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Periodic Solutions of the Forced Burgers Equation

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1 Problem

We consider the forced Burgers equation

$$u_t(x,t)+u(x,t)u_x(x,t)=F_x(x,t), \tag{1.1}$$

where the forcing term F_x is the partial derivative of a given C^2-function F which is periodic in both x and t with the period 1. The problem is to find $\mathbb{Z}^2$-periodic (weak) solutions of (1.1), namely periodic solutions with the period 1 in both x and t, in constructive ways. Our basic tool is the Lax-Friedrichs difference scheme. We present two methods of constructing $\mathbb{Z}^2$-periodic solutions of (1.1): The one is based on the long time behavior of the Lax-Friedrichs difference scheme. The other is based on Newton's method, regarding $\mathbb{Z}^2$-periodic solutions as fixed points of the Poincaré map derived from the Lax-Friedrichs scheme. We give convergence proofs to these methods and simulate $\mathbb{Z}^2$-periodic solutions.

It is known that there is an interesting connection between the forced Burgers equation and Hamiltonian dynamics. One of the central issues in the theory of Hamiltonian dynamics is to look for their invariant manifolds. The graph of a $\mathbb{Z}^2$-periodic solution to (1.1) plays an important role in the issue. We also simulate trajectories of the Hamiltonian dynamics corresponding to the forced Burgers equation and visualize their connection.

2 Background

In this section we state the history of our problem. Let us go back to *Boltzmann's statistical mechanics.* Boltzmann tried to derive thermodynamics from the dynamics of particles. The dynamics of particles is governed by Hamiltonian systems

$$x'(s)=\mathcal{H}_y(x(s),y(s)),\quad y'(s)=-\mathcal{H}_x(x(s),y(s)), \tag{2.1}$$

where the Hamiltonian $\mathcal{H}(x,y):\mathbb{R}^{2n}\to\mathbb{R}$ is the total energy of particles. Because of the energy conservation law, each trajectory $(x(s),y(s))$ of (2.1) is trapped on the $2n-1$-dimensional energy level set

$$\Sigma_h:=\mathcal{H}^{-1}(h),\quad h=\mathcal{H}((x(0),y(0))).$$

In order to derive thermodynamical variables such as entropy, temperature, pressure, etc., it is required that for each function $G(x,y)$ the following equality holds:

$<$ *time-average of* G *along any trajectory on* Σ_h $>$ $=$ $<$ *space-average of* G *on* Σ_h $>$.

This is possible, if *each trajectory on* Σ_h *passes through all the points of* Σ_h *(Boltzmann's ergodic hypothesis)*. After this hypothesis was presented, many mathematicians started to analyze the ergodic problems and improved Boltzmann's ergodic hypothesis (see [**4**]).

Although the consequences of the ergodic properties of the equations (2.1) come out, it is not easy to find Hamiltonian systems with the ergodic properties. As an example, let us consider harmonic lattices, which are a model of crystals. In the case of the 1-dimensional harmonic lattice with fixed end points, its total energy is given by

$$\mathcal{H}(x,y) = \sum_{i=1}^{n} \frac{1}{2} y_i^2 + \frac{1}{2}\kappa x_1^2 + \sum_{i=1}^{n-1} \frac{1}{2}\kappa (x_{i+1} - x_i)^2 + \frac{1}{2}\kappa x_n^2. \tag{2.2}$$

We can find the symplectic transform: $(x,y) \mapsto (\tilde{x},\tilde{y})$ which decomposes the motions of the system (2.1) into the normal modes. The new Hamiltonian $\tilde{\mathcal{H}}$ takes the simpler form

$$\tilde{\mathcal{H}}(\tilde{x},\tilde{y}) = \sum_{i=1}^{n} \frac{1}{2}\omega_i(\tilde{x}_i^2 + \tilde{y}_i^2) \quad (\omega = (\omega_1, \cdots, \omega_n) \text{ is constant}).$$

By the symplectic polar coordinates $(q,p) \in \mathbb{T}^n \times (\mathbb{R}_+)^n$, $\mathbb{T}^n := \mathbb{R}^n/2\pi\mathbb{Z}^n$ defined by

$$\tilde{x}_i = \sqrt{2p_i}\sin q_i, \quad \tilde{y}_i = \sqrt{2p_i}\cos q_i,$$

$\tilde{\mathcal{H}}$ changes into

$$H(q,p) = \sum_{i=1}^{n} \omega_i p_i : \mathbb{T}^n \times (\mathbb{R}_+)^n \to \mathbb{R}.$$

Therefore each trajectory is given by

$$(q(s),p(s)) = (\omega s + q(0), p(0)) \mod 2\pi,$$

which implies for any $s \in \mathbb{R}$

$$(q(s),p(s)) \in \mathcal{I} := \mathbb{T}^n \times \{p(0)\}.$$

Each trajectory is trapped on an n-dimensional torus and cannot be ergodic!

Fermi, Pasta and Ulam [**2**] made numerical simulations to anharmonic lattices which have non-quadratic potential energy in addition to (2.2). In the anharmonic cases, the above reduction yields the Hamiltonian of the form

$$H(q,p) = \sum_{i=1}^{n} \omega_i p_i + H_1(q,p) : \mathbb{T}^n \times (\mathbb{R}_+)^n \to \mathbb{R},$$

where H_1 is a perturbation due to the anharmonic potential energy. Since H_1 depends on q, it is difficult to give expression of each trajectory $(q(s),p(s))$. They expected that small anharmonic perturbations would make trajectories ergodic. Their numerical results, however, suggested strongly that *many trajectories of anharmonic lattices are still trapped on slightly deformed n-dimensional tori and not ergodic.*

Nishida [7] showed that the *KAM theory*, stated later, is applicable to the anharmonic lattices and proved that *there exist the deformed n-dimensional tori on which trajectories are trapped.*

Existence of such deformed n-dimensional tori is significant to an understanding of the stability or instability of Hamiltonian dynamics. Now we give a brief description of the problem of the search for such deformed n-dimensional tori.

We consider C^2-Hamiltonians

$$H(q,p) : \mathbb{T}^n \times D \to \mathbb{R}, \ \mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n, \ \ D \subset \mathbb{R}^n \tag{2.3}$$

and Hamiltonian systems

$$q'(s) = H_p(q(s), p(s)), \quad p'(s) = -H_q(q(s), p(s)). \tag{2.4}$$

The solution (trajectory) of (2.4) with an initial value $(\theta, I) \in \mathbb{T}^n \times D$ is denoted by

$$\phi_H^s(\theta, I),$$

where ϕ_H^s is the flow of (2.4). A manifold $\mathcal{I}$ is called a *ϕ_H^s-invariant manifold diffeomorphic to $\mathbb{T}^n$* or just *ϕ_H^s-invariant n-torus*, if $\mathcal{I}$ is an embedded n-dimensional torus by a smooth embedding: $\mathbb{T}^n \to \mathbb{T}^n \times D$ and satisfies for each s

$$\phi_H^s(\mathcal{I}) \subset \mathcal{I}.$$

First, we consider the simple case where Hamiltonians are of the form

$$H(q,p) = H_0(p).$$

It is proved that, in "general", Hamiltonians of integrable Hamiltonian systems can be brought into the above form by an appropriate symplectic transform (e.g., see [6]). The harmonic lattice is an example of integrable Hamiltonian systems. Each trajectory is represented by

$$\phi_H^s(\theta, I) = \big(\partial_p H_0(I)s + \theta, \ I\big) \mod 1,$$

where $\partial_p H_0$ is the gradient of H_0. Hence we can find ϕ_H^s-invariant n-torus for each $I \in D$

$$\mathcal{I} := \mathbb{T}^n \times \{I\},$$

which implies that the phase space $\mathbb{T}^n \times D$ is foliated by these tori, namely

$$\mathbb{T}^n \times D = \cup \mathcal{I}.$$

Each $\mathcal{I}$ carries trajectories $(q(s), p(s))$, which are just straight lines with the same slope

$$\lambda := \lim_{|s| \to \infty} \frac{\tilde{q}(s)}{s} = \partial_p H_0(I),$$

where $q(s) = \tilde{q}(s) \mod 1$. λ is called the frequency vector of the trajectory $(q(s), p(s))$.

Next we consider the perturbed Hamiltonians

$$H(q,p) = H_0(p) + H_1(q,p) : \mathbb{T}^n \times D \to \mathbb{R}, \tag{2.5}$$

where H_1 is a perturbation. Because of the so-called *small divisor problem*, people had difficulties in proving existence of ϕ_H^s-invariant n-tori for (2.5). In 1954, Kolmogorov brought the great progress. His arguments were completed by Arnold and Moser, which is now well-known as the KAM theory. One of the main assertion of the KAM theory is that there exist the ϕ_H^s-invariant n-tori carrying trajectories with the *strongly nonresonant frequency vectors*:

The KAM Theorem ([**5**][**1**]). *Let D be a bounded connected closed domain of $\mathbb{R}^n$. Suppose (A1)-(A3):*

(A1) $H(q,p) = H_0(p) + H_1(q,p)$ *is analytic on a complex neighborhood G of $\mathbb{T}^n \times D$,*

(A2) H_0 *is nondegenerate, namely the Hessian matrix of H_0 has rank n on D,*

(A3) $\| H_1 \| = \sup_G |H_1(q,p)|$ *is sufficiently small.*

Then there exists a family of KAM n-tori $\mathcal{I}$ whose union satisfies the following measure estimate:

$$\text{mes}\,[\cup\mathcal{I}] \to \text{mes}\,[\mathbb{T}^n \times D] \quad \textit{as } \| H_1 \| \to 0,$$

where a KAM n-torus $\mathcal{I}$ is a ϕ_H^s-invariant manifold diffeomorphic to $\mathbb{T}^n$ which is a Lagrangian sub-manifold and carries quasi-periodic trajectories with the same frequency vector λ satisfying the Diophantine condition.

The developments of the KAM theory within half a century are collected in [**9**]. The KAM theory leaves interesting questions: What is going on in the region $\mathbb{T}^n \times D \setminus \cup\mathcal{I}$? What happens for large perturbations? Numerical studies tell us that there are irregular trajectories called "chaos", which sometimes seem to be ergodic in Boltzmann's sense.

Finally we search for ϕ_H^s-invariant n-tori in the general case with C^2-Hamiltonians (2.3). We restrict ourselves to the manifolds diffeomorphic to $\mathbb{T}^n$ which are of the form

$$\{(q, \partial S(q)) \mid q \in \mathbb{T}^n\}, \tag{2.6}$$

where $S : \mathbb{R}^n \to \mathbb{R}$ is a C^2-function with the $\mathbb{Z}^n$-periodic gradient ∂S. Note that they are Lagrangian sub-manifolds. It is easily proved that each KAM n-torus takes the form (2.6) with a real analytic function S satisfying the Hamilton-Jacobi equation

$$H(q, \partial S(q)) = h \quad \text{in } \mathbb{R}^n \quad (h \text{ is constant}) \tag{2.7}$$

with the real analytic Hamiltonian $H(q,p) = H_0(p) + H_1(q,p)$. Let us consider the Hamilton-Jacobi equations (2.7) with general C^2-Hamiltonians. Note that each solution of the Hamiltonian system (2.4) is part of the characteristics of (2.7). The following assertion is well-known: *Let $H : \mathbb{T}^n \times D \to \mathbb{R}$ and $S : \mathbb{R}^n \to \mathbb{R}$ be a C^2-function. Then the graph of $\partial S(q)$*

$$\mathcal{I}_{\partial S} := \{(q, \partial S(q)) \mid q \in \mathbb{T}^n\}$$

is a ϕ_H^s-invariant manifold diffeomorphic to $\mathbb{T}^n$, if and only if S is a C^2-solution of the Hamilton-Jacobi equation (2.7) with the $\mathbb{Z}^n$-periodic gradient $\partial S(q)$. Particularly we have the following: *Let S be a C^2-solution of the Hamilton-Jacobi equation (2.7) with the $\mathbb{Z}^n$-periodic gradient $\partial S(q)$. If $n = 2$ and H satisfies the relation*

$$H(q_1, q_2, p_1, p_2) = h \Longleftrightarrow p_2 = f(q_1, q_2, p_1; h),$$

then $u(q) := S_{q_1}(q)$ is a $\mathbb{Z}^2$-periodic C^1-solution of the scalar conservation law

$$\partial_{q_2} u(q_1, q_2) = \partial_{q_1}\{f(q_1, q_2, u(q_1, q_2))\} \quad \text{in } \mathbb{R}^2 \tag{2.8}$$

and the graph $\mathcal{I}_{\partial S}$ is represented by

$$\mathcal{I}_{\partial S} = \{(q, u(q), f(q, u(q); h)) \mid q \in \mathbb{T}^2\}.$$

A C^1-solution of (2.8) yields a C^2-solution of the corresponding Hamilton-Jacobi equation. Of course we cannot always expect classical solutions of the Hamilton-Jacobi equations (2.7) or the scalar conservation laws (2.8). This implies that *we may have no universal methods of searching for the ϕ_H^s-invariant n-tori we are concerned with, even no such tori.*

An interesting question arises: what is the relation among *regular/chaotic properties of the Hamiltonian systems (2.4), viscosity solutions of the Hamilton-Jacobi equations (2.7)* and *entropy solutions of the scalar conservation laws (2.8).*

We consider this question taking a simple example of a nonlinear pendulum in the extended phase space with the Hamiltonian of the form

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}p_1^2 + p_2 - F(q_1, q_2) : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}. \tag{2.9}$$

We assume that F is a C^2-function. The corresponding scalar conservation law (2.8) becomes the forced Burgers equation (1.1):

$$u_t(x,t) + u(x,t)u_x(x,t) = F_x(x,t),$$

replacing the variables (q_1, q_2) with (x, t). The Hamiltonian system for (2.9) is reduced to the nonautonomous Hamiltonian system

$$X'(s) = U(s), \quad U'(s) = F_x(X(s), s), \tag{2.10}$$

which gives the characteristics of (1.1). We focus our attention on the connection between $\mathbb{Z}^2$-periodic entropy solutions of (1.1) and the dynamics of (2.10) .

Jauslin, Kreiss and Moser **[3]** obtained $\mathbb{Z}^2$-periodic solutions of (1.1) by the vanishing viscosity method. They also pointed out several interesting open problems on the forced Burgers equation and the corresponding Hamiltonian dynamics. They considered the parabolic equations with the periodic boundary condition

$$u_t^\nu(x,t) + u^\nu(x,t)u_x^\nu(x,t) = F_x(x,t) + \nu u_{xx}^\nu(x,t) \quad \text{in } \mathbb{T} \times \mathbb{R}_+, \tag{2.11}$$

where $\nu > 0$ is an artificial viscosity. Using the long time behavior of solutions to (2.11), they proved the following: *For each $\nu > 0$ and $C \in \mathbb{R}$ there exists the unique $\mathbb{Z}$-periodic in t solution $\bar{u}^\nu \in C^2$ to (2.11) such that for all $t \in \mathbb{R}$*

$$< \bar{u}^\nu > := \int_0^1 \bar{u}^\nu(x,t)dx = C.$$

Moreover there is a sequence

$$\bar{u}^{\nu_i} \in \{\bar{u}^{\nu} \mid \nu > 0, <\bar{u}^{\nu}>= C\}$$

with $\nu_i \to 0$ *which converges to a* $\mathbb{Z}^2$*-periodic entropy solution* $\bar{u}$ *of (1.1) with* $<\bar{u}>= C$ *in the* $C^0(\mathbb{T}; L^1(\mathbb{T}))$*-topology.*

Takeno **[10]** also obtained $\mathbb{Z}^2$-periodic solutions of (1.1) by the Lax-Friedrichs difference scheme and Brouwer's fixed point theorem. He regarded approximate $\mathbb{Z}^2$-periodic solutions of (1.1) as fixed points of the Poincaré map derived from the Lax-Friedrichs difference scheme and showed existence of these fixed points by Brouwer's fixed point theorem.

E **[11]** made clear the connection between $\mathbb{Z}^2$-periodic solutions of (1.1) and regular motions of the corresponding Hamiltonian system. His results include a *flow version of the Aubry-Mather theory for twist maps.* Let $\bar{u}$ be a $\mathbb{Z}^2$-periodic entropy solution of (1.1) with $<\bar{u}>= C$ and $c(s) := (\tilde{X}(s), s, U(s)) \mod 1$ be characteristic curves derived from the equations (2.10). He proved the following: *Each characteristic curve* $c(s)$ *which is defined on* $(-\infty, \tau]$ *with* $\tau \in [0,1]$ *and satisfies the initial condition*

$$c(\tau) \in graph(\bar{u}) := \{(x, t, \bar{u}(x,t)) \mid (x,t) \in \mathbb{T}^2\}$$

is trapped on $graph(\bar{u})$ *and never absorbed by the shocks of* $\bar{u}$*. The motion of* $c(s)$ *on* $graph(\bar{u})$ *is characterized by the asymptotic slope*

$$\lim_{s\to-\infty} \frac{\tilde{X}(s)}{s} = \alpha(C),$$

where $\alpha(C)$ *depends only on the average* C*. Moreover there exist characteristic curves* $c^*(s) = (\tilde{X}^*(s), s, U^*(s)) \mod 1$ *defined on* $\mathbb{R}$ *with the same asymptotic slope*

$$\lim_{|s|\to\infty} \frac{\tilde{X}^*(s)}{s} = \alpha(C)$$

which are trapped on $graph(\bar{u})$ *and never absorbed by the shocks. For any given* $\alpha \in \mathbb{R}$ *there exists* $C \in \mathbb{R}$ *such that* $\alpha(C) = \alpha$. Note that if $\bar{u}$ is a C^1-function, then any $c(s)$ with an initial condition on $graph(\bar{u})$ is trapped on it for any $s \in \mathbb{R}$. In the original phase space $\mathbb{T}^2 \times D$ with the Hamiltonian (2.9), these results mean that for each $h \in \mathbb{R}$ the set

$$\mathcal{I}_{\bar{u}} := \{(q, \bar{u}(q), h - \frac{1}{2}\bar{u}(q)^2 + F(q)) \mid q \in \mathbb{T}^2\}$$

is ϕ_H^s-backward-invariant. Moreover there exists a ϕ_H^s-invariant closed set $\Gamma^* \subset \mathcal{I}_{\bar{u}}$, on which each trajectory has the frequency vector

$$\lambda = (\alpha(C), 1).$$

Note that if $\bar{u}$ is a C^1-function, then $\mathcal{I}_{\bar{u}}$ is a ϕ_H^s-invariant manifold diffeomorphic to $\mathbb{T}^2$.

3 Main Results

Analytical results. We make use of the *two-step Lax-Friedrichs difference scheme on* $\mathbb{T} \times \mathbb{R}_{\geq 0} \ni (x,t)$: Let N, K be natural numbers. We define the mesh sizes as

$$\Delta x := \frac{1}{N}, \quad \Delta t := \frac{1}{K}, \quad \lambda := \frac{\Delta t}{\Delta x}.$$

We set $x_n := n\Delta x \in [0,1]$ $(n = 0,1,2,\cdots,N)$ and $t_k := k\Delta t \in [0,+\infty)$ $(k = 0,1,2,\cdots)$. The solution to the initial value problem of (1.1)

$$\begin{cases} u_t(x,t) + u(x,t)u_x(x,t) = F_x(x,t) & \text{in } \mathbb{T}\times\mathbb{R}_+, \\ u(x,0) = g(x) & \text{on } \mathbb{T} \end{cases}$$

is replaced with the vectors

$$u^k = (u_0^k, u_1^k, \cdots, u_{N-1}^k) \in \mathbb{R}^N \quad (k = 0,1,2,\cdots)$$

called the difference solution with the initial value

$$u^0 = (g(x_0), \cdots, g(x_{N-1})).$$

Each difference solution u^k with an initial value $u^0 \in \mathbb{R}^N$ is determined in the following way: Let $\Delta y := \frac{1}{2}\Delta x$, $y_m := m\Delta y \in [0,1]$ $(m = 0,1,2,\cdots,2N)$, $\Delta\tau := \frac{1}{2}\Delta t$ and $\tau_l := l\Delta\tau \in [0,+\infty)$ $(l = 0,1,2,\cdots)$. We define

$$u_n^k := W_{2n}^{2k},$$

where W_m^l are computed for $l + m = even$ by the difference equation

$$\begin{cases} \dfrac{W_{m+1}^{l+1} - \frac{(W_{m+2}^l + W_m^l)}{2}}{\Delta\tau} + \dfrac{1}{2}\dfrac{(W_{m+2}^l)^2 - (W_m^l)^2}{2\Delta y} = \dfrac{F(\tau_l, y_{m+2}) - F(\tau_l, y_m)}{2\Delta y}, \\ W_{2N\pm m}^l = W_{\pm m}^l, \\ W_{2n}^0 = u_n^0. \end{cases}$$

We put $u_{N\pm n}^k = u_{\pm n}^k$. The maps: $u^0 \mapsto u^k$ and $W^0 \mapsto W^l$ are denoted by

$$\psi^k : \mathbb{R}^N \to \mathbb{R}^N, \qquad \Psi^l : \mathbb{R}^N \to \mathbb{R}^N$$

respectively. Since F is $\mathbb{Z}^2$-periodic, we have the Poincaré map (or the time-1 map)

$$\phi := \psi^K = \Psi^{2K}.$$

Note that ψ^k, Ψ^l, ϕ are C^2 and $\psi^{KT+k} = \psi^k \circ \phi^T$ for each $T \in \mathbb{N}$, where ϕ^T is the T-iteration of ϕ. We call the following step function an approximate solution of (1.1):

$$u_\Delta(x,t) := u_n^k \quad \text{for } x \in [x_n, x_{n+1}),\ t \in [t_k, t_{k+1}),\ \Delta = (\Delta x, \Delta t).$$

It follows from a simple calculation that the average in x of each difference solution u^k at each k and therefore that of the approximate solution $u_\Delta(x,t)$ is conservative, namely

$$C(u^0) := \sum_{n=0}^{N-1} u_n^0 \Delta x \equiv \sum_{n=0}^{N-1} u_n^k \Delta x \equiv \int_0^1 u_\Delta(x,t)dx.$$

The value $C = C(u^0)$ is called the momentum of solution. $u^k(C), u_\Delta^C(x,t)$ denote $u^k, u_\Delta(x,t)$ with the momentum C. We say that u^k is a periodic difference solution, if for all $k = 0,1,2,\cdots$

$$u^{k+K} = u^k,$$

which is equivalent to the relation

$$\phi(u^0) = u^0.$$

For each $v = (v_0, \cdots, v_{N-1}) \in \mathbb{R}^N$ we set

$$\| v \|_\infty := \max_{1 \le n \le N-1} |v_n|, \quad \| v \|_1 := \sum_{n=0}^{N-1} |v_n|, \quad Var.[v] := \sum_{n=0}^{N-1} |v_{n+1} - v_n| \ \ (v_N = v_0).$$

We state analytical results.

Theorem. *Let* $M := \sqrt{\max_{(x,t)\in\mathbb{R}^2} F_{xx}(x,t)}$, $r > 0$, $\tilde{r} \ge M$ *and*

$$B_{r,\tilde{r}} := \left\{ v \in \mathbb{R}^N \mid -r \le \sum_{n=0}^{N-1} v_n \Delta x \le r, \max_{0 \le n \le N-1} \frac{v_{n+1} - v_n}{\Delta x} \le \tilde{r} \ \ (v_N = v_0) \right\}.$$

Initial values u^0 *are restricted to* $B_{r,\tilde{r}}$. *Fix arbitrarily* $\Delta x = \frac{1}{N}, \Delta t = \frac{1}{K}$ *so that*

$$0 < \lambda_0 \le \frac{\Delta t}{\Delta x} = \lambda < \left(r + \tilde{r}\right)^{-1}, \ \ \tilde{r} < K, \ \ \Delta t \le \Delta x \tag{3.1}$$

for some constant λ_0. *Then*

1. For each $u^0 \in B_{r,\tilde{r}}$, *there exists the unique difference solution* $u^k = \psi^k(u^0)$, *which satisfies for any* k

$$\max_{0 \le n \le N-1} \frac{u_{n+1}^k - u_n^k}{\Delta x} \le \tilde{r}, \quad \| u^k \|_\infty \le |C(u^0)| + \tilde{r}, \quad Var.[u^k] \le 2\tilde{r}.$$

2. For each $C \in [-r, r]$, *there exists the unique periodic difference solution* $\bar{u}^k(C)$ *with the momentum* C, *which satisfies for any* k

$$\max_{0 \le n \le N-1} \frac{\bar{u}_{n+1}^k(C) - \bar{u}_n^k(C)}{\Delta x} \le M, \quad \| \bar{u}^k(C) \|_\infty \le |C| + M, \quad Var.[\bar{u}^k(C)] \le 2M.$$

3. The stability of $\bar{u}^k(C)$: *For any other difference solution* $u^k(C)$ *with the momentum* C, *we have* $\| u^k(C) - \bar{u}^k(C) \|_1 \to 0 \quad (k \to \infty)$.

4. The asymptotic behavior: For any two difference solutions $u^k(C), v^k(C)$ *with the same momentum* C, *we have* $\| u^k(C) - v^k(C) \|_1 \to 0 \quad (k \to \infty)$.

5. The decay rate of the asymptotic behavior: There exist constants $a > 0$ *and* $\rho < 1$ *depending on* Δx *such that for any two difference solutions* $u^k(C), v^k(C)$ *with the same momentum* C *and* $T \in \mathbb{N}$, *we have* $\| u^{TK} - v^{TK} \|_1 = \| \phi^T(u^0) - \phi^T(v^0) \|_1 \le a\rho^T$.

6. Newton's method is applicable to the equation $\phi(u) = u$.

7. There exists a sequence $\bar{u}_{\Delta_i}^C \in \{\bar{u}_\Delta^C(t,x) \mid \Delta x > 0, \Delta t > 0, \ (3.1)\}$ *with* $\Delta_i \to 0$ *as* $i \to \infty$ *which converges in the* $C^0(\mathbb{T}; L^1(\mathbb{T}))$*-topology to a* $\mathbb{Z}^2$*-periodic entropy solution* $\bar{u}^C$ *of (1.1) having the momentum* C. *(The* $\mathbb{Z}^2$*-periodic entropy solution of (1.1) having the momentum* C *is not unique in general.)*

Idea for proof of Theorem. Basically we follow the same way as Oleinik's in [8], where the Δ-independent one-sided estimate for

$$\frac{u_{n+1}^k - u_n^k}{\Delta x}$$

is established and then the argument on the functions of bounded variation is used. However we need some modifications, since we deal with the long time behavior of our difference scheme in $\mathbb{T} \times \mathbb{R}_{\geq 0}$ with the fixed mesh $\Delta = (\Delta x, \Delta t)$, namely we consider the limit $t_k \to \infty$ with the fixed mesh Δ at first and then take the limit $\Delta \to 0$. The above difference scheme has the *numerical viscosity.* This causes, like the artificial viscosity in the parabolic equation (2.11), the $\| \cdot \|_1$-contraction for the difference scheme.

Numerical results. We simulate $\mathbb{Z}^2$-periodic solutions $\bar{u}$ of (1.1) and characteristic curves $c(s)$ derived from (2.10). We use the long time behavior of the two-step Lax-Friedrichs difference scheme for the computation of $\bar{u}$ and the Runge-Kutta method for $c(s)$. Note that Newton's method is also available for the computation, since we can calculate the derivative $D\phi(u^0)$ through the linearized difference equation along u^k. We take the following function as an example of the forcing term:

$$F(x,t) = -\frac{1}{10}\cos(4\pi x)\sin(2\pi t).$$

The following figures show the intersections of $graph(\bar{u}) = \{(x, t, \bar{u}(x,t)) \mid (x,t) \in \mathbb{T}^2\}$ or curves $c(s) = (\tilde{X}(s), s, U(s)) \bmod 1$ (of course approximate ones) onto the Poincaré section: $Y = 0$ in the three-dimensional space $\mathbb{T}^2 \times \mathbb{R} \ni (X, Y, Z)$, where (x,t), $(\tilde{X}(s), s) \bmod 1$ correspond to (X, Y) and $\bar{u}(x,t)$, $U(s)$ to Z.

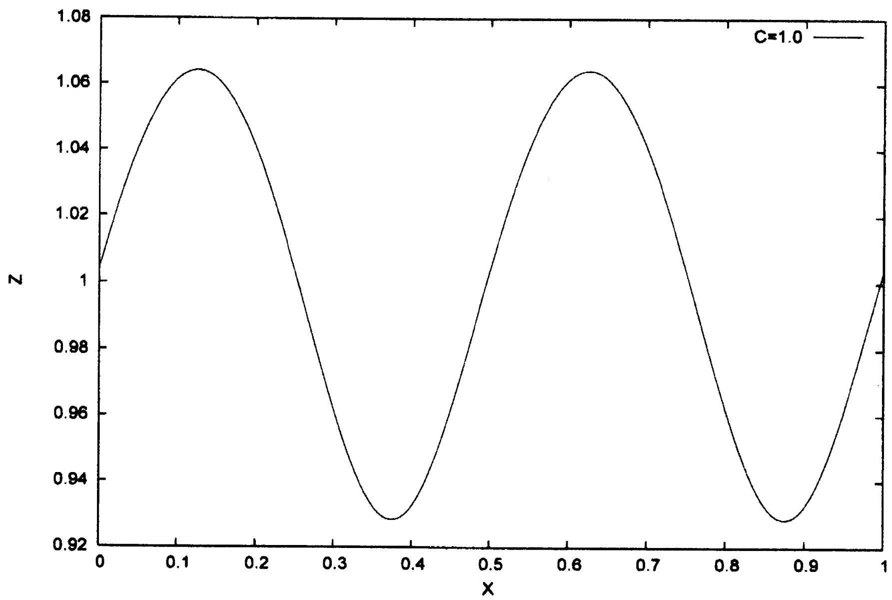

Figure 1.

Figure 1 shows a $\mathbb{Z}^2$-periodic solution $\bar{u}^C$ with the momentum $C = 1.0$. Since $graph(\bar{u})$ seems to be smooth, we expect that any characteristic curve $c(s)$ with the initial condition on $graph(\bar{u})$ is trapped on it forever.

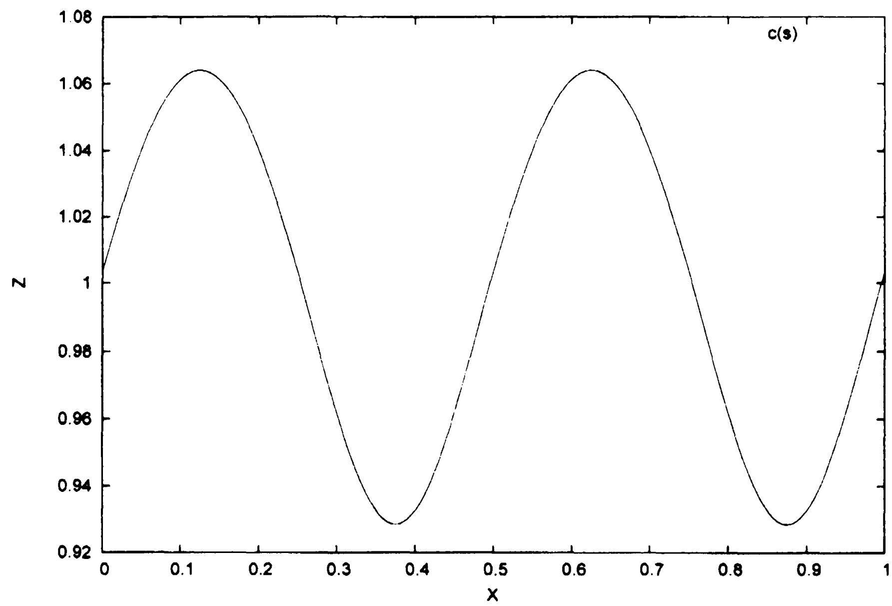

Figure 2.

Figure 2 is formed by a characteristic curve $c(s)$ with an initial condition on the graph in Figure 1. The set formed by $c(s)$ numerically coincides with the graph in Figure 1. This implies that $\bar{u}^C$ is really smooth.

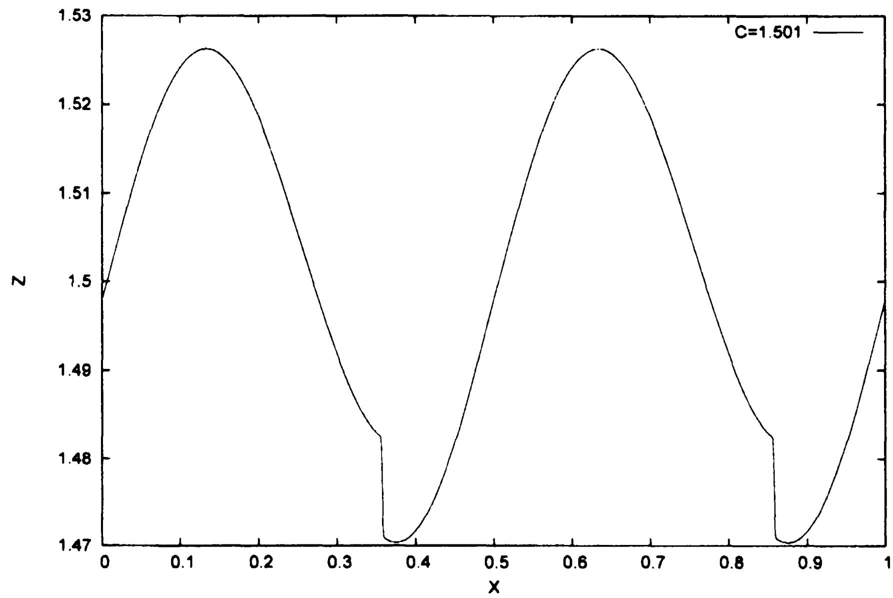

Figure 3.

Figure 3 shows a discontinuous $\mathbb{Z}^2$-periodic solution $\bar{u}^C$ with the momentum $C = 1.501$. We took $N = 60000$ as the number of meshes on x-axis in oder to make the shocks sharpen. The dynamics of characteristic curves around this graph is visualized in Figure 4.

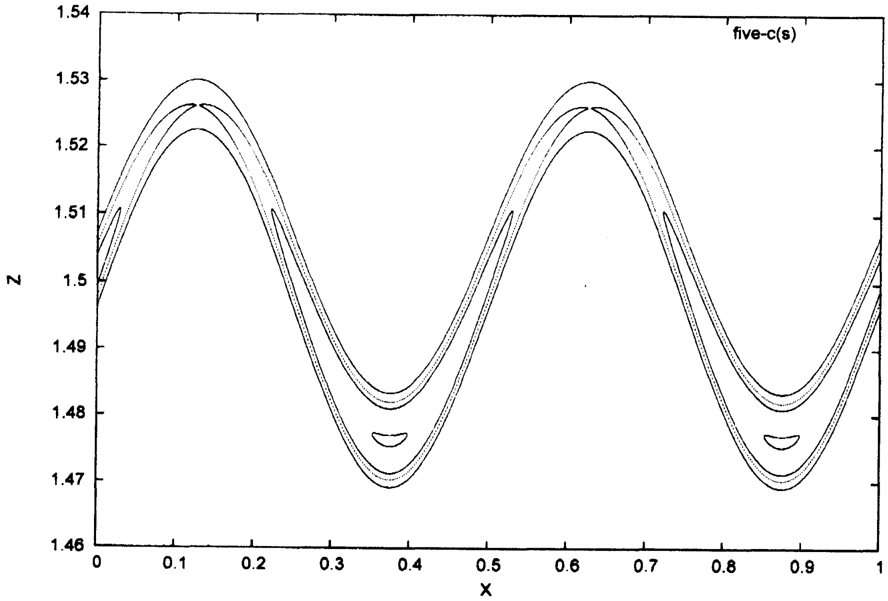

Figure 4.

In Figure 4, we see two characteristic curves forming a curve-like set and in between three characteristic curves forming a pair of islands. The dynamics may have on the Poincaré section a pair of elliptic points with elliptic islands and a pair of hyperbolic points with the stable/unstable curves. We put Figure 3 and 4 together in Figure 5.

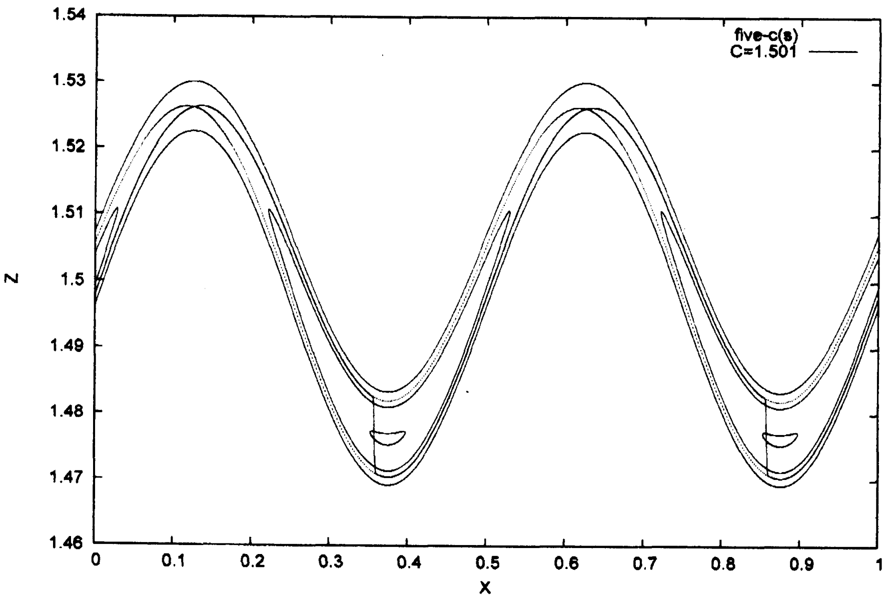

Figure 5.

We can say that Figure 5 indicates the situation where the smooth parts of the discontinuous graph are pieces of the unstable curves and the shocks are across the elliptic islands.

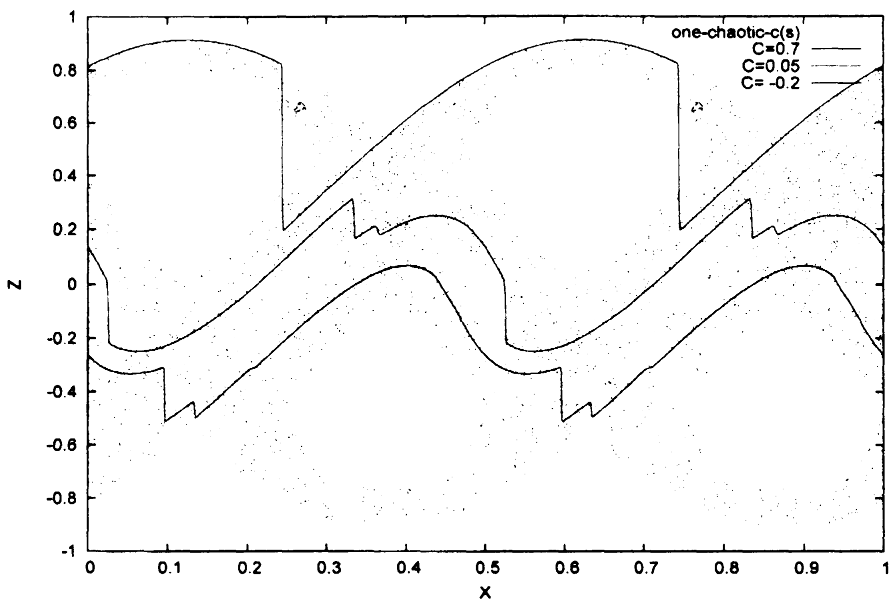

Figure 6.

Figure 6 illustrates discontinuous $\mathbb{Z}^2$-periodic solutions $\bar{u}^C$ with the momentum $C = 0.7,\ 0.05,\ -0.2$. The dynamics around their graph is "chaotic". The scattered dots are formed by a characteristic curve $c(s)$ wandering wide range of the space. The previous relation between the discontinuous graph and the unstable curves is not so clear.

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