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An extension of the existence theorem of a pure-strategy Nash equilibrium

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Abstract

In this paper, we study necessary and sufficient conditions for the existence of a pure-strategy Nash equilibrium. It is well known that any non-cooperative n-person game in strategic form has a mixed-strategy Nash equilibrium. On the other hand, a pure-strategy Nash equilibrium does not always exist. Wherein, there are few results considering sufficient conditions for the existence of a pure-strategy Nash equilibrium; see Topkis [9], Iimura [1] and Sato and Kawasaki [6]. These results imply that monotonicity of best responses is one of the most important assumptions for its existence. They, however, do not cover studies on necessary conditions for its existence. Hence, in this paper, we first extend the class of games having a monotonicity condition and show that the games belonging the class have a pure-strategy Nash equilibrium. Next, we prove that the extended monotonicity is a necessary condition for the existence of a pure-strategy Nash equilibrium in the case of two-person.

1 Introduction

In this paper, we study necessary and sufficient conditions for the existence of a pure-strategy Nash equilibrium. A Nash equilibrium is one of the most important solution concepts in non-cooperative games, and Nash [4], [5] has shown that if each player use mixed-strategy, then any non-cooperative game has it. A pure-strategy Nash equilibrium, on the other hand, does not always exist. Hence we consider how games have it. As answers of this problem, there are few known results. The first result is due to

Topkis [9]. He has introduced so-called supermodular games. He first got the monotonicity of the greatest and least element of each player's best response, by assuming the increasing differences for each player's payoff function. Next, relying on Tarski's fixed point theorem [8], he showed the existence of a pure-strategy Nash equilibrium in supermodular games. Sato and Kawasaki [6] has introduced so-called monotone games. They provided a discrete fixed point theorem, and as its application, showed that any monotone game has a pure-strategy Nash equilibrium. These papers' idea is that monotonicity of best responses ensures the existence of a pure-strategy Nash equilibrium. However, these results were concerned with only sufficiency for the existence of a pure-strategy Nash equilibrium. This paper, on the other hand, shall consider not only sufficiency but also necessity for the existence.

Also, Iimura [1] has given a class of games having a pure-strategy Nash equilibrium as an application of the discrete fixed point theorem [2]. The discrete fixed point theorem plays on integrally convex sets (see Murota [3]) and relies on Brouwer's fixed point theorem. However, these result also was concerned with only sufficiency for the existence of a pure-strategy Nash equilibrium.

Our paper is organized as follows: In Section 2, we shall discuss on the sufficient conditions for the existence of a pure-strategy Nash equilibrium. We first summarize the result of Sato and Kawasaki [6]. Next, we extend the result to deal with a wide range of non-cooperative n-person games. Here we remark that the result of this section is not only an extension of monotone games but also central rule of the next section. In Section 3, we shall show that the monotonicity condition is on necessity for the existence of a pure-strategy Nash equilibrium in the case of two-person. In order to show this fact, we use the directed graph representation of set-valued mappings.

Throughout this paper, we represent a strategic form game as $G = \{N, \{S_i\}_{i\in N}, \{p_i\}_{i\in N}\}$, where

- $N := \{1, \ldots, n\}$ is the set of players.
- For any $i \in N$, S_i denotes the finite set, with a total order $\leqq_i$, of player i's pure strategies. An element of this set is denoted by s_i.
- $p_i : S := \prod_{j=1}^{n} S_j \to \mathbb{R}$ denotes the payoff function of player i.

2 Sufficiency for the existence of a pure-strategy Nash equilibrium

2.1 Known results: Monotone game

In this subsection, we review the sufficient condition for the existence of a pure-strategy Nash equilibrium, which has been originally introduced by Sato and Kawasaki [6]. In the paper, the authors have provided the class of non-cooperative games that so-called *monotone games*, and shown that the games have a pure-strategy Nash equilibrium. Their crucial assumption is monotonicity for best responses of games. The definition of the class of games is as follows:

Definition 2.1 (Monotone game) G is said to be a *monotone game* if, for any $i \in N$, $s_{-i}^{0}, s_{-i}^{1} \in S_{-i}$ with $s_{-i}^{0} \preceq s_{-i}^{1}$ and for any $t_i^1 \in f_i(s_{-i}^{0})$, there exists $t_i^2 \in f_i(s_{-i}^{1})$ such that $\varepsilon_i t_i^1 \leqq \varepsilon_i t_i^2$.

Further, in the paper, the authors have shown that the games have a pure-strategy Nash equilibrium.

Proposition 2.2 ([6]) *Any monotone non-cooperative n-person game G has a Nash equilibrium of pure strategies.*

Here we present an example of monotone games in the case of two-person game, that is, bimatrix game. In the rest of this section, we use the following notation:

- $A = (a_{ij})$ is a payoff matrix of player 1 (P1), that is, $u_1(i,j) = a_{ij}$.
- $B = (b_{ij})$ is a payoff matrix of player 2 (P2), that is, $u_2(i,j) = b_{ij}$.
- $S_1 := \{1, \dots, m_1\}$ is the set of pure strategies of P1, where $m_1 \in \mathbb{N}$.
- $S_2 := \{1, \dots, m_2\}$ is the set of pure strategies of P2, where $m_2 \in \mathbb{N}$.
- For any $j \in S_2$, $I(j) := \{i^* \in S_1 \colon a_{i^*j} = \max_{i \in S_1} a_{ij}\}$ is the set of best responses of P1.
- For any $i \in S_1$, $J(i) := \{j^* \in S_2 \colon b_{ij^*} = \max_{j \in S_2} b_{ij}\}$ is the set of best responses of P2.
- $F(i,j) := I(j) \times J(i)$ denotes the set of best responses of $(i,j) \in S_1 \times S_2$.

- A pair (i^*, j^*) is a Nash equilibrium of pure strategies if $(i^*, j^*) \in F(i^*, j^*)$.

Then Definition 2.1 reduces to Definition 2.3 below.

Definition 2.3 (Monotone bimatrix game) A is said to be a *monotone matrix* if for any $j^0, j^1 \in S_2$ such that $\varepsilon_2 j^0 < \varepsilon_2 j^1$ and for any $i^1 \in I(j^0)$, there exists $i^2 \in I(j^1)$ such that $\varepsilon_1 i^1 \leqq \varepsilon_1 i^2$. Also, B is said to be a *monotone matrix* if for any i^0, i^1 such that $\varepsilon_1 i^0 < \varepsilon_1 i^1$ and for any $j^1 \in J(i^0)$, there exists $j^2 \in J(i^1)$ such that $\varepsilon_2 j^1 \leqq \varepsilon_2 j^2$. When both A and B are monotone matrices, the game is said to be a *monotone bimatrix game.*

Example 2.4 The following are monotone matrices for $(\varepsilon_1, \varepsilon_2) = (1, 1)$, where framed numbers correspond to best responses, and circled numbers correspond to the Nash equilibrium.

$$A = \left(\begin{array}{c|c|c} 5 & 6 & 1 \\ \boxed{8} & 2 & 3 \\ 4 & \boxed{7} & \boxed{4} \end{array} \right), \quad B = \left(\begin{array}{ccc} \boxed{7} & 2 & 6 \\ \hline 3 & \boxed{9} & 5 \\ \hline \boxed{8} & 6 & \boxed{8} \end{array} \right).$$

Indeed, the following inequalities show that A and B are monotone matrices.

$$\begin{array}{ccccc} I(1) = \{2\} & & I(2) = \{3\} & & I(3) = \{3\} \\ \Cup & & \Cup & & \Cup \\ 2 & \leqq & 3 & \leqq & 3 \\ J(1) = \{1\} & & J(2) = \{2\} & & J(3) = \{1, 3\} \\ \Cup & & \Cup & & \Cup \\ 1 & \leqq & 2 & \leqq & 3. \end{array}$$

Moreover, since $(i, j) = (3, 3)$ belongs to the set of best responses to itself, $(3, 3)$ is a pure-strategy Nash equilibrium.

Next, we show that structure of monotone games by introducing a directed graphic representation of set-valued mappings. Since S is the product of finite sets S_i's, it is also finite, say, $S = \{s^1, \ldots, s^m\}$. For any non-empty set-valued mapping F from S to itself, we define a directed graph $D_F = (S, A_F)$ by $A_F = \{(s^i, s^j) \colon s^j \in F(s^i),\ s^i, s^j \in S\}$. For any selection f of F, that is, $f(s) \in F(s)$ for all $s \in S$, we similarly define a directed graph D_f. For any $s \in S$, we denote by $\mathrm{od}(s)$ and $\mathrm{id}(s)$ the outdegree and indegree of s, respectively. Then, $\mathrm{od}(s) \geq 1$ for D_F, and $\mathrm{od}(s) = 1$ for D_f.

Definition 2.5 (Cycle of length l) A set-valued mapping F is said to have a *directed cycle of length* l if there exists l distinct points $\{s^{i_1}, s^{i_2}, \ldots, s^{i_l}\}$ of S such that $s^{i_1} \in F(s^{i_l})$ and $s^{i_{k+1}} \in F(s^{i_k})$ for all $k \in \{1, \ldots, l-1\}$.

Example 2.6 Let $S = \{s^1, s^2, s^3, s^4\}$ and define a non-empty set-valued mapping F as follows:

$$F(s^1) := \{s^2, s^4\}, \; F(s^2) := \{s^4\}, \; F(s^3) := \{s^1\}, \; F(s^4) := \{s^4\}.$$

Then the directed graph corresponding to F is given by Figure 1.

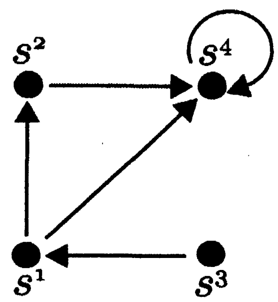

Fig. 1 The directed graph corresponding to F

Here we note that the existence of a Nash equilibrium corresponds to of a directed cycle of length 1. Hereafter, in particular, we call a directed cycle of length 1 a *loop*, for short. Then the directed graph corresponding to the game in Example 2.4 is given by Figure 2. From the figure, we can observe that monotone games are ensured to exist a loop on a pass starting from the minimum element $(1, 1)$; see Figure 3. However, the pass having a loop need not to start from the minimum element. Moreover, we can reorder the pure-strategies of each player. Thus, there is still room for extend the class of monotone games. This is discussed in the next subsection.

2.2 An extension of the class of monotone games

In this subsection, we extend the class of monotone games, which introduced by Sato and Kawasaki [6], and show the games belonging to the class have a pure-strategy Nash equilibrium. We call a game belonging to the class a *partially monotone game*.

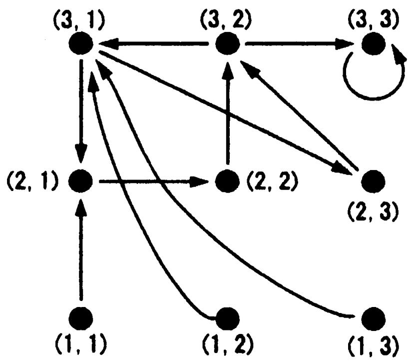

Fig. 2 The directed graph corresponding to the game in Example 2.4.

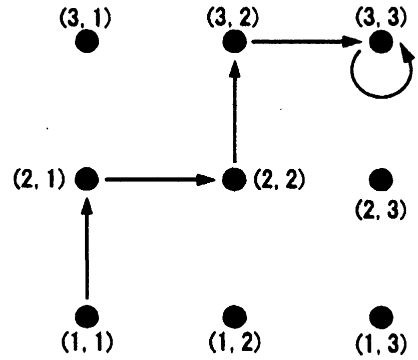

Fig. 3 The directed graph has a loop, which is a pure-strategy Nash equilibrium.

Definition 2.7 G is said to be a *partially monotone game* if there exist a selection f of F, non-empty subsets $T_i \subset S_i$, and bijections σ_i from T_i into itself ($i \in N$) such that at least one of T_i's has two or more elements, $f(T) \subset T$, and

$$s_{-i} \preceq_{\sigma_{-i}} t_{-i} \;\Rightarrow\; f_i(s_{-i}) \leqq_{\sigma_i} f_i(t_{-i}) \tag{2.1}$$

for any $i \in N$.

Theorem 2.8 ([7]) *Any partially monotone non-cooperative n-person game has a pure-strategy Nash equilibrium.*

Next, we show an example of the partially monotone game in the case of two-person, that is, the partially monotone bimatrix game. In the rest of this section, we use the notation defended in the last section.

Example 2.9 Let $S_1 = S_2 = \{1, 2, 3\}$, and let us consider the following bimatrix game:

$$A = \left(\begin{array}{c|c|c} 4 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 1 & 4 \end{array}\right), \quad B = \left(\begin{array}{ccc} 2 & 1 & 3 \\ \hline 1 & 4 & 2 \\ \hline 3 & 3 & 2 \end{array}\right).$$

The game defined by A and B is not a monotone game, since B is not a monotone matrix. By interchanging the second and third columns, A and B are transformed into

A' and B', respectively, as given below:

$$A' = \left(\begin{array}{c|c|c} ④ & 3 & 2 \\ 2 & ④ & ⑤ \\ 3 & ④ & 1 \end{array} \right), \quad B' = \left(\begin{array}{ccc} 2 & ③ & 1 \\ \hline 1 & 2 & ④ \\ \hline ③ & 2 & ③ \end{array} \right).$$

Here the game defined by A' and B' is not also a monotone game, since both A' and B' are not monotone matrices. However, we remove the third row, A' and B' are transformed into A'' and B'', respectively:

$$A'' = \left(\begin{array}{c|c|c} ④ & 3 & 2 \\ 2 & ④ & ⑤ \end{array} \right), \quad B'' = \left(\begin{array}{ccc} 2 & ③ & 1 \\ \hline 1 & 2 & ④ \end{array} \right).$$

Then the bimatrix game defined by A'' and B'' is now a monotone game for $(\varepsilon_1, \varepsilon_2) = (1,1)$, so we can know that the game has a pure-strategy Nash equilibrium $(3,3)$ from Proposition 2.2. In the original bimatrix game, the equilibrium is $(2,2)$.

The above procedure is equivalent to taking $T_1 := \{1,2\} \subset S_1$, $\sigma_1 :=$ id, $T_2 := S_2$ and σ_2 permutation $(2,3)$ in Definition 2.7. Thus, the original game is a partially monotone one. Further, from this example, we can immediately see the class of partially monotone games is an extension of the class of monotone games.

The extension of this section is central rule to show that the monotonicity condition is not only sufficiency but also necessity for the existence of a pure-strategy Nash equilibrium in the case of two-person.

3 Necessity of the monotonicity condition

In this section, we show that the partial monotonicity is necessary for the existence of a pure-strategy Nash equilibrium in the case of bimatrix games. The main result of this section is stated by the next theorem:

Theorem 3.1 ([7]) *Assume that a bimatrix game has a pure-strategy Nash equilibrium s^*. If one reaches s^* following a sequence $s^1, \ldots, s^m = s^*$ in S such that $s^{k+1} \in F(s^k)$ and $F(s^k)$ is a singleton for all $k = 1, \ldots, m-1$, then there exist non-empty subsets T_i $(i = 1,2)$ and bijections σ_i $(i = 1,2)$ from T_i into itself such that*

$$s^1 \preceq_\sigma s^2 \preceq_\sigma \cdots \preceq_\sigma s^m = s^*. \tag{3.1}$$

We need the following lemma to prove Theorem 3.1:

Lemma 3.2 ([7]) *If D_f is connected in the sense of the undirected graph, then f has only one directed cycle.*

Here we remark that, in Theorem 3.1, the fact that the number of player is two is a crucial assumption. Indeed, if the number of player is more than three, we can present an counter example as follows:

Example 3.3 Let P1, P2 and P3 be players; let the player's strategies be $i \in \{1,2\}$, $j \in \{1,2\}$ and $k \in \{1,2\}$, respectively; also let each player's best responses be the following:

P1	$k=1$	$k=2$
$j=1$	$i=2$	$i=1$
$j=2$	$i=2$	$i=2$

P2	$k=1$	$k=2$
$i=1$	$j=2$	$j=2$
$i=2$	$j=1$	$j=2$

P3	$j=1$	$j=2$
$i=1$	$k=2$	$k=1$
$i=2$	$k=1$	$k=2$

Then, since

$$f(2,2,2) = f_1(2,2) \times f_2(2,2) \times f_3(2,2) = (2,2,2),$$

this game has a pure-strategy Nash equilibrium $(2,2,2)$. However, this game is not a partially monotone game. Because, if we focus on player 3's best responses, then there are only four combinations of two bijections on S_1 and S_2. The above table on P3 corresponds to $(\sigma_1, \sigma_2) = (id, id)$. Three tables below correspond to $((1,2), id)$, $(id, (1,2))$, and $((1,2),(1,2))$, respectively. However, in any case, the best response does not satisfy (2.1).

P3	$j=1$	$j=2$
$i=2$	$k=1$	$k=2$
$i=1$	$k=2$	$k=1$

P3	$j=2$	$j=1$
$i=1$	$k=1$	$k=2$
$i=2$	$k=2$	$k=1$

P3	$j=2$	$j=1$
$i=2$	$k=2$	$k=1$
$i=1$	$k=1$	$k=2$

Indeed, for example, when we take $(\sigma_1, \sigma_2) = (id, id)$, for $(i,j) = (1,1)$ and $(1,2)$, we have

$$(1,1) \leq (1,2) \text{ but } f_3(1,1) = 2 \nleq 1 = f_3(1,2),$$

which implies that the best response does not satisfy (2.1).

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