A Hardy type inequality and application to the stability of degenerate stationary waves

Shuichi Kawashima
Faculty of Mathematics, Kyushu University
Fukuoka 812-8581, Japan

Kazuhiro Kurata
Department of Mathematics and Information Sciences
Tokyo Metropolitan University
Hachioji, Tokyo 192-03, Japan

1 Introduction
This note is a survey of our joint paper [2] on the stability problem of degenerate stationary waves for viscous conservation laws in the half space \( x > 0 \):

\[
\begin{align*}
 u_t + f(u)_x &= u_{xx}, \\
 u(0, t) &= -1, \quad u(x, 0) = u_0(x).
\end{align*}
\] (1.1)

Here \( u_0(x) \to 0 \) as \( x \to \infty \), and \( f(u) \) is a smooth function satisfying

\[
f(u) = \frac{1}{q}(-u)^{q+1}(1 + g(u)), \quad f''(u) > 0 \text{ for } -1 \leq u < 0,
\] (1.2)

where \( q \) is a positive integer (degeneracy exponent) and \( g(u) = O(|u|) \) for \( u \to 0 \). Notice that \( 1 + g(u) > 0 \) for \(-1 \leq u \leq 0\). It is known that the corresponding stationary problem

\[
\begin{align*}
 \phi_x &= f(\phi), \\
 \phi(0) &= -1, \quad \phi(x) \to 0 \text{ as } x \to \infty,
\end{align*}
\] (1.3)
admits a unique solution $\phi(x)$ (called \textit{degenerate stationary wave}) which verifies $\phi(x) \sim -(1 + x)^{-1/q}$. In particular, we have $\phi(x) = -(1 + x)^{-1/q}$ when $g(u) \equiv 0$.

To discuss the stability of the degenerate stationary wave $\phi(x)$, it is convenient to introduce the perturbation $v$ by $u(x, t) = \phi(x) + v(x, t)$ and rewrite the problem (1.1) as

$$
v_t + (f(\phi + v) - f(\phi))_x = v_{xx},
$$

$$
v(0, t) = 0, \quad v(x, 0) = v_0(x),
$$

where $v_0(x) = u_0(x) - \phi(x)$, and $v_0(x) \to 0$ as $x \to \infty$. The stability of degenerate stationary waves has been studied recently in [14, 2]. The paper [14] proved the following stability result: If the initial perturbation $v_0(x)$ is in the weighted space $L^2_\alpha$, then the perturbation $v(x, t)$ decays in $L^2$ at the rate $t^{-\alpha/4}$ as $t \to \infty$, provided that $\alpha < \alpha_*(q)$, where

$$
\alpha_*(q) := (q + 1 + \sqrt{3q^2 + 4q + 1})/q.
$$

The decay rate $t^{-\alpha/4}$ obtained in [14] would be optimal but the restriction $\alpha < \alpha_*(q)$ was not very sharp. This restriction has been relaxed to $\alpha < \alpha_c(q) := 3 + 2/q$ in our joint paper [2] by employing the space-time weighted energy method in [14] and by applying a Hardy type inequality with the best possible constant. Notice that $\alpha_*(q) < \alpha_c(q)$. This new stability result will be reviewed in this note.

It is interesting to note that a similar restriction on the weight is imposed also for the stability of degenerate shock profiles (see [9]). We remark that our stability result for degenerate stationary waves is completely different from those for non-degenerate case. In fact, for non-degenerate stationary waves, we have the better decay rate $t^{-\alpha/2}$ for the perturbation without any restriction on $\alpha$. See [4, 5, 13, 15] for the details. See also [6, 8, 10] for the related stability results for stationary waves.

To check the validity of our restriction $\alpha < \alpha_c(q) := 3 + 2/q$, it is important to discuss the dissipativity of the following linearized operator associated with (1.4):

$$
Lv = v_{xx} - (f'(\phi)v)_x.
$$

In a simpler situation including the case $g(u) \equiv 0$ in (1.2), we showed in [2] that the operator $L$ is uniformly dissipative in $L^2_\alpha$ for $\alpha < \alpha_c(q)$ but can not be dissipative for $\alpha > \alpha_c(q)$. This suggests that the exponent $\alpha_c(q)$ is the critical exponent of the stability problem of degenerate stationary waves. This result on the characterization of the dissipativity of $L$ is an improvement on the previous one in [14] and has been established again by using a Hardy
type inequality with the best possible constant. This result will be also reviewed in this note.

Notations. For $1 \leq p \leq \infty$ and a nonnegative integer $s$, $L^p$ and $W^{s,p}$ denote the usual Lebesgue space on $\mathbb{R}_+ = (0, \infty)$ and the corresponding Sobolev space, respectively. When $p = 2$, we write $H^s = W^{s,2}$. We introduce weighted spaces. Let $w = w(x) > 0$ be a weight function defined on $[0, \infty)$ such that $w \in C^0[0, \infty)$. Then, for $1 \leq p < \infty$, we denote by $L^p(w)$ the weighted $L^p$ space on $\mathbb{R}_+$ equipped with the norm
\[
\|u\|_{L^p(w)} := \left( \int_0^\infty |u(x)|^p w(x) \, dx \right)^{1/p}.
\]
(16)
The corresponding weighted Sobolev space $W^{s,p}(w)$ is defined by $W^{s,p}(w) = \{ u \in L^p(w); \partial_x^k u \in L^p(w) \text{ for } k \leq s \}$ with the norm $\| \cdot \|_{W^{s,p}(w)}$. Also, we denote by $W^{1,p}_0(w)$ the completion of $C^\infty_0(\mathbb{R}_+)$ with respect to the norm
\[
\|u\|_{W^{1,p}_0(w)} := \|\partial_x u\|_{L^p(w)} = \left( \int_0^\infty |\partial_x u(x)|^p w(x) \, dx \right)^{1/p}.
\]
(1.7)
When $p = 2$, we write $H^s(w) = W^{s,2}(w)$ and $H^1_0(w) = W^{1,2}_0(w)$. In the special case where $w = (1 + x)^\alpha$ with $\alpha \in \mathbb{R}$, these weighted spaces are abbreviated as $L^p_\alpha$, $W^{s,p}_\alpha$, $W^{1,p}_0$, $H^s_\alpha$ and $H^{1,0}_\alpha$, respectively.

2 Hardy type inequality

Our Hardy type inequality used in [2] is a simple modification of the original Hardy’s inequality introduced in [1, 7] (see also [12]).

Proposition 2.1. Let $\psi \in C^1[0, \infty)$ and assume either
1) $\psi > 0$, $\psi_x > 0$ and $\psi(x) \to \infty$ for $x \to \infty$; or
2) $\psi < 0$, $\psi_x > 0$ and $\psi(x) \to 0$ for $x \to \infty$.

Then we have
\[
\int_0^\infty v^2 \psi_x \, dx \leq 4 \int_0^\infty v_x^2 \psi^2/\psi_x \, dx
\]
(2.1)
for $v \in C^\infty_0(\mathbb{R}_+)$ and hence for $v \in H^1_0(w)$ with $w = \psi^2/\psi_x$. Here 4 is the best possible constant, and there is no function $v \in H^1_0(w)$, $v \neq 0$, which attains the equality in (2.1).

Proof. The proof is quite simple. Let $v \in C^\infty_0(\mathbb{R}_+)$. A simple calculation gives
\[
(v^2 \psi)_x = v^2 \psi_x + 2v v_x \psi \\
= \frac{1}{2} v^2 \psi_x + \frac{1}{2} (v + 2v_x \psi/\psi_x)^2 \psi_x - 2v_x^2 \psi^2/\psi_x.
\]
(2.2)
Integrating (2.2) in $x$, we obtain
\[
\int_0^\infty v^2 \psi_x \, dx + \int_0^\infty (v + 2v_x \psi / \psi_x)^2 \, dx = 4 \int_0^\infty v_x^2 \psi^2 / \psi_x \, dx,
\]
which gives the desired inequality (2.1). It follows from (2.3) that the equality in (2.1) holds if and only if $v + 2v_x \psi / \psi_x \equiv 0$. But we find that such a $v$ in $H_0^1(w)$ must be $v \equiv 0$.

We show the best possibility of the constant 4 in (2.1). We consider the case (1). Let us fix $a > 0$. Let $\epsilon > 0$ be a small parameter and put
\[
v^\epsilon(x) = \begin{cases} 0, & 0 \leq x < a, \\ (x - a)\psi(x)^{-1/2 - \epsilon}, & a < x < a + 1, \\ \psi(x)^{-1/2 - \epsilon}, & a + 1 < x. \end{cases}
\]
Then we have after straightforward computations that
\[
\frac{\int_0^\infty (v_x^\epsilon)^2 \psi^2 / \psi_x \, dx}{\int_0^\infty (v^\epsilon)^2 \psi_x \, dx} = \frac{O(1) + (1/2 + \epsilon)^2 \frac{1}{2\epsilon} \psi(a + 1)^{-2\epsilon}}{O(1) + \frac{1}{2\epsilon} \psi(a + 1)^{-2\epsilon}} \to \frac{1}{4}
\]
for $\epsilon \to 0$. This shows that 4 in (2.1) is the best possible constant. The case (2) can be treated similarly if we take a test function $v^\epsilon(x)$ as
\[
v^\epsilon(x) = \begin{cases} 0, & 0 \leq x < a, \\ (x - a)(-\psi(x))^{-1/2 - \epsilon}, & a < x < a + 1, \\ (-\psi(x))^{-1/2 - \epsilon}, & a + 1 < x. \end{cases}
\]
but we omit the details. This completes the proof of Proposition 2.1.

The $L^p$ version of Proposition 2.1 is given as follows.

**Proposition 2.2.** Let $\psi$ be the same as in Proposition 2.1. Let $1 < p < \infty$. Then we have
\[
\int_0^\infty |v|^p \psi_x \, dx \leq p^p \int_0^\infty |v_x|^p |\psi|^p / \psi_x^{p-1} \, dx
\]
for $v \in C_0^\infty(\mathbb{R}_+)$ and hence for $v \in W_0^{1,p}(w)$ with $w = |\psi|^p / \psi_x^{p-1}$. Here $p^p$ is the best possible constant, and there is no function $v \in W_0^{1,p}(w)$, $v \neq 0$, which attains the equality in (2.5).
Proof. We only prove the inequality (2.5) and omit the other discussions. Let \( 1 < p < \infty \) and \( \psi \in C_0^\infty(\mathbb{R}_+) \). A simple calculation gives

\[
(|v|^p \psi)_x = |v|^p \psi_x + p|v|^{p-2} vv_x \psi = \frac{1}{p} (|v|^p \psi_x - p^p |v_x|^p |\psi|^{p/p-1}) + R,
\]

where

\[
R = (1 - \frac{1}{p})|v|^p \psi_x + \frac{1}{p} p^p |v_x|^p |\psi|^{p/p-1} + p|v|^{p-2} vv_x \psi.
\]

Integrating (2.6) in \( x \), we obtain

\[
\int_0^\infty |v|^p \psi_x \, dx + p \int_0^\infty R \, dx = p^p \int_0^\infty |v_x|^p |\psi|^{p/p-1} \, dx.
\]

By applying the Young inequality \( AB \leq (1 - 1/p)A^{p/(p-1)} + (1/p)B^p \) for \( A = |v|^{p-1} \psi_x^{(p)} \) and \( B = p|v_x||\psi|^{p/p-1} \), we find that \( R \geq 0 \), which together with (2.7) gives the desired inequality (2.5).

The following variant of Proposition 2.1 is useful in our application.

**Proposition 2.3.** Let \( \phi \in C^1[0, \infty) \), \( \phi < 0 \), \( \phi_x > 0 \), and \( \phi(x) \to 0 \) for \( x \to \infty \). Let \( \sigma \in \mathbb{R} \) with \( \sigma \neq 0 \), and define the weight functions \( w \) and \( w_1 \) by

\[
w = (-\phi)^{-\sigma+1}/\phi_x, \quad w_1 = (-\phi)^{-\sigma-1} \phi_x.
\]

Then we have

\[
\int_0^\infty v^2 w_1 \, dx \leq \frac{4}{\sigma^2} \int_0^\infty v_x^2 w \, dx
\]

for \( v \in H^1_0(w) \). Here \( 4/\sigma^2 \) is the best possible constant, and there is no function \( v \in H^1_0(w) \), \( v \neq 0 \), which attains the equality in (2.9).

**Proof.** We put \( \psi = (-\phi)^{-\sigma} \) for \( \sigma > 0 \) and \( \psi = -(-\phi)^{-\sigma} \) for \( \sigma < 0 \), and apply Proposition 2.1. This gives the desired conclusion.

As a simple corollary of Proposition 2.3, we have:

**Corollary 2.4.** Let \( \alpha \in \mathbb{R} \) with \( \alpha \neq 1 \). Then we have

\[
\|v\|_{L^2_{\alpha-2}} \leq \frac{2}{|\alpha - 1|} \|v_x\|_{L^2_\alpha}
\]

for \( v \in H^1_{\alpha,0} \). Here the constant \( 2/|\alpha - 1| \) is the best possible, and there is no function \( v \in H^1_{\alpha,0}, v \neq 0 \), which attains the equality in (2.10).

**Proof.** Let \( \phi = -(1 + x)^{-1/q} \) with \( q > 0 \). We apply Proposition 2.3 for this \( \phi \) and \( \sigma = (\alpha - 1)q \). This gives the proof.
3 Dissipativity of the linearized operator

Following [2], we discuss the dissipativity of the operator $L$ defined by (1.5) in the weighted space $L^2(w)$, where $w$ is given by (2.8) with $\phi$ being the degenerate stationary wave. Note that our degenerate stationary wave $\phi$ is a smooth solution of (1.3) and verifies

$$-1 \leq \phi(x) < 0, \quad \phi_x(x) > 0, \quad \phi(x) \to 0 \text{ for } x \to \infty,$$

(3.1)

$$c(1 + x)^{-1/q} \leq -\phi(x) \leq C(1 + x)^{-1/q}.$$  

(3.2)

Now, letting $w > 0$ be a smooth weight function depending only on $x$, we calculate the inner product $\langle Lv, v \rangle_{L^2(w)}$ for $v \in C^\infty_0(\mathbb{R}_+)$, where

$$\langle u, v \rangle_{L^2(w)} := \int_0^\infty uvw dx.$$  

(3.3)

We multiply (1.5) by $v$. Then a simple computation gives

$$(Lv)v = (vv_x - \frac{1}{2}f'(\phi)v^2)_x - v_x^2 - \frac{1}{2}f''(\phi)\phi_xv^2.$$  

Multiplying this equality by $w$, we obtain

$$(Lv)vw = \{(vv_x - \frac{1}{2}f'(\phi)v^2)w - \frac{1}{2}v^2w_x\}_x$$

$$- v_x^2w + \frac{1}{2}v^2(w_{xx} + w_xf'(\phi) - wf''(\phi)\phi_x).$$  

(3.4)

Now we choose the weight function $w$ and the corresponding $w_1$ in terms of our degenerate stationary wave $\phi$ by (2.8), where $\sigma \in \mathbb{R}$. Then we have $w = (-\phi)^{-\sigma+1}/f(\phi)$ and $w_1 = (-\phi)^{-\sigma-1}f(\phi)$ by $\phi_x = f(\phi)$. After straightforward computations, we find that

$$w_{xx} + w_xf'(\phi) - wf''(\phi)\phi_x = 2(c_1(\sigma) - r(\phi))w_1,$$  

(3.5)

where

$$c_1(\sigma) := \sigma(\sigma - 1)/2 - q(q + 1),$$

$$r(\phi) := (-\phi)^2f''(\phi)/f(\phi) - q(q + 1).$$  

(3.6)

Substituting (3.5) into (3.4) and integrating with respect to $x$, we get the following conclusion.
**Claim 3.1.** Let $\phi$ be the degenerate stationary wave and define the weight functions $w$ and $w_1$ by (2.8) with $\sigma \in \mathbb{R}$. Then the operator $L$ defined in (1.5) verifies

$$\langle Lv, v \rangle_{L^2(w)} = -\|v_x\|^2_{L^2(w)} + c_1(\sigma)\|v\|^2_{L^2(w_1)} - \int_0^\infty v^2 r(\phi) w_1 \, dx \quad (3.7)$$

for $v \in C_0^\infty(\mathbb{R}_+)$ and hence for $v \in H^1_0(w)$, where $c_1(\sigma)$ and $r(\phi)$ are given in (3.6).

The term $r(\phi)$ in (3.7) can be regarded as a small perturbation. In fact, a straightforward computation gives

$$r(u) = (-u)(-g''(u) - 2(q+1)g'(u))/(1 + g(u)) \quad (3.8)$$

which shows that $r(u) = O(|u|)$ for $u \to 0$. In particular, we have $r(u) \equiv 0$ if $g(u) \equiv 0$. With these preparations, we have the following result on the characterization of the dissipativity of $L$.

**Theorem 3.2.** Assume (1.2). Let $\phi$ be the degenerate stationary wave and $L$ be the operator defined in (1.5). Let $w$ and $w_1$ be the weight functions in (2.8) with the parameter $\sigma \in \mathbb{R}$. Then we have:

1. Let $-2q < \sigma < 2(q+1)$. Then, under the additional assumption that $r(u) \geq 0$ for $-1 \leq u \leq 0$, the operator $L$ is uniformly dissipative in $L^2(w)$. Namely, there is a positive constant $\delta$ such that

$$\langle Lv, v \rangle_{L^2(w)} \leq -\sigma^2(\|v_x\|^2_{L^2(w)} + \|v\|^2_{L^2(w_1)}) \quad (3.9)$$

for $v \in H^1_0(w)$.

2. Let $\sigma > 2(q+1)$ or $\sigma < -2q$. Then the operator $L$ can not be dissipative in $L^2(w)$. Namely, we have $\langle Lv, v \rangle_{L^2(w)} > 0$ for some $v \in H^1_0(w)$ with $v \neq 0$.

**Proof.** The proof is based on the equality (3.7) in Claim 3.1 and the Hardy type inequality (2.9) in Proposition 2.3.

Let $-2q < \sigma < 2(q+1)$. This is equivalent to $c_1(\sigma) < \sigma^2/4$. Therefore we can choose $\delta > 0$ so small that $\delta(1 + \sigma^2/4) \leq \sigma^2/4 - c_1(\sigma)$. Since $r(\phi) \geq 0$ by the additional assumption on $r(u)$ and since $(\sigma^2/4)\|v\|^2_{L^2(w_1)} \leq \|v_x\|^2_{L^2(w)}$ by the Hardy type inequality (2.9), we have from (3.7) that

$$\langle Lv, v \rangle_{L^2(w)} \leq -\|v_x\|^2_{L^2(w)} + c_1(\sigma)\|v\|^2_{L^2(w_1)}$$

$$= -\delta\|v_x\|^2_{L^2(w)} - (1 - \delta)\|v_x\|^2_{L^2(w)} + c_1(\sigma)\|v\|^2_{L^2(w_1)}$$

$$\leq -\delta\|v_x\|^2_{L^2(w)} - \{(1 - \delta)\sigma^2/4 - c_1(\sigma)\}\|v\|^2_{L^2(w_1)}$$

$$\leq -\delta(\|v_x\|^2_{L^2(w)} + \|v\|^2_{L^2(w_1)}) \quad (3.10)$$
for $v \in C^\infty_0(\mathbb{R}_+)$ and hence for $v \in H^1_0(w)$, where we have used the fact that $(1-\delta)\sigma^2/4 - c_1(\sigma) \geq \delta$. This completes the proof of the uniform dissipative case (1).

Next we consider the case where $\sigma > 2(q+1)$; the case $\sigma < -2q$ can be treated similarly and we omit the argument in this latter case. When $\sigma > 2(q+1)$, we have $c_1(\sigma) > \sigma^2/4$. Then we choose $\delta > 0$ so small that $c_1(\sigma) \geq \sigma^2/4 + 3\delta$. Since $r(u) = O(|u|)$ for $u \to 0$ and $\phi(x) \to 0$ for $x \to \infty$, we take $a = a(\delta) > 0$ so large that $|r(\phi)| \leq \delta$ for $x \geq a$. For this choice of $a$ and for $\epsilon > 0$, we take a test function $v^\epsilon$ as in (2.4):

$$v^\epsilon(x) = \begin{cases} 0, & 0 \leq x < a, \\ (x-a)(-\phi(x))^{\sigma(1/2+\epsilon)}, & a < x < a+1, \\ (-\phi(x))^{\sigma(1/2+\epsilon)}, & a+1 < x. \end{cases}$$ (3.11)

Then we have

$$\left| \int_0^\infty (v^\epsilon)^2 r(\phi) w_1 \, dx \right| \leq \delta \int_a^\infty (v^\epsilon)^2 w_1 \, dx = \delta \|v^\epsilon\|_{L^2(w_1)}^2,$$

so that we have from (3.7) that

$$\langle Lv^\epsilon, v^\epsilon \rangle_{L^2(w)} \geq -\|v_x^\epsilon\|_{L^2(w)}^2 + (c_1(\sigma) - \delta)\|v^\epsilon\|_{L^2(w_1)}^2.$$ (3.12)

Also, by straightforward computations, we find that

$$\frac{\|v_x^\epsilon\|_{L^2(w)}^2}{\|v^\epsilon\|_{L^2(w_1)}^2} = \frac{O(1) + \sigma^2(1/2 + \epsilon)^2 \frac{1}{2\sigma}(\phi(a+1))^{2\sigma}}{O(1) + \frac{1}{2\sigma}(\phi(a+1))^{2\sigma}} \to \frac{\sigma^2}{4}$$

for $\epsilon \to 0$. Thus we have $\|v_x^\epsilon\|_{L^2(w)}^2/\|v^\epsilon\|_{L^2(w_1)}^2 \leq \sigma^2/4 + \delta$ for a suitably small $\epsilon = \epsilon(\delta) > 0$. Consequently, we have from (3.12) that

$$\frac{\langle Lv^\epsilon, v^\epsilon \rangle_{L^2(w)}}{\|v^\epsilon\|_{L^2(w_1)}^2} \geq -\frac{\|v_x^\epsilon\|_{L^2(w)}^2}{\|v^\epsilon\|_{L^2(w_1)}^2} + c_1(\sigma) - \delta \geq -\frac{\sigma^2}{4} + c_1(\sigma) - \delta \geq \delta.$$

This completes the proof of the non-dissipative case (2). Thus the proof of Theorem 3.2 is complete. \square
In the special case where $g(u) \equiv 0$ so that $f(u) = \frac{1}{q}(-u)^{q+1}$, we have $\phi = -(1 + x)^{-1/q}$ and the operator $L$ in (1.5) is reduced to

$$L_0v = v_{xx} + \frac{q+1}{q} \left( \frac{v}{1+x} \right)_{x}.$$  (3.13)

In this simplest case, we have the complete characterization of the dissipativity of the operator $L_0$.

**Theorem 3.3.** Let $\alpha_c(q) := 3 + 2/q$. Then we have the complete characterization of the dissipativity of the operator $L_0$ given in (3.13):

1. Let $-1 < \alpha < \alpha_c(q)$. Then $L_0$ is uniformly dissipative in $L^2_{\alpha}$. Namely, there is a positive constant $\delta$ such that

$$\langle L_0v, v \rangle_{L^2_{\alpha}} \leq -\delta(\|v_x\|_{L^2_{\alpha}}^2 + \|v\|_{L^2_{\alpha-2}}^2) \quad \text{for } v \in H^1_{\alpha,0}. \quad (3.14)$$

2. Let $\alpha = \alpha_c(q)$ or $\alpha = -1$. Then $L_0$ is strictly dissipative in $L^2_{\alpha}$. Namely, we have $\langle L_0v, v \rangle_{L^2_{\alpha}} < 0$ for $v \in H^1_{\alpha,0}$ with $v \neq 0$.
3. Let $\alpha > \alpha_c(q)$ or $\alpha < -1$. Then $L_0$ can not be dissipative in $L^2_{\alpha}$. Namely, we have $\langle L_0v, v \rangle_{L^2_{\alpha}} > 0$ for some $v \in H^1_{\alpha,0}$ with $v \neq 0$.

**Proof.** In this case, we have $\phi = -(1 + x)^{-1/q}, \ L = L_0$ and $r(u) \equiv 0$. Therefore, (3.7) is reduced to

$$\langle L_0v, v \rangle_{L^2_{\alpha}} = -\|v_x\|_{L^2_{\alpha}}^2 + c_1(\sigma)\|v\|_{L^2_{\alpha-2}}^2,$$  (3.15)

where $w$ and $w_1$ are the weight functions defined in (2.8) with $\phi = -(1 + x)^{-1/q}$ and $\sigma = (\alpha - 1)q$. The desired conclusions easily follow from (3.15) by applying the same argument as in Theorem 3.2. We omit the details. \(\Box\)

### 4 Nonlinear stability

The following stability result for the nonlinear problem (1.4) was obtained in [2] as a refinement of the result in [14].

**Theorem 4.1.** Assume (1.2). Suppose that $v_0 \in L^2_{\alpha} \cap L^\infty$ for some $\alpha$ with $1 \leq \alpha < \alpha_c(q) := 3 + q/2$. Then there is a positive constant $\delta_1$ such that if $\|v_0\|_{L^2_{\alpha}} \leq \delta_1$, then the problem (1.4) has a unique global solution $v \in C^0([0, \infty); L^2_{\alpha} \cap L^p)$ for each $p$ with $2 \leq p < \infty$. Moreover, the solution verifies the decay estimate

$$\|v(t)\|_{L^p} \leq C(\|v_0\|_{L^2_{\alpha}} + \|v_0\|_{L^\infty})(1 + t)^{-\alpha/4 - \nu} \quad (4.1)$$

for $t \geq 0$, where $2 \leq p < \infty$, $\nu = (1/2)(1/2 - 1/p)$, and $C$ is a positive constant.
Proof. A key to the proof of this theorem is to show the following space-time weighted energy inequality:

\[
(1 + t)^\gamma \| v(t) \|_{L^2_\beta}^2 + \int_0^t (1 + \tau)^\gamma (\| v_x(\tau) \|_{L^2_\beta}^2 + \| v(\tau) \|_{L^2_{\beta-2}}^2) \, d\tau \leq C \| v_0 \|_{L^2_\beta}^2 + \gamma C \int_0^t (1 + \tau)^{\gamma-1} \| v(\tau) \|_{L^2_\beta}^2 \, d\tau + CS_\beta^\gamma(t)
\]  

(4.2)

for any \( \gamma \geq 0 \) and \( \beta \) with \( 0 \leq \beta \leq \alpha \), where \( 1 \leq \alpha < \alpha_c(q) := 3 + 2/q \), \( C \) is a constant independent of \( \gamma \) and \( \beta \), and

\[
S_\beta^\gamma(t) = \int_0^t (1 + \tau)^\gamma \| v(\tau) \|_{L^3_{\beta-1}}^3 \, d\tau.
\]

(4.3)

Here we give an outline of the proof of (4.2) and omit the other discussions. We refer to [2, 14] for the complete proof of Theorem 4.1.

Proof of (4.2) for \( \beta = 0 \). The proof is based on the time weighted \( L^2 \) energy method. First we note that

\[
\| v(t) \|_{L^\infty} \leq M_\infty,
\]

(4.4)

where \( M_\infty = \| v_0 \|_{L^\infty} + 2 \). This is an easy consequence of the maximum principle (see [5] for the details). Now we multiply the equation (1.4) by \( v \). This yields

\[
\left( \frac{1}{2} v^2 \right)_t + (F - vv_x)_x + v_x^2 + G = 0,
\]

(4.5)

where

\[
F = (f(\phi + v) - f(\phi))v - \int_0^v (f(\phi + \eta) - f(\phi)) \, d\eta,
\]

(4.6)

\[
G = \int_0^v (f'(\phi + \eta) - f'(\phi)) \, d\eta \cdot \phi_x.
\]

We note that

\[
F = \frac{1}{2} f'(\phi)v^2 + O(|v|^3), \quad G = \frac{1}{2} f''(\phi)\phi_xv^2 + \phi_xO(|v|^3)
\]

(4.7)

for \( v \to 0 \). Here, a careful computation, using (3.2) and (4.4), shows that

\[
G \geq c(1 + x)^{-2}v^2 - C(1 + x)^{-1-1/q}|v|^3
\]

(4.8)

for any \( x \in \mathbb{R}_+ \). We integrate (4.5) over \( \mathbb{R}_+ \) and substitute (4.8) into the resulting equality, obtaining

\[
\frac{1}{2} \frac{d}{dt} \| v \|_{L^2_2}^2 + \| v_x \|_{L^2_2}^2 + c \| v \|_{L^2_{-2}}^2 \leq C \| v \|_{L^2_1}^3.
\]
We multiply this inequality by \((1 + t)^\gamma\) and integrate with respect \(t\). This yields the desired inequality (4.2) for \(\beta = 0\).

**Proof of (4.2) for \(\beta > 0\).** We apply the space-time weighted energy method employed in [14, 2] (see also [3]). Let \(w > 0\) be a smooth weight function depending only on \(x\), which will be specified later. We multiply (4.5) by \(w\), obtaining

\[
\left(\frac{1}{2}v^2w\right)_t + \left\{(F - \mu vv_x)w + \frac{1}{2}v^2w_x\right\}_x
+ v_x^2 w - \left(\frac{1}{2}v^2 w_{xx} + Fw_x - Gw\right) = 0.
\]  

Here, using (4.7), we have

\[
\frac{1}{2}v^2w_{xx} + Fw_x - Gw = \frac{1}{2}v^2(w_{xx} + w_x f'(\phi) - wf''(\phi)\phi_x) + R,
\]  

where \(R = w_xO(|v|^3) - w\phi_xO(|v|^3)\) for \(v \to 0\). Notice that the coefficient \(w_{xx} + w_x f'(\phi) - wf''(\phi)\phi_x\) in (4.10) is just the same as that appeared in (3.4).

Now we choose the weight function \(w\) and the corresponding \(w_1\) by (2.8) with \(\sigma = (\beta - 1)q\), where \(0 \leq \beta \leq \alpha\) and \(1 \leq \alpha < \alpha_c(q) := 3 + 2/q\). Then we have (3.5) with \(\sigma = (\beta - 1)q\). Substituting these expressions into (4.9) and integrating over \(\mathbb{R}_+\), we obtain

\[
\frac{1}{2}\frac{d}{dt}\|v\|_{L^2(w)}^2 + \|v_x\|_{L^2(w)}^2 - c_1(\sigma)\|v\|_{L^2(w_1)}^2 + \int_0^\infty v^2 r(\phi) w_1 dx = \int_0^\infty R dx,
\]  

where \(c_1(\sigma)\) and \(r(\phi)\) are given in (3.6) with \(\sigma = (\beta - 1)q\). Here our weight functions verify

\[
w \sim (1 + x)^\beta, \quad w_1 \sim (1 + x)^{\beta - 2},
\]  

where the symbol \(\sim\) means the equivalence. This implies that the norms \(\|\cdot\|_{L^2(w)}\) and \(\|\cdot\|_{L^2(w_1)}\) are equivalent to \(\|\cdot\|_{L^2_\beta}\) and \(\|\cdot\|_{L^2_{\beta - 2}}\), respectively.

We estimate (4.11) similarly as in (1) of Theorem 3.2. To this end, we note that \(\sigma_1 \leq \sigma \leq \sigma_2\), where \(\sigma_1 = -q\) and \(\sigma_2 = (\alpha - 1)q\). Since \(c_1(\sigma) < \sigma^2/4\) for \(-2q < \sigma < 2(q + 1)\) and since \(-2q < \sigma_1 < \sigma_2 < 2(q + 1)\), we can choose \(\delta > 0\) so small that

\[
\delta \leq \min_{\sigma_1 \leq \sigma \leq \sigma_2} \frac{\sigma^2/4 - c_1(\sigma)}{2 + \sigma^2/4}.
\]  

Notice that this \(\delta\) is independent of \(\beta\). For this choice of \(\delta\), we take \(a = a(\delta) > 0\) so large that \(|r(\phi)| \leq \delta\) for \(x \geq a\). Then we have

\[
\left|\int_0^\infty v^2 r(\phi) w_1 dx\right| \leq \delta\|v\|_{L^2(w_1)}^2 + C\|v\|_{L^2_{\beta - 2}}^2,
\]
where $C$ is a constant satisfying $C \geq (1 + x)^2 |r(\phi)|w_1$ for $0 \leq x \leq a$. Also, using the Hardy type inequality $(\sigma^2/4)\|v\|_{L^2(w_1)}^2 \leq \|v_x\|_{L^2(w)}^2$ in (2.9) and estimating similarly as in (3.10), we have

$$\|v_x\|_{L^2(w)}^2 - c_1(\sigma)\|v\|_{L^2(w_1)}^2 \geq \delta\|v_x\|_{L^2(w)}^2 + 2\delta\|v\|_{L^2(w_1)}^2,$$

where we have used the fact that $(1 - \delta)\sigma^2/4 - c_1(\sigma) \geq 2\delta$. On the other hand, using (4.4), we see that $|R| \leq C(|w_x| + w\phi_x)|v|^3$. Moreover, a straightforward computation shows that $|w_x| + w\phi_x \leq C(1 + x)^{\beta-1}$. Substituting all these estimates into (4.11), we obtain

$$\frac{1}{2} \frac{d}{dt}\|v\|_{L^2(w)}^2 + \delta(\|v_x\|_{L^2(w)}^2 + \|v\|_{L^2(w_1)}^2) \leq C\|v\|_{L^2_2}^2 + C\|v\|_{L^3_{\beta-1}}^3,$$

(4.13)

where $\delta$ and $C$ are independent of $\beta$. We multiply this inequality by $(1 + t)^\gamma$ and integrate with respect to $t$. By virtue of (4.12), we have

$$(1 + t)^\gamma\|v(t)\|_{L^2_\beta}^2 + \int_0^t (1 + \tau)^\gamma(\|v_x(\tau)\|_{L^2_\beta}^2 + \|v(\tau)\|_{L^2_{\beta-2}}^2)\,d\tau$$

$$\leq C\|v_0\|_{L^2_\beta}^2 + \gamma C\int_0^t (1 + \tau)^{\gamma-1}\|v(\tau)\|_{L^2_\beta}^2\,d\tau$$

(4.14)

$$+ C\int_0^t (1 + \tau)^\gamma\|v(\tau)\|_{L^2_{\beta-2}}^2\,d\tau + CS^\gamma_\beta(t),$$

where the constant $C$ is independent of $\gamma$ and $\beta$. Here the third term on the right hand side of (4.14) was already estimated by (4.2) with $\beta = 0$. Hence we have proved (4.2) also for $0 < \beta \leq \alpha$. This completes the proof. \qed

References


