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# 行列関数とJENSEN'S INEQUALITY

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## 1. Introduction

In 1980 Hansen showed [7] that if $f$ is a operator convex function on $[0, \infty)$ with $f(0)=0$,

$$f(a^{*}xa) \leq a^{*}f(x)a$$

for every positive operator $x$ and a contraction $a$.

Hansen and Pedersen in 1982 [8] showed that if $f$ is a continuous, real function on $[0, \alpha)$ $(\alpha \leq \infty)$, the followings are equivalent:

(1) $f$ is operator convex and $f(0) \leq 0$,

(2) For an operator $a$ with its spectrum in $[0, \alpha)$ and a contraction $c$,

$$f(c^{\star}ac) \leq c^{\star}f(a)c,$$

(3) For two operators $a, b$ with their spectra in $[0, \alpha)$ and two contractions $c, d$ such that $c^{\star}c+d^{\star}d \leq 1$ we have the inequality

$$f(c^{\star}ac+d^{\star}bd) \leq c^{\star}f(a)c+d^{\star}f(b)d,$$

(4) For an operator $a$ with its spectrum in $[0, \alpha)$ and a projection $p$ we have the inequality,

$$f(pap) \leq pf(a)p$$

(5) The function $g(t)=f(t)/t$ is operator monotone in the open interval $(0, \alpha)$.

We call a function $f$ defined in an interval $I$ *matrix monotone of order* $n$ or $n$-*monotone* in short whenever the inequality $f(a) \leq f(b)$ holds for every pair of self-adjoint matrices $a, b \in M_{n}$ such that $a \leq b$ and all eigenvalues of $a$ and $b$ are contained in $I$.

A function $f$ is said to be *operator monotone* whenever the inequality $f(a) \leq f(b)$ holds for every pair of self-adjoint elements $a, b$ in $B(H)$ for an infinite dimensional Hilbert space.

*Matrix convex (concave) functions* on $I$ are similarly defined as above as well as operator convex (concave) functions.

We denote the spaces of operator monotone functions and of operator convex functions by $P_{\infty}(I)$ and $K_{\infty}(I)$ respectively.

The spaces for $n$-monotone functions and $n$-convex functions are written as $P_{n}(I)$ and $K_{n}(I)$.

We note that $P_{n+1}(I) \subseteq P_{n}(I)$ and $\cap_{n=1}^{\infty}P_{n}(I)=P_{\infty}(I)$.

Similarly, we have $K_{n+1}(I) \subseteq K_{n}(I)$ and $\cap_{n=1}^{\infty}K_{n}(I)=K_{\infty}(I)$.

These notions were introduced and discussed by K. Loewner and his two students O. Dobsch, F. Kraus more than 70 years ago but the piling structure of $P_{n}(I)$ and $K_{n}(I)$ down to $P_{\infty}(I)$ and $K_{\infty}(I)$ are investigated only recently in spite of the great necessity of these notions for many fields such as operator theory, electric networks, quantum mechanics etc.

In this talk we consider the following assertions.

(i) $f(0) \le 0$, and $f$ is n-convex,

(ii) For each positive semi-definite element a with its spectrum in $[0, \alpha)$ and a contraction c in $M_n$, we have

$$f(c^*ac) \le c^*f(a)c,$$

(iii) $g$ $(g(t) = f(t)/t)$ is n-monotone in the interval $(0, \alpha)$.

## 2. Preliminary discussions

The first question is whether
$P_{n+1}(I)$ (resp. $K_{n+1}(I)$) is strictly contained in $P_n(I)$ (resp. $K_n(I)$) for every $n$.
For example,

- $e^t$ is a good monotone increasing functions but it is not 2-monotone.
- $\log t$ is an operator monotone in the interval $(0, \infty)$.
- $t \log t$, $\frac{1}{t}$ is operator convex in $(0, \infty)$.
- (Loewner-Heinz) For $0 \le p \le 1$, the function $t^p$ is operator monotone in $[0, \infty)$.
- For $p > 1$ $t^p$ is not 2-monotone.

Although most of literatures assert the existence of such gaps, no explicit example was given in case $n \ge 3$ in spite of the longtime since the paper [15] of Loewner in 1934.

Here are criteria for $n$-monotone functions/n-convex functions on an open interval $I$. For them we need to introduce the notion of divided difference of order $n$.

For a sufficiently smooth function $f(t)$ we denote its n-th divided difference for n-tuple of points $\{t_1, t_2, \dots, t_n\}$ defined as, when they are all different,

$$[t_1, t_2]_f = \frac{f(t_1) - f(t_2)}{t_1 - t_2}, \text{and inductively}$$

$$[t_1, t_2, \dots, t_n]_f = \frac{[t_1, t_2, \dots, t_{n-1}]_f - [t_2, t_3, \dots, t_n]_f}{t_1 - t_n}.$$

And when some of them coincides such as $t_1 = t_2$ and so on, we put as

$$[t_1, t_1]_f = f'(t_1) \quad \text{and inductively}$$

$$[t_1, t_1, \dots, t_1]_f = \frac{f^{(n)}(t_1)}{n!}.$$

When there appears no confusion we often skip the referring function $f$. We notice here the most important property of divided differences is that it is free from permutations of $\{t_1, t_2, \dots, t_n\}$ in $I$.

**Proposition 1.** (1) • Monotonicity(Loewner 1934)

$$f \in P_n(I) \Longleftrightarrow ([t_i, t_j]) \ge 0 \text{ for any } \{t_1, t_2, \dots, t_n\}$$

• Convexity (Kraus 1936)

$$f \in K_n(I) \Longleftrightarrow ([t_1, t_i, t_j]) \ge 0 \text{ for any } \{t_1, t_2, \dots, t_n\},$$

where $t_1$ can be replaced by any (fixed) $t_k$.

(2) • Monotonicity (Loewner 1934, Dobsch 1937-Donoghue 1974)
For $f \in C^{2n-1}(I)$

$$f \in P_n(I) \Longleftrightarrow M_n(f;t) = \left( \frac{f^{(i+j-1)}(t)}{(i+j-1)!} \right) \geq 0 \ \forall t \in I$$

• Convexity (Hansen-Tomiyama 2007) For $f \in C^{2n}(I)$

$$f \in K_n(I) \Longrightarrow K_n(f;t) = \left( \frac{f^{(i+j)}(t)}{(i+j)!} \right) \geq 0 \ \forall t \in I.$$

In particular, for $n = 2$ the converse is also true.

Note that to prove the converse implication in the criterion of convexity in (2) in the above proposition we need **the local property theorem for the convexity**, that is, if $f$ is $n$-convex in the intervals $(a,b) and (c,d)$ $(a < c < b < d)$, then $f$ is $n$-convex on $(a,d)$. Now we have only a partial sufficiently such that $K_n(f;t_0)$ is positive, then there exists a neighborhood of $T_0$ on which $f$ is $n$-convex.

In [9] Hansen, Ji and Tomiyama in 2002 presented an explicit example with the gap between $P_{n+1}(I)$ and $P_n(I)$ for every $n$ and an interval $I$.

For example,

$$g_n(t) = t + \frac{1}{3}t^3 + \cdots + \frac{1}{2n-1}t^{2n-1} \in P_n([0,\alpha_n)) \setminus P_{n+1}([0,\alpha_n))$$

for some $\alpha_n > 0$.

More general discussions are treated in [13] by Osaka, Silvestrov and Tomiyama in 2006 about gaps of $\{P_n(I)\}_{n \in \mathbf{N}}$.

**Theorem 2.** [13] Let $f(t) = c + b_0 t + b_1 t^2 + \cdots + b_{2n-2} t^{2n-1} + b_{2n-1} t^{2n} + \ldots$ be a polynomial of degree at least $2n-1$. Then

a) $M_n(f;0) > 0$ if and only if there is a Borel measure $\mu$ on $\mathbb{R}$ with at least $n$ points in the support, and such that

$$b_k = \int_{\mathbb{R}} t^k d\mu < \infty, \qquad (0 \leq k \leq 2n-2).$$

Moreover, in this case there exists $\alpha_n > 0$ such that $f \in P_n([0,\alpha_n[)$.

b) If $\det M_n(f;0) = 0$, and $r$ is the smallest number such that the submatrix $M_{r+1}(f;0)$ is not invertible, then there exists a Borel measure $\mu$ such that

$$b_k = \int_{\mathbb{R}} t^k d\mu < \infty, \qquad (0 \leq k \leq 2r-2),$$

and there exists $\alpha > 0$ such that $f \in P_r([0;\alpha[)$.

**Demonstration:**

Suppose that the measure $\mu$ has at least $n$ points in the support. Take arbitrary $n$ points $t_1, \ldots, t_n$ in the support of $\mu$. Then $\mu(I_i) > 0$ for any family of $n$ non-overlapping open intervals such that $t_i \in I_i$ for $i = 1, \ldots, n$. Choose inside each of these open intervals a closed interval $J_i$ such that $t_i \in J_i \subset I_i$ and hence also $\mu(J_i) > 0$ for $i = 1, \ldots, n$.

Note that $f^{(i+j-1)}(0) = (i+j-1)! \cdot b_{i+j-2}$, for $i,j = 1, 2, \ldots, n$.

For any vector $\vec{c} = (c_1, \dots, c_n) \in \mathbb{C}^n$, the following holds for the quadratic form

$$\begin{aligned}(M_n(f;0)\vec{c} \mid \vec{c}) &= \sum_{i=1}^{n}\sum_{j=1}^{n} b_{i+j-2}c_j\bar{c}_i \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\mathbb{R}} t^{i+j-2}d\mu c_j\bar{c}_i \\ &= \frac{1}{2}\int_{\mathbb{R}}\left|\sum_{i=1}^{n} c_i t^{i-1}\right|^2 d\mu \\ &\geq \frac{1}{2}\left|\sum_{i=1}^{n} c_i \tilde{t_k}^{i-1}\right|^2 \mu(J_k) > 0,\end{aligned}$$

where $\tilde{t_k}$ is the minimum point for the continuous function $|\sum_{i=1}^{n} c_i t^{i-1}|^2$ on the closed interval $J_k$. □

We have now abundant examples of polynomials in $P_n(I)\backslash P_{n+1}(I)$ using the truncated moment problems for Hankel matrices in [3] of Curto and Fialkow.

In [10] Hansen and Tomiyama in 2006 also discussed about gaps of $\{K_n(I)\}_{n\in\mathbf{N}}$, and constructed abundant examples of polynomials in $K_n(I)\backslash K_{n+1}(I)$.

For example,

$$h_n(t) = t + \frac{1}{2}t^2 + \cdots + \frac{1}{2n}t^{2n} \in K_n([0,\beta_n))\backslash K_{n+1}([0,\beta_n))$$

for some $\beta_n > 0$.

## 3. Jensen's type Inequality

Now we return to Hansen-Pedersen's Theorem:

**Theorem 3.** [8] If $f$ is a continuous, real function on $[0,\alpha)$ $(\alpha \leq \infty)$, the followings are equivalent:

(1) $f$ is operator convex and $f(0) \leq 0$,

(2) For an operator $a$ with its spectrum in $[0,\alpha)$ and a contraction $c$,

$$f(c^\star ac) \leq c^\star f(a)c,$$

(3) For two operators $a, b$ with their spectra in $[0,\alpha)$ and two contractions $c, d$ such that $c^\star c + d^\star d \leq 1$ we have the inequality

$$f(c^\star ac + d^\star bd) \leq c^\star f(a)c + d^\star f(b)d,$$

(4) For an operator $a$ with its spectrum in $[0,\alpha)$ and a projection $p$ we have the inequality,

$$f(pap) \leq pf(a)p$$

(5) The function $g(t) = f(t)/t$ is operator monotone in the open interval $(0,\alpha)$.

This theorem is proved in the following way:

$$(1)_{2n} \prec (2)_n \prec (5)_n \prec (4)_n, (2)_{2n} \prec (3)_n \prec (4)_n, \text{and } (4)_{2n} \prec (1)_n.$$

**Demonstration for the implication** $(1)_{2n} \prec (2)_n$:

Let $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $U = \begin{pmatrix} c & (I-cc^*)^{\frac{1}{2}} \\ (I-c^*c)^{\frac{1}{2}} & -c^* \end{pmatrix}$, and $V = \begin{pmatrix} c & -(I-cc^*)^{\frac{1}{2}} \\ (I-c^*c)^{\frac{1}{2}} & -c^* \end{pmatrix}$.

Then we have

$$\begin{aligned}
\begin{pmatrix} f(c^*ac) & 0 \\ 0 & f((I-cc^*)^{\frac{1}{2}}a(I-cc^*)^{\frac{1}{2}}) \end{pmatrix} &= f(\begin{pmatrix} c^*ac & 0 \\ 0 & (I-cc^*)^{\frac{1}{2}}a(I-cc^*)^{\frac{1}{2}} \end{pmatrix}) \\
&= f(\frac{1}{2}U^*AU + \frac{1}{2}V^*AV) \\
&\le \frac{1}{2}f(U^*AU) + \frac{1}{2}f(V^*AV) \\
&= \begin{pmatrix} c^*f(a)c & 0 \\ 0 & (I-cc^*)^{\frac{1}{2}}f(a)(I-cc^*)^{\frac{1}{2}} \end{pmatrix}.
\end{aligned}$$

□

Therefore, those assertions become equivalent when $f$ is operator convex and $g$ is operator monotone by the piling structure, where the assertions (A) and (B) is in a relation $m \prec n$ if (A) holds for the matrix algebra $M_m$ then (B) holds for the matrix algebra $M_n$, and write $(A)_m \prec (B)_n$.

The basic problem for double piling structure is to find the minimum difference of degrees between those gaped assertions. Since however even single piling problems are clarified recently, as we have mentioned above, in spite of a long history of monotone matrix functions and convex matrix functions, little is known for the double piling structure except the result by Mathias ([16]), which asserts that a $2n$-monotone function in the positive half line $[0, \infty)$ becomes $n$-concave.

We consider the following assertions.

(i) $f(0) \le 0$, and $f$ is n-convex,

(ii) For each positive semi-definite element a with its spectrum in $[0, \alpha)$ and a contraction c in $M_n$, we have

$$f(c^*ac) \le c^*f(a)c,$$

(iii) $g$ $(g(t) = f(t)/t)$ is n-monotone in the interval $(0, \alpha)$.

**Theorem 4.** [14] For each $n \in \mathbf{N}$ the assertions (ii) and (iii) are equivalent.

**Theorem 5.** [14] Let $n \ge 2$. The assertion (i) implies that $g(t)$ is $(n-1)$-monotone in $(0, \alpha)$.

*Proof.* We may assume that $f(t)$ is twice continuously differentiable. Now take a point $s$ in $[0, \alpha)$ and fix. We consider the function $h_s(t) = [t, s]_f$, then for two points $\{r_1, r_2\}$ we have

$$[r_1, r_2, s]_f = [r_1, s, r_2]_f = [r_1, r_2]_{h_s}$$

Let $\{t_1, t_2, \ldots, t_{n-1}\}$ be an arbitrary $n-1$ tuple of points in the interval $(0, \alpha)$. Then by Proposition 1(1) we see that the matrix $([t_i, t_j, s])$ is positive semidefinite for $n$-points $\{t_1.t_2, \ldots, t_{n-1}, s\}$. Hence its submatrix $([t_i, t_j]_{h_s})$ is positive semidefinite, which means by Proposition 1(1) that the function $h_s(t)$ is a monotone function of degree $n-1$. Thus, in particular, $h_0(t) = \frac{f(t)-f(0)}{t}$ becomes $n-1$-monotone. From asimple observation we have conclusion. □

**Remark 6.** In connection with this theorem it would be important to note that for a finite interval we can never get the result of Mathias' type mentioned before. In fact, in such an interval for any $2n$ we can always find a $2n$-monotone and $2n$-convex polynomial $f(t)$ by [10, Proposition 1.3]. Therefore, if $f(t)$ become $n$-concave it had to be a constant.

Now whether there exists an exact gap from (i) to (iii) we confirm first the following observation. Though it is almost trivial, we state it as a proposition for completeness sake of our arguments.

**Proposition 7.** For n = 1, the assertion (i) implies (iii) but the converse does not hold.

**Problem 8.** Is it true that $f$ is $n$-convex with $f(0) \leq 0$ implies that $g(t) = f(t)/t$ is $n$-monotone ?

## 4. Interpolation class and matrix monotone functions

An interpolation function $h$ relative to a positive operator $A$ in a Hilbert space $H$ is a positive continuous function defined on the spectrum of $A$ fulfilling the condition

$$||h(A)^{\frac{1}{2}}Th(A)^{-\frac{1}{2}}|| \leq \max(||T||, ||A^{\frac{1}{2}}TA^{-\frac{1}{2}}||)$$

for every bounded operator on $H$. By a theorem of Donoghue [4][5], it is known that the class of interpolation functions relative to $A$ coincides precisely with the class of restriction on $\sigma(A)$ of positive Pick function, that is,

$$h(\lambda) = \int_{[0,\infty]} \frac{(1+t)\lambda}{1+t\lambda} d\rho, \ \lambda > 0$$

where $\rho$ is some positive Radon measure on $[0, \infty]$.

**Definition 9.** [2] Let $I$ be a finite interval (open, closed, or open-closed). For $n \in \mathbf{N}$ we denote $C_n(I)$ be the set of all positive real-valued continuous interpolation functions $f$ over $I$ such that for any $\{\lambda_i\}_{i=1}^n \subset I^\circ$ there is a Pick function $h\colon I^\circ \to \mathbf{R}$ such that $f(\lambda_i) = h(\lambda_i)$ for $1 \leq i \leq n$, where $I^\circ$ denotes the set of inner points in $I$.

For two finite intervals of the same type such as an open, half-open like $[\alpha, \beta)$ and $[\gamma, \delta)$ one can easily find an monotone increasing linear function $h\colon [\gamma, \delta) \to [\gamma, \delta)$ with the inverse function $h^{-1}\colon [\gamma, \delta) \to [\alpha, \beta)$ having the same property. As both functions $h$ and $h^{-1}$ are operator monotone and operator convex functions the set $C_n([\alpha, \beta))$ and $C_n([\gamma, \delta))$ is easily transferred each other. So we consider the case that $I = [0, 1]$.
Therefore, we consider the class $C_n([0, 1])$ of real valued continuous interpolations functions $f$ on $[0, 1]$ such that for any $\{\lambda_i\}_{i=1}^n \subset (0, 1)$ there is a Pick function $h\colon (0, 1) \to \mathbf{R}$ such that $f(\lambda_i) = h(\lambda_i)$ for $1 \leq i \leq n$, which was studied in the case of $[0, \infty)$ by [1] and [2].

The following is the characterization of a class $C_n([0, 1])$

**Theorem 10.** Let $f\colon [0, 1] \to \mathbf{R}$ be a continuous function. The followings are equivalent.
(1) $f \in C_n([0, 1])$.

(2) For any $\{\lambda_i\}_{i=1}^n \subset (0,1)$ if

$$\sum_{i=1}^n a_i \frac{(1+t)\lambda_i}{1+(t-1)\lambda_i} \geq 0$$

for any $\{a_i\}_{i=1}^n \subset \mathbf{R}$ we have

$$\sum_{i=1}^n a_i f(\lambda_i) \geq 0.$$

(3) For any $A, T \in M_n(\mathbf{C})$ with $T^*T \leq 1$ and $\sigma(A) \subset (0,1)$

$$T^*AT \leq A \Longrightarrow T^*f(A)T \leq f(A).$$

Note that by [2, Theorem 3.1] for any $n \in \mathbf{N}$

$$P_{n+1}^+([0,\infty)) \subseteq C_{2n+1}([0,\infty)) \subseteq C_{2n}([0,\infty)) \subseteq P_n^+([0,\infty)).$$

The following is a partial answer to [2, conjecture].

**Proposition 11.** For each $n \in \mathbf{N}$ $C_{2n}([0,\infty) \subsetneq P_n^+([0,\infty))$.

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