\cdot,0) + o(1) \quad \text{locally uniformly in } \overline{\Omega}\backslash\{0\}, \tag{28}$$

where $P_{H_0^1(\Omega)}v_k$ denote the projection of v_k to $H_0^1(\Omega)$, that is, $v = P_{H_0^1(\Omega)}f$ is the solution of the following problem:

$$-\Delta v = -\Delta f \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega.$$

Indeed,

$$\begin{aligned}
P_{H_0^1(\Omega)}v_k &= \int_\Omega G_\Omega(x,y)\lambda_k e^{u_k(x)} v_k(x)dx \\
&= \int_{\Omega_k} G_\Omega(x, x_k + \delta_k y) e^{\tilde{u}_k(y)} \tilde{v}_k(y)dy \\
&= \int_{\Omega_k} G_\Omega(x, x_k + \delta_k y) \left\{ f_k(y) + 64b\frac{8-|x|^2}{(8+|x|^2)^3} \right\} dy \\
&= (\mathrm{I}) + (\mathrm{II}),
\end{aligned}$$

where

$$f_k(x) = e^{\tilde{u}_k}\tilde{v}_k - 64b\frac{8-|x|^2}{(8+|x|^2)^3} \left(= e^{\tilde{u}_k}\tilde{v}_k - e^{\tilde{u}_\infty}\tilde{v}_\infty\right).$$

The calculation

$$(\mathrm{II}) = o(\delta_k)$$

is established in [5, (3.12)]. For the other part (I), Gladiali-Grossi applying the following fact with $f = f_k$ and get

$$(\mathrm{I}) = \delta_k \left\{ 2\pi\, (\mathbf{a}\cdot\nabla_2)\, G(x,0) + o(1) \right\}$$

[5, (3.11)] when $P_{H_0^1(\Omega)}v_k = v_k$.

Fact 3.2 ([5, Lemma 6]). *Let $f \in C^1(\mathbf{R}^2)$ be a function of $\mathbf{x} = (x_1, x_2)$ satisfying $f(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|^4}\right)$ at infinity and set*

$$w(\mathbf{x}) = \int_\infty^{\frac{1}{|\mathbf{a}|^2}(P_\mathbf{a}\mathbf{x}\cdot\mathbf{a})} f(t\mathbf{a} + P_{\mathbf{a}^\perp}\mathbf{x})dt, \tag{29}$$

where $P_\mathbf{b}\mathbf{x} = \frac{(\mathbf{b}\cdot\mathbf{x})}{|\mathbf{b}|^2}\mathbf{b}$ denote the projection of $\mathbf{x} \in \mathbf{R}^2$ to $\mathbf{b} \in \mathbf{R}^2$ and $a^\perp = (a_2, -a_1)$. Then w satisfies

$$(\mathbf{a}\cdot\nabla)\, w(\mathbf{x}) = f(\mathbf{x}). \tag{30}$$

We note that, assuming $f \in \mathcal{D}(\mathbf{R}^2)$, we are able to prove the above lemma easily because $\frac{1}{|\mathbf{a}|^2}(P_\mathbf{a}\mathbf{x}\cdot\mathbf{a})\mathbf{a} = P_\mathbf{a}\mathbf{x}$. Moreover, under appropriate rotation, we are able to assume $\mathbf{a} = (a, 0)$ $(a > 0)$ without loss of generality. Then the above formula (29) becomes simply

$$w(\mathbf{x}) = \int_{\infty}^{\frac{x_1}{a}} f(ta, x_2)dt.$$

Therefore it is easy to see that we are able to get the conclusion (30) under more weaker condition than $f \in C^1(\mathbf{R}^2)$ and $f(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|^4}\right)$ at infinity if we weaken the meaning of the equality (30), e.g., (30) holds for a.e.$x \in \mathbf{R}^2$.

In our cases, $\tilde{v}_k$ is not necessarily continuously at $\partial\Omega_k$ because of the *non-homogeneous* boundary condition of (21). Therefore we must use the above lemma after slight modifications but then we would get (28).

3.5 Step.2: $\mathbf{a} = 0$ if (22) is non-degenerate.

Suppose we get the asymptotic behaviour (28). Then the following Pohozaev type identity for $G_\Omega(x, y)$ is applicable:

Fact 3.3 ([5, Lemma 7]). *It holds*

$$\frac{\partial^2 R(y)}{\partial y_i \partial y_j} = -2\int_{\partial\Omega} \frac{\partial G_\Omega(x, y)}{\partial x_i} \frac{\partial}{\partial y_j}\left(\frac{\partial G_\Omega(x, y)}{\partial \nu_x}\right) dS_x.$$

for every $y \in \Omega$.

Then we have

$$\begin{aligned}
\sum_{j=1}^{2} \partial_i\partial_j R(0)a_j &= -2\int_{\partial\Omega} \frac{\partial G_\Omega(x, 0)}{\partial x_i} \frac{\partial}{\partial \nu_x} \{(\mathbf{a}\cdot\nabla_2)\, G_\Omega(x, 0)\}\, dS_x \\
&= -2\lim_{k\longrightarrow\infty}\int_{\partial\Omega} \frac{1}{8\pi}\partial_i u_k \frac{\partial}{\partial \nu_x}\left(-\frac{1}{2\pi}\frac{P_{H_0^1(\Omega)}v_k}{\delta_k}\right) dS_x \\
&= \frac{1}{8\pi^2}\lim_{k\longrightarrow\infty}\frac{1}{\delta_k}\int_{\partial\Omega} \partial_i u_k \frac{\partial}{\partial \nu_x} P_{H_0^1(\Omega)}v_k dS_x. \qquad (31)
\end{aligned}$$

Here we have

$$\begin{aligned}
&\int_{\partial\Omega} \partial_i u_k \frac{\partial}{\partial \nu_x} P_{H_0^1(\Omega)}v_k dS_x, \\
&= \int_{\Omega} \partial_i u_k \Delta P_{H_0^1(\Omega)}v_k - \int_{\Omega} (\Delta\partial_i u_k) P_{H_0^1(\Omega)}v_k. \qquad (32)
\end{aligned}$$

Recall that

$$
\begin{aligned}
&-\Delta P_{H_0^1(\Omega)} v_k = -\Delta v_k = \lambda_k e^{u_k} v_k, \\
&-\Delta \partial_i u_k = \lambda_k e^{u_k} \partial_i u_k .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(32) &= -\lambda_k \int_\Omega e^{u_k} (\partial_i u_k) v_k + \lambda_k \int_\Omega e^{u_k} (\partial_i u_k) P_{H_0^1(\Omega)} v_k \\
&= -\lambda_k \int_\Omega e^{u_k} (\partial_i u_k) \left(v_k - P_{H_0^1(\Omega)} v_k \right) \\
&= -\lambda_k \int_\Omega \partial_i e^{u_k} \left(v_k - P_{H_0^1(\Omega)} v_k \right) \\
&= -\lambda_k \left\{ \int_{\partial\Omega} v_k \nu_i - \int_\Omega e^{u_k} \partial_i \left(v_k - P_{H_0^1(\Omega)} v_k \right) \right\}
\end{aligned}
$$

and

$$
(31) = -\frac{1}{8\pi^2} \lim_{k \longrightarrow \infty} \left\{ \frac{\lambda_k}{\delta_k} \int_{\partial\Omega} v_k \nu_i - \frac{\lambda_k}{\delta_k} \int_\Omega e^{u_k} \partial_i \left(v_k - P_{H_0^1(\Omega)} v_k \right) \right\}. \tag{33}
$$

Here if $v_k - P_{H_0^1(\Omega)} v_k \equiv 0$, that is, the homogeneous boundary condition $v_k|_{\partial\Omega} \equiv 0$ case of Gladiali-Grossi, we get $(33) \equiv 0$. Consequently

$$
\sum_{j=1}^{2} \partial_i \partial_j R(0) a_j = 0
$$

for each $i = 1, 2$ and we get $\mathbf{a} = \mathbf{0}$ if (22) is non-degenerate. This is a contradiction. For *the one point blow-up mean field Dirichlet case* considered in 3.2, the conclusion is obtained easily, too. Indeed in this case, we have

$$
v_k - P_{H_0^1(\Omega)} v_k \equiv v_k|_{\partial\Omega} =: -c_k \quad \text{in } \overline{\Omega}
$$

and

$$
\int_{\partial\Omega} v_k \nu_i = c_k \int_{\partial\Omega} \nu_i = 0.
$$

Consequently $(33) = 0$.

For general boundary conditions of (21), we should recall the asymptotic behaviour (26) of δ_k. Since $v_k|_{\partial\Omega}$ is uniformly bounded with respect to k, we have

$$
\frac{\lambda_k}{\delta_k} \int_{\partial\Omega} v_k \nu_i = \lambda_k^{\frac{1}{2}} O(1) \longrightarrow 0.
$$

On the other hand, we know $v_k - P_{H_0^1(\Omega)} v_k$ is a harmonic function satisfying

$$\left(v_k - P_{H_0^1(\Omega)} v_k\right)\Big|_{\partial\Omega} = v_k|_{\partial\Omega} \longrightarrow c_\infty \quad \text{uniformly,}$$

which guarantees

$$\partial_i \left(v_k - P_{H_0^1(\Omega)} v_k\right) \longrightarrow 0 \quad \text{uniformly near } 0.$$

Nevertheless, we have only

$$\begin{aligned}\left|\frac{\lambda_k}{\delta_k}\int_\Omega e^{u_k}\partial_i\left(v_k - P_{H_0^1(\Omega)} v_k\right)\right| &\le \frac{1}{\delta_k}\left(\lambda_k \int_\Omega e^{u_k}\right)\left\|\partial_i\left(v_k - P_{H_0^1(\Omega)} v_k\right)\right\|_{L^\infty(\Omega)} \\ &= \frac{1}{\delta_k}\left(8\pi + o(1)\right)\left\|\partial_i\left(v_k - P_{H_0^1(\Omega)} v_k\right)\right\|_{L^\infty(\Omega)}.\end{aligned}$$

Therefore, to get the same conclusion $\mathbf{a} = \mathbf{0}$ for general boundary condition, it seems necessary to prove

$$\left\|\nabla\left(v_k - P_{H_0^1(\Omega)} v_k\right)\right\|_{L^\infty(\Omega)} = o(\delta_k). \tag{34}$$

3.6 Step 3: $b = 0$.

Assume $\mathbf{a} = \mathbf{0}$, that is,

$$\tilde{v}_k \longrightarrow \tilde{v}_\infty = b\frac{8 - |x|^2}{8 + |x|^2} \quad \text{locally uniformly in } \mathbf{R}^2.$$

To get the precise value of b, several calculations might be considered. For example,

$$\int_{\Omega_k} e^{\tilde{u}_k}\tilde{v}_k \longrightarrow 64b\int_{\mathbf{R}^2}\frac{8 - |x|^2}{(8 + |x|^2)^3}$$

is able to be obtained by using the estimate of Y. Y. Li [6]. It holds, however, that

$$\int_{\mathbf{R}^2}\frac{8 - |x|^2}{(8 + |x|^2)^3} = 0.$$

Gladiali-Grossi used the following quantity:

$$\int_{\Omega_k} e^{\tilde{u}_k}\tilde{v}_k\tilde{u}_k \longrightarrow 64b\int_{\mathbf{R}^2}\frac{8 - |x|^2}{(8 + |x|^2)^3}\log\frac{1}{\left(1 + \frac{|x|^2}{8}\right)^2} = 8\pi b. \tag{35}$$

In this case, we have

$$\int_{\Omega_k} e^{\tilde{u}_k}\tilde{v}_k\tilde{u}_k = \lambda_k \int_{\Omega} e^{u_k} v_k u_k + \|u_k\|_{L^\infty(\Omega)}\lambda_k \int_{\Omega_k} e^{u_k} v_k \tag{36}$$

and

$$\begin{aligned}\lambda_k \int_{\Omega} e^{u_k} v_k u_k &= -\int_{\Omega} \Delta v_k u_k = \int_{\Omega} \nabla v_k \cdot \nabla u_k \\ &= \int_{\partial\Omega} v_k \frac{\partial}{\partial\nu} u_k - \int_{\Omega} v_k \Delta u_k \\ &= \int_{\partial\Omega} v_k \frac{\partial}{\partial\nu} u_k + \lambda_k \int_{\Omega} e^{u_k} v_k.\end{aligned}$$

As a result, we have

$$(36) = \int_{\partial\Omega} v_k \frac{\partial}{\partial\nu} u_k + (\|u_k\|_{L^\infty(\Omega)} + 1)\lambda_k \int_{\Omega_k} e^{u_k} v_k. \tag{37}$$

From our assumption (21), we have

$$\int_{\partial\Omega} v_k \frac{\partial}{\partial\nu} u_k \longrightarrow 8\pi \int_{\partial\Omega} c_\infty \frac{\partial}{\partial\nu} G_\Omega(\cdot, 0) = -8\pi c_\infty.$$

On the other hand, similar to deriving (28), the following estimate seems to hold from the argument of [5, (3.22-23)]:

$$\frac{\partial P_{H_0^1(\Omega)} v_k}{\partial x_i} = o(\delta_k) \tag{38}$$

for each $i = 1, 2$. Then we have

$$\lambda_k \int_{\Omega} e^{u_k} v_k = -\int_{\Omega} \Delta P_{H_0^1(\Omega)} v_k = -\int_{\partial\Omega} \frac{\partial}{\partial\nu} P_{H_0^1(\Omega)} v_k = o(\delta_k). \tag{39}$$

From (35-39) and previous (25-26), we have

$$\begin{aligned}8\pi b &= -8\pi c_\infty + (\|u_k\|_{L^\infty(\Omega)} + 1)o(\delta_k) + o(1) \\ &= -8\pi c_\infty + (-2\log\lambda_k + O(1))o(\lambda_k^{\frac{1}{2}}) + o(1) \\ &= -8\pi c_\infty + o(1),\end{aligned}$$

that is,

$$b = -c_\infty. \tag{40}$$

Therefore, when $c_\infty = 0$ is a priori known, e.g. , the homogeneous boundary condition case of Gladiali-Grossi [5], this step has finished.

For the general boundary condition, I have not finished the proof of this step. The following calculation using the Pohozaev identity [5, (2.20)], however, seems to suggest one approach.

Let $\eta_k = (x\cdot\nabla)u_k$. Then η_k satisfies

$$-\Delta\eta_k = 2\lambda_k e^{u_k} + \lambda_k e^{u_k}\eta_k.$$

On the other hand, v_k satisfies (21) and we have

$$-\int_\Omega v_k\Delta\eta_k = 2\lambda_k\int_\Omega e^{u_k}v_k - \int_\Omega \eta_k\Delta v_k,$$

that is,

$$\int_{\partial\Omega} v_k\frac{\partial}{\partial\nu}\eta_k = -2\lambda_k\int_\Omega e^{u_k}v_k + \int_{\partial\Omega}\eta_k\frac{\partial}{\partial\nu}v_k. \tag{41}$$

Under our assumption, we get

$$\int_{\partial\Omega} v_k\frac{\partial}{\partial\nu}\eta_k \longrightarrow c_\infty\int_{\partial\Omega}\frac{\partial}{\partial\nu}(x\cdot\nabla)G_\Omega(\cdot,0)$$

since

$$\frac{\partial}{\partial x_i}\eta_k = \frac{\partial}{\partial x_i}(x\cdot\nabla)u_k \longrightarrow \frac{\partial}{\partial x_i}(x\cdot\nabla)G_\Omega(\cdot,0)\quad\text{uniformly on } \partial\Omega.$$

So it seems that we have caught c_∞, but unfortunately not. Indeed

$$\int_{\partial\Omega}\frac{\partial}{\partial\nu}(x\cdot\nabla)G_\Omega(\cdot,0) = 0,$$

more precisely,

$$\int_{\partial\Omega}\frac{\partial}{\partial\nu}(x\cdot\nabla)G_\Omega(\cdot,0) = \lim_{k\longrightarrow\infty}\int_{\partial\Omega}\frac{\partial}{\partial\nu}\eta_k$$

and

$$\begin{aligned}
\int_{\partial\Omega}\frac{\partial}{\partial\nu}\eta_k &= \int_\Omega\Delta\eta_k = -2\lambda_k\int_\Omega e^{u_k} - \lambda_k\int_\Omega e^{u_k}\eta_k\\
&= -2\lambda_k\int_\Omega e^{u_k} - \lambda_k\int_\Omega e^{u_k}(x\cdot\nabla)u_k\\
&= -2\lambda_k\int_\Omega e^{u_k} - \lambda_k\int_\Omega (x\cdot\nabla)e^{u_k}\\
&= -2\lambda_k\int_\Omega e^{u_k} - \lambda_k\int_{\partial\Omega}(x\cdot\nu)e^{u_k} + \lambda_k\int_\Omega(\nabla\cdot x)e^{u_k}\\
&= -\lambda_k\int_{\partial\Omega}(x\cdot\nu) = -\lambda_k\int_\Omega\nabla\cdot x = -2\lambda_k|\Omega| \longrightarrow 0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
c_\infty &= \lim_{k\longrightarrow\infty}\frac{1}{-2\lambda_k|\Omega|}c_\infty\int_{\partial\Omega}\frac{\partial}{\partial\nu}\eta_k \\
&= -\frac{1}{2|\Omega|}\lim_{k\longrightarrow\infty}\frac{1}{\lambda_k}\left(\int_{\partial\Omega}(c_\infty - v_k)\frac{\partial}{\partial\nu}\eta_k + \int_{\partial\Omega} v_k\frac{\partial}{\partial\nu}\eta_k\right).
\end{aligned}$$

Here

$$\left|\int_{\partial\Omega}(c_\infty - v_k)\frac{\partial}{\partial\nu}\eta_k\right| \le \|c_\infty - v_k\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}\left|\frac{\partial}{\partial\nu}\eta_k\right| = O(1)\|c_\infty - v_k\|_{L^\infty(\partial\Omega)}.$$

On the other hand, from (41) we have

$$\begin{aligned}
\int_{\partial\Omega} v_k\frac{\partial}{\partial\nu}\eta_k &= -2\lambda_k\int_\Omega e^{u_k}v_k + \int_{\partial\Omega}\eta_k\frac{\partial}{\partial\nu}v_k \\
&= 2\int_\Omega \Delta P_{H_0^1(\Omega)}v_k + \int_{\partial\Omega}\eta_k\frac{\partial}{\partial\nu}\left(v_k - P_{H_0^1(\Omega)}v_k\right) + \int_{\partial\Omega}\eta_k\frac{\partial}{\partial\nu}P_{H_0^1(\Omega)}v_k \\
&= 2\int_{\partial\Omega}\frac{\partial}{\partial\nu}P_{H_0^1(\Omega)}v_k + \int_{\partial\Omega}\eta_k\frac{\partial}{\partial\nu}\left(v_k - P_{H_0^1(\Omega)}v_k\right) + \int_{\partial\Omega}\eta_k\frac{\partial}{\partial\nu}P_{H_0^1(\Omega)}v_k.
\end{aligned}$$

Assuming (38) we are able to conclude

$$\left|\int_{\partial\Omega} v_k\frac{\partial}{\partial\nu}\eta_k\right| \le o(\delta_k) + O(1)\left\|\frac{\partial}{\partial\nu}\left(v_k - P_{H_0^1(\Omega)}v_k\right)\right\|_{L^\infty(\partial\Omega)},$$

where $o(\delta_k)$ is determined only in (38).

Consequently the following facts seem to be necessary to prove $b = 0$ for the general boundary condition of (21):

$$\frac{\partial P_{H_0^1(\Omega)}v_k}{\partial x_i} = o(\lambda_k)(= o(\delta_k^2)) \quad \text{instead of (38)}, \tag{42}$$

$$\|c_\infty - v_k\|_{L^\infty(\partial\Omega)} = o(\lambda_k), \tag{43}$$

$$\left\|\frac{\partial}{\partial\nu}\left(v_k - P_{H_0^1(\Omega)}v_k\right)\right\|_{L^\infty(\partial\Omega)} = o(\lambda_k). \tag{44}$$

We note that the last condition (44) is stronger condition than (34).

3.7 Step 4: Finish

Suppose $\mathbf{a} = \mathbf{0}$ and $b = 0$. Then we have proved that

$$\tilde{v}_k \longrightarrow \tilde{v}_\infty \equiv 0 \quad \text{locally uniformly in } \mathbf{R}^2. \tag{45}$$

Suppose

$$\limsup_{k\longrightarrow\infty} \|v_k\|_{L^\infty(\Omega)} = M > 0.$$

Then, taking subsequence if necessary, we are able to assume

$$\lim_{k\longrightarrow\infty} \|v_k\|_{L^\infty(\Omega)} = M.$$

Let $\tilde{x}_k \in \Omega_k \subset \mathbf{R}^2$ such that

$$|\tilde{v}_k(\tilde{x}_k)| = \|\tilde{v}_k\|_{L^\infty(\Omega_k)}.$$

Then it must hold that

$$\tilde{x}_k \longrightarrow \infty$$

from (45).

Here let us define the functions

$$\hat{u}_k(x) := \tilde{u}_k\left(\frac{x}{|x|^2}\right), \quad \hat{v}_k(x) := \tilde{v}_k\left(\frac{x}{|x|^2}\right) \quad \text{in } x \in \mathbf{R}^2\backslash\{0\},$$

that is, the Kelvin transformed function of $\tilde{u}_k$ and $\tilde{v}_k$ with respect to $B_1(0)$. We also set

$$\hat{x}_k = \frac{\tilde{x}_k}{|\tilde{x}_k|^2}.$$

Clearly

$$|\hat{v}_k(\hat{x}_k)| = |\tilde{v}_k(\tilde{x}_k)| \longrightarrow M > 0 \quad \text{as } k \longrightarrow \infty \tag{46}$$

and $\hat{v}_k$ satisfies

$$-\Delta\hat{v}_k = \frac{1}{|x|^4}e^{\hat{u}_k}\hat{v}_k.$$

From Y. Y. Li's estimate [6], we know

$$\frac{1}{|x|^4}e^{\hat{u}_k} \leq C$$

for some constant C. Moreover $|\hat{v}_k| \leq 2M$ for sufficiently large M and $\hat{v}_k \longrightarrow 0$ locally uniformly in $\mathbf{R}^2\backslash\{0\}$. Consequently we have

$$\|\hat{v}_k\|_{L^2(B_1(0))} \longrightarrow 0 \quad \text{as } k \longrightarrow 0$$

for example. Here we identify $\hat{v}_k$ with its 0-extension to $\mathbf{R}^2$.

Since we are able to assume that $\delta_k\tilde{x}_k + x_k$ is uniformly away from $\partial\Omega$. Therefore

$$|\tilde{x}_k| = O(\delta_k^{-1}) \quad \text{and} \quad |\hat{x}_k| = O(\delta_k).$$

Then the local elliptic estimate ([4, Theorem 8.17]) seems applicable on $B_{2C\delta_k}(\hat{x}_k)$ for some appropriate C if

$$\|\hat{v}_k\|_{L^2(B_1(0))} = o(\delta_k). \tag{47}$$

Then we would get

$$|\hat{v}_k(\hat{x}_k)| \leq \frac{M}{2}$$

for $k \gg 1$, which contradicts to (46), that is, $M = 0$.

4 Concluding remarks

So far we have observed how to apply the Gladiali-Grossi's argument to our Conjecture 1.3. As a conclusion I must say that we have not finished the proof even for the simplified case considered in section 3. The gap seems to filled with more detailed analysis of the asymptotic behaviour of v_k, see (34), (42-44), and (47).

I would like to continue further study of this topic.

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