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METRIC CONVEXITY OF #-SYMMETRIC CONES

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ABSTRACT. In this paper we introduce a general notion of a symmetric cone, valid for the finite and infinite dimensional case, and prove that one can deduce the seminegative curvature of the Thompson part metric in this general setting: the distance function between points evolving in time on two geodesics is a convex function.

1. SYMMETRIC SETS WITH MIDPOINTS

A *#-symmetric set* consists of a binary system $(X, \bullet)$, with left translation $S_x y := x \bullet y$ representing the point symmetry through x, satisfying for all $a, b, c \in X$:

(S1) $a \bullet a = a$ $(S_a a = a)$;

(S2) $a \bullet (a \bullet b) = b$ $(S_a S_a = \mathrm{id}_X)$;

(S3) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c)$ $(S_a S_b = S_{S_a b} S_a)$;

(S4) the equation $x \bullet a = b$ $(S_x a = b)$ has a unique solution $x \in X$, called the *midpoint* or *mean* of a and b, and denoted by $a\#b$.

The axioms bear close resemblance to the Loos axioms for a symmetric space. A binary system $(X, \bullet)$ satisfying (S1), (S2), and (S3) also satisfies (S4) if and only if it is a quasigroup. Thus the preceding structures are also referred to as *symmetric quasigroups*. Systems satisfying only Axioms (1)-(3) are called *symmetric sets* (or *involutive quandles* in knot theory circles)

A pointed #-symmetric set is a triple $(X, \bullet, \varepsilon)$, where $(X, \bullet)$ is a #-symmetric set and $\varepsilon \in X$ is some distinguished point, called the base point. In this setting we define

$$x^0 = \varepsilon, \quad x^{-1} := S_\varepsilon x, \quad x^2 := S_x \varepsilon, \quad x^{1/2} := \varepsilon \# x$$

and inductively from these definitions all dyadic powers are defined so that the following rules are satisfied:

$$(x^r)^s = x^{rs}, \quad x^r \# x^s = x^{\frac{(r+s)}{2}}.$$

If we consider the dyadic rationals $\mathbb{D}$ endowed with the $\#$-symmetric structure $a \bullet b = 2a - b$ (the reflection of b through a), then $a\#b = (a+b)/2$, the usual midpoint, and the map $t \mapsto x^t : \mathbb{D} \to X$ is both a $\bullet$-homomorphism and $\#$-homomorphism. From this fact the preceding rules (and others) easily follow.

The displacement group $G(X)$ (also called the transvection group) of a $\#$-symmetric set X is the group generated under the composition by all transformations of the form $S_x S_y$, $x, y \in X$. If X is pointed with base point ε, then $G(X)$ is generated by all $S_x S_\varepsilon$ and X embeds into $G(X)$ as a twisted subgroup (closed under $g \bullet h = gh^{-1}g$) via the *quadratic representation* $Q : X \to G(X)$ defined by $Q(x) = S_x S_\varepsilon$. The image $Q(X)$ is a pointed $\#$-symmetric set under the preceding $\bullet$-operation and the quadratic representation is an isomorphism between X and $Q(X)$. In particular, $Q(X)$ is uniquely 2-divisible and $Q(x\#y) = Q(x)\#Q(y), Q(x^{1/2}) = Q(x)^{1/2}$ ([1, 2]). For $x, y \in X$, we write interchangeably as convenient

$$x.y = Q(x)y = Q(x)(y).$$

Example 1.1. *Let $\mathbb{R}$ be equipped with the standard $\#$-symmetric operation $x \bullet y := 2x - y$ and the usual metric. Then $x\#y = (x+y)/2$, the usual midpoint operation, and the metric is convex. Thus $(\mathbb{R}, \bullet, 0)$ is a pointed symmetric space with convex metric.*

Definition 1.2. *A* pointed symmetric space with convex metric *is a pointed $\#$-symmetric set P equipped with a complete metric $d(\cdot,\cdot)$ satisfying for all $x, y \in P$ and $g \in G(P)$*

(i) $d(g.x, g.y) = d(x, y)$,
(ii) $d(x^{-1}, y^{-1}) = d(x, y)$,
(iii) $d(x^{1/2}, y^{1/2}) \leq \frac{1}{2} d(x, y)$,
(iv) $x \mapsto x^2 : P \to P$ *is continuous.*

A symmetric space with convex metric *is a $\#$-symmetric set equipped with a complete metric that is a pointed symmetric space with convex metric with respect to some pointing.*

We recall some basic results about symmetric spaces with convex metrics from [3].

Theorem 1.3 ([3]). *Let P be a symmetric space with convex metric. Then for distinct $x, y \in P$, there exists a unique continuous homomorphism $\alpha_{x,y}$ (called an $\mathfrak{s}$-geodesic) of $\#$-symmetric sets from $\mathbb{R}$ into P satisfying $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$. Furthermore, the maps*

$$(x, y) \mapsto x \bullet y : P \times P \to P, \quad (t, x, y) \mapsto \alpha_{x,y}(t) := x\#_t y : \mathbb{R} \times P \times P \to P$$

are continuous.

The element $x\#_t y$ is called the *t-weighted mean* of x and y. Note that $x\#y = x\#_{1/2}y$.

Theorem 1.4 ([3]). *Let P be a symmetric space with convex metric. For every pair (β, γ) of $\mathfrak{s}$-geodesics, the real function $t \mapsto d(\beta(t), \gamma(t))$ is a convex function.*

Remark 1.5. *We note that the unique $\mathfrak{s}$-geodesic line satisfying $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$ is*

$$\alpha_{x,y}(t) = x^{1/2}.(x^{-1/2}.y)^t$$

and $\alpha_{y,x}(1-t) = \alpha_{x,y}(t), t \in \mathbb{R}$ ([3]). In particular,

$$(Q(y)x)^t = Q(y)Q(x^{1/2})(Q(x^{1/2})y^2)^{t-1}. \tag{1.1}$$

2. #-SYMMETRIC CONES

Let V be a real Banach space and let Ω henceforth denote a non-empty open convex cone of V: $t\Omega \subset \Omega$ for all $t > 0$, $\Omega + \Omega \subset \Omega$, and $\overline{\Omega} \cap -\overline{\Omega} = \{0\}$, where $\overline{\Omega}$ denotes the closure of Ω. We consider the partial order on V defined by

$$x \leq y \text{ if and only if } y - x \in \overline{\Omega}.$$

We further assume that Ω is a *normal* cone: there exists a constant K with $||x|| \leq K||y||$ for all $x, y \in \Omega$ with $x \leq y$. Any member a of Ω is an order unit for the ordered space $(V, \leq)$, and hence $|x|_a := \inf\{\lambda > 0 : -\lambda a \leq x \leq \lambda a\}$ defines a norm. By Proposition 1.1 in [6], for a normal cone $\overline{\Omega}$, the order unit norm $|\cdot|_a$ is equivalent to $||\cdot||$.

A. C. Thompson [7] (cf. [5], [6]) has proved that Ω is a complete metric space with respect to the *Thompson part metric* defined by

$$d(x,y) = \max\{\log M(x/y), \log M(y/x)\}$$

where $M(x/y) := \inf\{\lambda > 0 : x \leq \lambda y\} = |x|_y$. Furthermore, the topology induced by the Thompson metric agrees with the relative Banach space topology.

Theorem 2.1 ([4]). *Let Ω be an open convex normal cone in a Banach space V. Suppose that there is a pointed $\#$-symmetric structure on Ω satisfying*

(i) $2x \leq \varepsilon + x^2$
(ii) *the squaring map $x \mapsto x^2 = Q(x)\varepsilon$ is continuous,*
(iii) *every basic displacement $Q(x)$ is continuous and linear on Ω.*

Then Ω is a pointed symmetric space with convex metric, the Thompson metric whose metric topology agrees with the relative topology.

A JB-algebra V is a Jordan algebra with unit e endowed with a complete norm $||\cdot||$ such that

$$\begin{aligned} ||zw|| &\leq ||z||\,||w||, \\ ||z^2|| &= ||z||^2, \\ ||z||^2 &\leq ||z^2 + w^2||. \end{aligned}$$

Example 2.2. (1) *The positive cone of hermitian elements of a C^*-algebra.*

(2) *Spin factors and Lorentz cones: Let $(H, \langle\cdot|\cdot\rangle)$ be a real Hilbert space with* $\dim H \geq 2$ *and let $V = \mathbb{R} \times H$ equipped with the Banach space norm $||(t,x)|| = |t| + \sqrt{\langle x|x\rangle}$. We define the Jordan product on V by*

$$(s,y) \circ (t,x) = (st + \langle y|x\rangle, sx + ty).$$

The element $e = (1,0) \in V$ acts as a unit element. The corresponding symmetric cone is given by (the Lorentz cone, forward light cone)

$$\Omega = \{(t,x) \in V : t > ||x|| = \sqrt{\langle x|x\rangle}\}.$$

For $x \in V$ we write $L(x)(y) = xy$, the multiplication operator. We consider the set

$$\Omega := \{x \in V : \mathrm{Spec}(L(x)) \subset (0,\infty)\}.$$

Then Ω is an open convex cone of V and is realized as $\Omega = \exp(V) := \{\exp(x) : x \in V\}$.

The Banach algebra norm agrees with the order unit norm $|x|_\varepsilon := \inf\{t > 0 : t\varepsilon \pm x \geq 0\}$, or equivalently Ω is a normal cone. The quadratic representation of the Jordan algebra is defined by $P(z) = 2L(z)^2 - L(z^2)$. It is well-known that for each $z \in \Omega$, $P(z) \in G(\Omega)$ the linear automorphism group of Ω. In fact, there is a polar decomposition $G(\Omega) = P(\Omega)\mathrm{Aut}(V)$ where $\mathrm{Aut}(V)$ denotes the Jordan automorphism group of V. We further note that $\mathrm{Aut}(V) = \{g \in G(\Omega) : g(e) = e\}$. The basic properties

$$P(z)z^{-1} = z,\ P(z)^{-1} = P(z^{-1}),\ P(P(z)w) = P(z)P(w)P(z)$$

yield a pointed symmetric set structure $x \bullet y = P(x)y^{-1}$ with $e = \varepsilon$ as base point on the set of invertible elements, in particular on the cone Ω. In symmetric set notation, $P(a) = Q(a)$ and the symmetric set inverse $a^{-1} := e \bullet a$ agrees with the Jordan inverse of a.

Next, we show that the pointed symmetric space $(\Omega, \varepsilon = e)$ is #-symmetric. Let $x, y \in \Omega$ such that $x^2 = y^2$. Then by the commutativity of Jordan products, $0 = x^2 + y^2 = L(x+y)(x-y)$. Since $L(z)$ is invertible for all $z \in \Omega$, $x - y = 0$. This implies that each element of Ω has a unique square root. Note that if $a = \exp(x), x \in V$ then $a^{1/2} = \exp(\frac{1}{2}x)$. Moreover, if $a, b \in \Omega$ then the quadratic equation $P(x)a^{-1} = b$ has a unique solution in Ω. Note that $x = P(a^{1/2})(P(a^{-1/2})b)^{1/2} \in \Omega$ solves the equation. Suppose that x and y are solutions in Ω. Then

$$\begin{aligned}
(P(a^{-1/2})x)^2 &= P(P(a^{-1/2})x)\varepsilon = P(a^{-1/2})P(x)P(a^{-1/2})\varepsilon \\
&= P(a^{-1/2})(P(x)a^{-1}) \\
&= P(a^{-1/2})b = P(a^{-1/2})(P(y)a^{-1}) \\
&= P(a^{-1/2})P(y)P(a^{-1/2})\varepsilon \\
&= (P(a^{-1/2})y)^2
\end{aligned}$$

and hence $P(a^{-1/2})x = P(a^{-1/2})y$, so $x = y$. We conclude that the open convex cone Ω is a #-symmetric set under the operation $x \bullet y = P(x)y^{-1}$. In this case the dyadic power a^t of $a = \exp(x)$ agrees with $\exp(tx)$ and the geometric mean $a\#b$ of a and b is

$$a\#b = P(a^{1/2})(P(a^{-1/2})b)^{1/2}.$$

Corollary 2.3. *Let V be a JB-algebra and let Ω be the associated symmetric cone. Then Ω is a symmetric space with convex metric with respect to the Thompson metric. In particular, the distance function between points evolving in time on two geodesics is a convex function.*

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