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EXISTENCE AND ASYMPTOTICS FOR A CAHN-HILLIARD/ALLEN-CAHN PARABOLIC EQUATION

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ABSTRACT. We consider the existence and asymptotic behavior of solutions to a mean field partial differential equation of Cahn-Hilliard/Allen-Cahn type. This equation arises in the description of pattern formation mechanisms for a prototypical model of surface processes that involves multiple microscopic mechanisms.

1. INTRODUCTION

This note is concerned with the mathematical study of the following mean field partial differential equation which was recently derived in [4]:

$$\begin{cases} u_t = \epsilon^2 D(-\Delta)(\Delta u + \dfrac{f(u)}{\epsilon^2}) + \Delta u + \dfrac{f(u)}{\epsilon^2} \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $f(u) = -W'(u)$, W is a double-well potential with wells ± 1, $D > 0$ is the diffusion constant and $0 < \epsilon \ll 1$ is a small parameter. A typical choice for W is $W(u) = (u^2 - 1)^2$. Equation (1.1) is associated with the effect of multiple microscopic mechanisms such as surface diffusion and adsorption/desorption which are typically involved in surface processes, on macroscopic cluster interface morphology and evolution. We note that equation (1.1) may be viewed as a combination of the well-known Cahn-Hilliard (CH) and Allen-Cahn (AC) equations. We recall that the former model can describe surface diffusion including particle/particle interactions, while the latter describes a simplified model of adsorption to and desorption from the surface. It is worth mentioning that in the model described by (1.1) the mobility is completely different from the one of the AC equation. This implies in particular that the diffusion speeds up the mean curvature flow, see [4]. It is known that the AC and CH equations can serve as diffuse interface models for limiting sharp interface motion. The AC equation serves as a diffuse interface model for antiphase grain boundary coarsening in the sense that the singular limit of the equation yields a geometric problem in which a sharp interface separating two phase variants evolves according to motion by mean curvature ($V = k$), see [1, 2, 6]. On the other hand, the CH equation was constructed to describe mass conservative phase separation. By considering an appropriate singular limit ($\epsilon \to 0$) it can describe the motion of interphase boundaries separating two phases of differing composition during the later stages of coarsening.

Here, we are interested in the mathematical structure of the fourth-order evolution equation (1.1). As ϵ will not play any role in our considerations, we set $\epsilon = 1$. In this case, equation (1.1) takes the form

$$u_t = -D\Delta(\Delta u + f(u)) + \Delta u + f(u). \tag{1.2}$$

We note that setting $D = 0$ in (1.1) we obtain the standard second-order AC equation. Hence, a natural question is: do solutions to the CH/AC equation resemble to solutions of the AC equation, at least when $D \ll 1$? We observe that the leading differential operator in (1.2) is given by $-D\Delta^2 + \Delta$. Therefore, the limit $D \to 0$ corresponds to dropping the higher order derivative, and therefore the asymptotic behavior of solutions as $D \to 0$ is not a priori obvious. An analogous situation was considered also in [5] in the context of Maxwell-Chern-Simons vortices, where it is shown that due to "good signs", the formal limit may be rigorously justified. Moreover, such asymptotics is used to extend certain properties of the limit second order equation deriving from the maximum principle to the whole fourth-order equation (for small values of D), which are used to prove a multiplicity result by techniques typical of second order problems. In Section 2 we show that, actually, for any fixed $D > 0$ we can construct a sequence of stationary "genuine" CH/AC solutions which converge to an AC solution. We do not know whether a similar result holds for the full evolution equation. On the other hand, in Section 3, as a first step towards the analysis of (1.1), we show that (1.1) admits a nice structure which allows to approximate solutions by a Galerkin ansatz, adapting some ideas from [3]. Similarly as in [5], a relevant feature of (1.2) is that it may be formulated as a system of two second order equations with "good signs". Namely, setting $v = \Delta u + f(u)$ in (1.2), we see that (1.2) is equivalent to the following system of second order equations:

$$\begin{cases} u_t = -D\Delta v + v \\ v = \Delta u + f(u). \end{cases} \tag{1.3}$$

Several estimates in the sequel, as well as the Galerkin ansatz, rely on the equivalence of (1.2) and (1.3).

2. The stationary problem: convergence of CH/AC to AC

It is readily seen that under doubly periodic or Neumann boundary conditions, the stationary solutions to the CH/AC equation are exactly the stationary solutions to the AC equation obtained by taking $D = 0$. Indeed, stationary solutions to (1.3) satisfy

$$\begin{cases} -D\Delta v + v = 0 \\ v = \Delta u + f(u). \end{cases}$$

Multiplying by v and integrating, under periodic or Neumann boundary conditions we have that $D \int_\Omega |\nabla v|^2 + \int_\Omega v^2 = 0$. It follows that $v = 0$ and u satisfies the stationary AC equation $\Delta u + f(u) = 0$.

Therefore, in this section we focus on Dirichlet boundary conditions. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We consider the Dirichlet problem

$$\begin{cases} -D\Delta(\Delta u + f(u)) + \Delta u + f(u) = 0 & \text{in} \quad \Omega \\ u = 1 & \text{on} \quad \partial\Omega. \end{cases}$$

By setting $v = \Delta u + f(u)$, we are led to consider the system

$$(2.4) \qquad \begin{cases} -D\Delta v + v = 0 & \text{in } \Omega \\ -\Delta u = -v + f(u) & \text{in } \Omega \\ u = 1 & \text{on } \partial\Omega \\ v = \psi & \text{on } \partial\Omega. \end{cases}$$

The main result in this section is the following.

Proposition 2.1. *For any $D > 0$ there exists a sequence of solutions $(u_n)_{n\in\mathbb{N}}$ to the stationary CH/AC equation*

$$(2.5) \qquad \begin{cases} -D\Delta(\Delta u + f(u)) + \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 1 & \text{on } \partial\Omega \end{cases}$$

and a solution u to the stationary AC equation

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 1 & \text{on } \partial\Omega \end{cases}$$

such that $u_n \to u$ strongly in $H^1(\Omega)$.

In order to prove Proposition 2.1 we need some lemmas.

Lemma 2.1. *Let (u, v) be a solution to system (2.4). Then $\|v\|_\infty = \|\psi\|_\infty$ and there exists a continuous function $A : [0, +\infty) \to [1, +\infty)$, $A(0) = 1$ such that $\|u\|_\infty \le A\|\psi\|_\infty$.*

Proof. Let $\bar{y} \in \Omega : v(\bar{y}) = \max\limits_{\bar{\Omega}} v$. Then $-\Delta v(\bar{y}) \ge 0$ which implies

$$0 = -D\Delta v(\bar{y}) + v(\bar{y}) \ge v(\bar{y}).$$

Hence, v cannot attain a positive interior maximum and therefore $v \le \|\psi\|_\infty$. Similarly, let $\underline{y} \in \Omega$: $v(\underline{y}) = \min\limits_{\bar{\Omega}} v$. Then $-\Delta v(\underline{y}) \le 0$ which implies

$$0 = -D\Delta v(\underline{y}) + v(\bar{y}) \le v(\underline{y}).$$

That is, v cannot attain a negative interior minimum and $v \ge -\|\psi\|_\infty$. Hence the estimate for v is established.

In order to obtain the estimate for u we recall that $f(u) = -4u(u^2 - 1)$ and $f(u) = -f(-u)$. Since $u = 1$ on $\partial\Omega$ we have that $\max\limits_{\bar{\Omega}} u \ge 1$.

Let $\bar{x} \in \Omega : u(\bar{x}) = \max\limits_{\bar{\Omega}} u$. Then $0 \le -\Delta u(\bar{x}) = -v(\bar{x}) + f(u(\bar{x}))$, which implies

$$f(u(\bar{x})) \ge v(\bar{x}) \ge -\|\psi\|_\infty.$$

Let $g : (-\infty, 0] \to [1, +\infty)$ be such that $f(g(t)) = t$ for all $t \in (-\infty, 0)$. Then, since $u(\bar{x}) \geq 1$, we have $u(\bar{x}) \leq g(-\|\psi\|_\infty)$. Similarly, let $\underline{x} \in \Omega$: $u(\underline{x}) = \min_{\bar{\Omega}} u$. Then $0 \geq -\Delta u(\underline{x}) = -v(\underline{x}) + f(u(\underline{x}))$. We have

$$f(u(\underline{x})) \leq v(\underline{x}) \leq \|\psi\|_\infty, \; f(-u(\underline{x})) = -f(u(\underline{x})) \geq -\|\psi\|_\infty$$

and therefore

$$-u(\underline{x}) \leq g(-\|\psi\|_\infty), \; u(\underline{x}) \geq -g(-\|\psi\|_\infty).$$

In conclusion, we have

$$\|u\|_\infty \leq g(-\|\psi\|_\infty)$$

and the proof is completed by taking $A(t) = g(-t)$. □

Lemma 2.2. *For all $\psi : \partial\Omega \to \mathbf{R}$ sufficiently smooth, there exists a solution to system* (2.4).

Proof. For all $\psi \in C^\infty(\partial\Omega)$ there exists a unique solution v to the problem

$$\begin{cases} -D\Delta v + v = 0 & \text{in} \quad \Omega \\ v = \psi & \text{on} \quad \partial\Omega. \end{cases}$$

Now we need to solve

$$\begin{cases} -\Delta u = -v - W'(u) & \text{in} \quad \Omega \\ u = 1 & \text{on} \quad \partial\Omega. \end{cases}$$

Let $w = u - 1$. Then w satisfies

$$\begin{cases} -\Delta w = -v - W'(w+1) & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Solutions to the problem above correspond to critical points in $H_0^1(\Omega)$ for the functional

$$I(w) := \int_\Omega \left\{ \frac{1}{2}|\nabla w|^2 + W(w+1) + vw \right\}.$$

We have

$$I(w) \geq \tfrac{1}{2} \int_\Omega |\nabla w|^2 - \|v\|_2 \|w\|_2 \geq a\|\nabla w\|^2 - C$$

for some $a, C > 0$. Therefore, I is bounded below and coercive. Hence, I admits a global minimum corresponding to a solution for (2.4). □

Lemma 2.3. *There exists $C = C(\|\psi\|_\infty, |\Omega|)$ such that*

$$\int_\Omega |\nabla u|^2 \leq C(\|\psi\|_\infty).$$

Proof. Since $u = 1$ on $\partial\Omega$, multiplying by u and integrating (2.4), we obtain

$$-\int_\Omega u\Delta u = -\int_{\partial\Omega} \frac{\partial u}{\partial n} + \int_\Omega |\nabla u|^2 = -\int_\Omega vu + \int_\Omega f(u)u,$$

so that

$$\int_\Omega |\nabla u|^2 = \int_{\partial\Omega} \frac{\partial u}{\partial n} - \int_\Omega vu + \int_\Omega f(u)u.$$

On the other hand, integrating over Ω we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} = \int_\Omega \Delta u = \int_\Omega v - \int_\Omega f(u). \tag{2.6}$$

Therefore, we derive

$$\int_\Omega |\nabla u|^2 = \int_\Omega v(1-u) - \int_\Omega f(u)(1-u).$$

Now, in view of the L^∞-estimates of Lemma 2.1, we derive

$$\int_\Omega |\nabla u|^2 \le |\Omega| \|v\|_\infty \|1-u\|_\infty + \|f(u)(1-u)\|_\infty |\Omega| \le C(\|\psi\|_\infty, |\Omega|),$$

as asserted. □

Now we can prove Proposition 2.1.

Proof of Proposition 2.1. We write (2.5) in a system form

$$\begin{cases} v = \Delta u + f(u) & \text{in} \quad \Omega \\ -D\Delta v + v = 0 & \text{in} \quad \Omega \\ u = 1 & \text{on} \quad \partial\Omega. \end{cases}$$

In view of Lemma 2.2, there exist solutions (u_n, v_n) to the problem

$$\begin{cases} -\Delta u_n = -v_n + f(u_n) & \text{in} \quad \Omega \\ -D\Delta v_n + v_n = 0 & \text{in} \quad \Omega \\ u_n = 1 & \text{on} \quad \partial\Omega \\ v_n = \frac{1}{n} & \text{on} \quad \partial\Omega. \end{cases}$$

Then by elliptic regularity, $v_n \to 0$ in C^k $\forall k \ge 0$. In view of Lemma 2.1 and Lemma 2.3, the sequence u_n is bounded in H^1. Hence, there exists $u \in H^1$, $u = 1$ on $\partial\Omega$ in the sense of H_0^1, such that $u_n \to u$ weakly in H^1, strongly in L^2 and a.e. Consequently, for any $\varphi \in H^1(\Omega)$, we obtain:

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega f(u)\varphi.$$

That is, u satisfies the AC equation. Finally, by similar arguments as above, we note that

$$\begin{aligned}\int_\Omega |\nabla u_n|^2 &= \int_{\partial\Omega} \frac{\partial u_n}{\partial n} - \int_\Omega u_n \Delta u_n \\ &= \int_\Omega v_n - \int_\Omega f(u_n) - \int_\Omega u_n v_n + \int_\Omega u_n f(u_n) = -\int_\Omega f(u) + \int_\Omega u f(u) + o(1).\end{aligned}$$

On the other hand,

$$\int_\Omega |\nabla u|^2 = \int_{\partial\Omega} \frac{\partial u}{\partial n} - \int_\Omega u\Delta u = \int_\Omega \Delta u(1-u) = -\int_\Omega f(u) + \int_\Omega u f(u).$$

Therefore, $\|u_n\|_{H^1} \to \|u\|_{H^1}$ and the H^1-convergence is strong. □

3. Existence of solutions: a Galerkin approximation

In this section for simplicity we restrict ourselves to the case where Ω is a bounded interval, and $f \in C^2(\mathbb{R})$ is a general nonlinearity such that $\|f\|_{C^2} < \infty$. We consider Neumann boundary conditions on Ω. We set $S_T = \partial\Omega \times (0,T)$, $\Omega_T = \Omega \times (0,T)$. We prove the following.

Theorem 3.1. *Let $T > 0$, $\|f\|_{C^2} < \infty$ and suppose that $u_0 \in H^1(\Omega)$. There exists a pair of functions (u,v) such that $u, v \in L^\infty(0,T;H^1(\Omega)) \cap C([0,T];H^\lambda)$, $\lambda < 1$, $u_t \in L^2(0,T;H^{-1}(\Omega))$, $u(0) = u_0$ in $L^2(\Omega)$, $u_x|_{S_T} = u_x|_{S_T} = 0$ in $L^2(S_T)$, and (u,v) satisfies (1.3) in the following weak sense:*

$$\begin{cases} \displaystyle\iint_{\Omega_T} v\varphi = -\iint_{\Omega_T} \nabla u \nabla\varphi + \iint_{\Omega_T} f(u)\varphi \\ \displaystyle\iint_{\Omega_T} u_t\varphi = d\iint_{\Omega_T} \nabla v\nabla\varphi + \iint_{\Omega_T} v\varphi \end{cases}$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$.

Let ψ_i, $i \in \mathbb{N}$ denote the eigenfunction of $-d^2/dx^2$ on Ω corresponding to the eigenvalue λ_i with Neumann boundary conditions such that

$$-\frac{d^2}{dx^2}\psi_i = \lambda_i\psi_i \text{ in } \Omega, \qquad \partial_x\psi_i = 0 \text{ on } \partial\Omega.$$

Then, $\int_\Omega \psi_{i,x}\psi_{j,x} = 0$ and we further assume that $\int_\Omega \psi_i\psi_j = \delta_{ij}$ for $0 = \lambda_1 < \lambda_2 \leq \cdots$.
For every $N \in \mathbb{N}$ we consider (u^N, v^N) defined by the Galerkin ansatz

$$(3.7) \qquad u^N(x,t) = \sum_{i=1}^N a_i^N(t)\psi_i(x), \qquad v^N(x,t) = \sum_{i=1}^N b_i^N(t)\psi_i(x),$$

where a_i, b_i are subject to the conditions

$$(3.8) \qquad \begin{cases} \int_\Omega v^N\psi_j = \int_\Omega \Delta u^N\psi_j + \int_\Omega f(u^N)\psi_j, & j = 1,2,\ldots,N \\ \int_\Omega u_t^N\psi_j = -D\int_\Omega \Delta v^N\psi_j + \int_\Omega v^N\psi_j, & j = 1,2,\ldots,N \\ \int_\Omega u^N(x,0)\psi_j = \int_\Omega u_0\psi_j, & j = 1,2,\ldots,N. \end{cases}$$

System (3.8) yields the following initial value problem for $a_j^N(t)$, $j = 1,2,\ldots,N$:

$$(3.9) \qquad \begin{cases} \dfrac{da_j^N(t)}{dt} = (D\lambda_j + 1)[-\lambda_j a_j^N(t) + \int_\Omega f(u^N)\psi_j] \\ a_j^N(0) = \int_\Omega u_0\psi_j, \end{cases}$$

while b_j^N is determined by a_j^N, $j = 1,2,\ldots,N$, by the equation

$$(3.10) \qquad b_j^N(t) = -\lambda_j a_j^N(t) + \int_\Omega f(u^N)\psi_j.$$

By standard arguments, it is readily seen that problem (3.9) has a local solution. We want to show that a global solution $(a_j^N)_{j=1,2,\ldots,N}$ exists on $(0,T)$ for any $T > 0$. Namely, we have the following.

Proposition 3.1. *Let $T > 0$. There exists a solution $(a_j^N, b_j^N)_{j=1,2,\ldots,N}$ globally defined on $(0,T)$.*

The main ingredients for the proof of Proposition 3.1 are the following estimates.

Proposition 3.2. *Let u^N be defined by (3.7)–(3.9). Then, the following identity holds:*

$$(3.11) \qquad \frac{1}{2}\frac{d}{dt}\int_\Omega (u_x^N)^2 + D\int_\Omega (u_{xxx}^N)^2 + \int_\Omega (u_{xx}^N)^2 = D\int_\Omega f(u^N)u_{xxxx}^N - \int_\Omega f(u^N)u_{xx}^N.$$

In particular, we have the following estimates:

(i) $\sup_{t\in(0,T)} \int_\Omega (u_x^N)^2 \leq e^{2C_0T}\int_\Omega u_{0,x}^2$;

(ii) $D\int_0^T\int_\Omega (u^N_{xxx})^2 + 2\int_0^T\int_\Omega u_{xx}^2 \le e^{2C_0T}\int_\Omega u_{0,x}^2$,

where $C_0 = \|f\|_{C_1}(1+\frac{D}{2}\|f\|_{C_1})$.

We first prove a lemma.

Lemma 3.1. *The following identities hold:*

(i) $\displaystyle\int_\Omega \psi_{i,x}^2 = \lambda_i;$

(ii) $\displaystyle\int_\Omega (u_x^N)^2 = \sum_{i=1}^N \lambda_i (a_i^N)^2;$

(iii) $\displaystyle\int_\Omega (u_{xx}^N)^2 = \sum_{i=1}^N \lambda_i^2 (a_i^N)^2;$

(iv) $\displaystyle\int_\Omega (u_{xxx}^N)^2 = \sum_{i=1}^N \lambda_i^3 (a_i^N)^2;$

(v) $\displaystyle\sum_{j=1}^N \lambda_j^2 a_j^N \psi_j = u_{xxxx}^N.$

Proof. (i). We readily have that

$$\int_\Omega \psi_{i,x}^2 = -\int_\Omega \psi_i\psi_{i,xx} = \lambda_i\int_\Omega \psi_i^2 = \lambda_i.$$

(ii). Using the orthogonality conditions on ψ_i and (i), we have

$$\int_\Omega (u_x^N)^2 = \int_\Omega\left(\sum_{i=1}^N a_i^N(t)\psi_{i,x}\right)^2 = \sum_{i=1}^N (a_i^N)^2(t)\int_\Omega(\psi_{i,x})^2 = \sum_{i=1}^N \lambda_i(a_i^N)^2(t).$$

(iii). Similarly as above, we have

$$\int_\Omega (u_{xx}^N)^2 = \int_\Omega (\sum_{i=1}^N a_i^N\psi_{ixx})^2 = \int_\Omega(-\sum_{i=1}^N \lambda_i a_i^N\psi_i)^2 = \int_\Omega \sum_{i=1}^N \lambda_i^2(a_i^N)^2\psi_i^2 = \sum_{i=1}^N \lambda_i^2 (a_i^N)^2.$$

(iv). We note that

$$u_{xxx}^N = \sum_{i=1}^N a_i^N\psi_{ixxx} = -\sum_{i=1}^N \lambda_i a_i^N\psi_{ix}.$$

Therefore, recalling (i) and the orthogonality conditions we obtain

$$\int_\Omega (u^N_{xxx})^2 = \int_\Omega (-\sum_{i=1}^N \lambda_i a_i^N \psi_{ix})^2 = \sum_{i=1}^N \int_\Omega \lambda_i^2 (a_i^N)^2 \psi_{ix}^2 = \sum_{i=1}^N \lambda_i^3 (a_i^N)^2.$$

(v). Note that $\psi_{jxxxx} = -\lambda_j \psi_{jxx} = \lambda_j^2 \psi_j$. Therefore,

$$\sum_{i=1}^N (\lambda_j)^2 a_j^N \psi_j = \sum_{j=1}^N a_j^N \psi_{jxxxx} = (\sum_{j=1}^N a_j^N \psi_j)_{xxxx} = u^N_{xxxx}.$$

□

Proof of Proposition 3.2. Multiplying (3.9) by $-\lambda_j a_j^N(t)$ and adding over $j = 1, 2, \cdots, N$, we have

$$-\sum_{j=1}^N \lambda_j \frac{da_j^N}{dt} a_j^N = \sum_{j=1}^N (D\lambda_j + 1)\lambda_j^2 (a_j^N)^2 - \sum_{j=1}^N \lambda_j a_j^N (D\lambda_j + 1) \int_\Omega f(u^N)\psi_j.$$

We have in view of Lemma 3.1–(ii) that

$$-\sum_{j=1}^N \lambda_j \frac{da_j^N}{dt} a_j^N = -\sum_{j=1}^N \frac{1}{2}\lambda_j \frac{d}{dt}(a_j^N)^2 = -\frac{1}{2}\frac{d}{dt}\sum_{j=1}^N \lambda_j (a_j^N)^2 = -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u^N|^2.$$

By making use of Lemma 3.1–(iv) we obtain

$$\sum_{j=1}^N \lambda_j^3 (a_j^N)^2 = \sum_{j=1}^N \lambda_j^3 (a_j^N)^2 = \int_\Omega (u^N_{xxx})^2.$$

In view of Lemma 3.1–(iii)

$$\sum_{j=1}^N \lambda_j^2 (a_j^N)^2 = \int_\Omega (u^N_{xx})^2.$$

Also by Lemma 3.1–(i)

$$\sum_{j=1}^N \lambda_j^2 (a_j^N) \int_\Omega f(u^N)\psi_j = \int_\Omega \sum_{j=1}^N \lambda_j^2 \psi_j a_j^N f(u^N) = \int_\Omega u^N_{xxxx} f(u^N)$$

and furthermore

$$\sum_{j=1}^N \lambda_j a_j^N \int_\Omega f(u^N)\psi_j = \int_\Omega \sum_{j=1}^N \lambda_j a_j^N \psi_j f(u^N) = -\int_\Omega \sum_{j=1}^N a_j^N \psi_{jxx} f(u^N) = -\int_\Omega f(u^N) u^N_{xx}.$$

Hence, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u^N|^2 + D\int_\Omega (u^N_{xxx})^2 + \int_\Omega (u^N_{xx})^2 = D\int_\Omega f(u^N)(u^N_{xxxx}) - \int_\Omega f(u^N)u^N_{xx}$$

and hence (3.11) is established.

In order to obtain the estimates we use a Gronwall argument. Integrating by parts, we may write:

$$\int_\Omega f(u^N)u^N_{xxxx} = -\int_\Omega f'(u^N)u^N_x u^N_{xxx}$$

and

$$\int_\Omega f(u^N)u^N_{xx} = -\int_\Omega f'(u^N)(u^N_x)^2.$$

Hence, for any $m \neq 0$ we have:

$$\left|\int_\Omega f(u^N)u^N_{xxxx}\right| \leq \|f\|_{C^1}\left[\frac{m^2}{2}\int_\Omega (u^N_x)^2 + \frac{1}{2m^2}\int_\Omega (u^N_{xxx})^2\right].$$

and consequently

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega (u^N_x)^2 + D\int_\Omega (u^N_{xxx})^2 + \int_\Omega u^2_{xx}\\ \leq & D\|f\|_{C^1}\frac{m^2}{2}\int_\Omega (u^N_x)^2 + D\frac{\|f\|_{C^1}}{2m^2}\int_\Omega (u^N_{xxx})^2 + \|f\|_{C^1}\int_\Omega (u^N_x)^2.\end{aligned}$$

Choosing $m^2 = \|f\|_{C^1}$, we derive

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^N_x)^2 + \frac{D}{2}\int_\Omega (u^N_{xxx})^2 + \int_\Omega u^2_{xx} \leq \|f\|_{C^1}\left(\frac{D}{2}\|f\|_{C^1} + 1\right)\int_\Omega (u^N_x)^2.$$

At this point a standard Gronwall argument concludes the proof. □

Proof of Proposition 3.1. Now we observe that, since $\lambda_1 = 0$ and $\psi_1 = \text{const} > 0$, the initial value problem for $a^N_1(t)$ takes the form

$$\dot{a}^N_1(t) = \int_\Omega f(u^N)\psi_1, \qquad a^N_1(0) = \int_\Omega u_0\psi_1 = \psi_1\int_\Omega u_0.$$

In particular,

$$|\dot{a}^N_1(t)| \leq \|f\|_{C^1}\psi_1|\Omega| =: C_1$$

and we derive that

$$|a^N_1(t)| \leq \psi_1\left|\int_\Omega u_0\right| + C_1 T$$

for all $t \in (0,T)$. On the other hand, since $\int_\Omega u^N = a_1^N(t) \int_\Omega \psi_1$, we have that

$$|\int_\Omega u^N| \leq \psi_1 |\Omega| \left(\psi_1 \left| \int_\Omega u_0 \right| + C_1 T \right).$$

In view of Proposition 3.2–(i) and the Poincaré inequality, we conclude that

$$\|u^N\|_{H^1(\Omega)} \leq C_2 e^{2C_0 T},$$

for some $C_2 > 0$ independent of N. In view of Lemma 3.1–(ii), we conclude in particular that $\|a_j^N\|_{L^\infty(0,T)} \leq C_2 e^{2C_0 T}$. Consequently, $a_j^N(t)$ exists globally in $(0,T)$. In turn, in view of (3.10), $b_j^N(t)$ also exists globally in $(0,T)$. □

In order to prove Theorem 3.1 we need some estimates on v^N and u_t^N.

Lemma 3.2. *Suppose that* $\|f\|_{C^2} < +\infty$. *Let* (u^N, v^N) *be defined by* (3.7)-(3.9)-(3.10). *Then, the following estimates hold:*

(i) $\int_0^T \int_\Omega (v_x^N)^2 + \int_0^T \int_\Omega (v^N)^2 \leq C$

(ii) $\|u_t^N\|_{L^2(0,T;H^{-1}(\Omega))} \leq C$

where $C = C(T)$ does not depend on N.

Proof. To this end, we recall that $v^N = \sum_{j=1}^N b_j(t)\psi_j(x)$, where b_j, $j = 1,2,\ldots,N$ is defined by

$$b_j = -\lambda_j a_j + \int_\Omega f(u^N)\psi_j.$$

Moreover, by similar arguments as in Lemma 3.1, we have

$$\int_\Omega (v^N)^2 = \sum_{j=1}^N (b_j^N)^2, \qquad \int_\Omega (v_x^N)^2 = \sum_{j=1}^N \lambda_j (b_j^N)^2, \qquad \int_\Omega v_N = b_1(t) = \psi_1 \int_\Omega f(u^N).$$

Therefore, we have

$$\int_\Omega (v_x^N)^2 = -\sum_{j=1}^N \lambda_j^2 a_j^N b_j^N + \sum_{j=1}^N \lambda_j b_j^N \int_\Omega f(u^N)\psi_j.$$

In view of Proposition 3.2, we estimate:

$$\int_0^T \left|\sum_{j=1}^N \lambda_j^2 a_j^N b_j^N\right| \le \int_0^T \left(\sum_{j=1}^N \lambda_j b_j^2\right)^{1/2} \left(\sum_{j=1}^N \lambda_j^3 (a_j^N)^2\right)^{1/2}$$

$$\le \left(\int_0^T \sum_{j=1}^N \lambda_j b_j^2\right)^{1/2} \left(\int_0^T \sum_{j=1}^N \lambda_j^3 (a_j^N)^2\right)^{1/2} \le C\left(\int_0^T \int_\Omega (v_x^N)^2\right)^{1/2}.$$

Similarly, we have

$$\left|\sum_{j=1}^N \lambda_j b_j \int_\Omega f(u^N)\psi_j\right| \le \left(\sum_{j=1}^N \lambda_j (b_j^N)^2\right)^{1/2} \left(\sum_{j=1}^N \lambda_j (\int_\Omega f(u^N)\psi_j)^2\right)^{1/2}.$$

We note that $|\int_\Omega f(u^N)\psi_j)| \le C$ and therefore we may estimate

$$\sum_{j=1}^N \lambda_j (\int_\Omega f(u^N)\psi_j)^2 \le C \sum_{j=1}^N \lambda_j \left|\int_\Omega f(u^N)\psi_j\right|.$$

Integration by parts yields

$$\lambda_j \int_\Omega f(u^N)\psi_j = -\int_\Omega f(u^N)\psi_{j,xx} = -\int_\Omega f''(u^N)(u_x^N)^2\psi_j - \int_\Omega f'(u^N)u_{xx}^N\psi_j.$$

Consequently, recalling Proposition 3.2,

$$\left|\int_0^T \lambda_j \int_\Omega f(u^N)\psi_j\right| \le \|f\|_{C^2}\left(\int_0^T\int_\Omega (u_x^N)^2 + \int_0^T\int_\Omega |u_{xx}^N|\right) \le C\|f\|_{C^2}.$$

It follows that

$$\left|\int_0^T \sum_{j=1}^N \lambda_j b_j \int_\Omega f(u^N)\psi_j\right| \le C\int_0^T \left(\sum_{j=1}^N \lambda_j (b_j^N)^2\right)^{1/2} \le C\left(\int_0^T \sum_{j=1}^N \lambda_j (b_j^N)^2\right)^{1/2}$$

$$= C\left(\int_0^T\int_\Omega (v_x^N)^2\right)^{1/2}.$$

We have obtained that

$$\int_0^T\int_\Omega (v_x^N)^2 \le C\left(1 + \left(\int_0^T\int_\Omega (v_x^N)^2\right)^{1/2}\right)$$

and hence $\int_0^T \int_\Omega (v_x^N)^2 \leq C$. Now we observe that since $\lambda_1 = 0$ we have

$$|\int_\Omega v^N| = |b_1 \int_\Omega \psi_1| = |\int_\Omega f(u^N)\psi_1|| \int_\Omega \psi_1| \leq C\|f\|_{L^\infty}.$$

Hence, we may estimate

$$\int_\Omega (v^N)^2 = \sum_{j=1}^N (b_j^N)^2 \leq (b_1^N)^2 + \sum_{j=1}^N \lambda_j (b_j^N)^2.$$

It follows that

$$\int_0^T \int_\Omega (v^N)^2 \leq C\left(1 + \int_0^T \int_\Omega (v_x^N)^2\right) \leq C$$

and hence (i) is established.

In order to prove (ii), we denote by $\Pi_N : L^2(\Omega) \to \mathrm{span}\{\psi_1, \psi_2, \ldots, \psi_N\}$ the projection operator. Let $\psi \in L^2(0,T;H^1(\Omega))$. Then, we have:

$$\int_0^T \int_\Omega u_t^N \psi = D \int_0^T \int_\Omega v_x^N (\Pi_N \psi)_x + \int_0^T \int_\Omega v^N \Pi_N \psi.$$

Therefore, in view of Proposition 3.2, we conclude that

$$\left| \int_0^T \int_\Omega u_t^N \psi \right| \leq C(T) \|\psi\|_{L^2(0,T;H^1(\Omega))}.$$

Hence, (ii) is also established. □

Proof of Theorem 3.1. In view of the estimates in Proposition 3.2 and Lemma 3.2, the proof of Theorem 3.1 readily follows by standard compactness results, as may be found, e.g., in [7]. □

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