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ON RANK-ONE PERTURBATIONS OF DIAGONAL OPERATORS AND INVARIANT SUBSPACES

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Abstract

This note is concerned with operators on Hilbert space of the form $T = D + u \otimes v$, where D is a diagonalizable normal operator and $u \otimes v$ is a rank-one operator. We discuss point spectra of such operator T and also an open problem: does every rank-one perturbation $T = D + u \otimes v$ have a nontrivial hyperinvariant subspace?

1. INTRODUCTION

This is based on the joint work with C. Foias, E. Ko, and C. Pearcy ([3], [4]) and was presented at the 2008 RIMS conference which was held at Kyoto University on Oct. 18-19, 2008. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we write $\{T\}'$ for the commutant of T (i.e., for the algebra of all $S \in \mathcal{L}(\mathcal{H})$ such that $TS = ST$) and $\{T\}'' = (\{T\}')'$ for the double commutant of T. We choose an ordered orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$. If $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ is any bounded sequence in $\mathbb{C}$, we write D_Λ for the normal operator in $\mathcal{L}(\mathcal{H})$ determined by the equations

$$D_\Lambda(e_n) = \lambda_n e_n, \qquad n \in \mathbb{N}. \tag{1.1}$$

This notation for $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ and D_Λ will also remain fixed throughout, as well the notation Λ' the derived set of Λ. By definition, we shall say that an operator T in $\mathcal{L}(\mathcal{H})$ is a *rank-one perturbation of a diagonal normal operator* if there exist nonzero vectors

$$u = \sum_{n\in\mathbb{N}} \alpha_n e_n \quad \text{and} \quad v = \sum_{n\in\mathbb{N}} \beta_n e_n \tag{1.2}$$

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in $\mathcal{H}$ and a bounded sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that T is unitarily equivalent to the operator $D_\Lambda + u\otimes v$, where, as usual, $u\otimes v$ is the operator of rank one defined by

$$(u\otimes v)(x) = \langle x, v\rangle\, u,\ \ x\in\mathcal{H}.$$

In this note, we discuss the following problem:

Problem 1.1. Does every rank-one perturbation $T = D_\Lambda + u\otimes v \in \mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ of a diagonal normal operator D_Λ have a nontrivial invariant subspace (n.i.s.), or better yet, a nontrivial hyperinvariant subspace (n.h.s.)?

which is one of the most annoying unsolved problems in operator theory (on Hilbert space) for (at least) 30 years duration ([7]). And it is discussed in section 3 that if $T \notin \mathbb{C}1$ and the vectors u and v have Fourier coefficients $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ with respect to an orthonormal basis that diagonalizes D that satisfy

$$\sum_{n=1}^{\infty}(|\alpha_n|^{2/3} + |\beta_n|^{2/3}) < \infty,$$

then T has a nontrivial hyperinvariant subspace.

2. POINT SPECTRA

The ideal of compact operators in $\mathcal{L}(\mathcal{H})$ will be denoted by $\mathbf{K}$ and the Calkin map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbf{K}$ by π. For T in $\mathcal{L}(\mathcal{H})$ we denote by $\sigma(T)$ the spectrum of T, by $\sigma_{le}(T)$ $[\sigma_{re}(T)]$ the left essential [right essential] spectrum of T, and

$$\sigma_e(T) = \sigma(\pi(T)) = \sigma_{le}(T)\cup\sigma_{re}(T), \qquad \sigma_{lre}(T) = \sigma_{le}(T)\cap\sigma_{re}(T).$$

Moreover, we write, as usual, $\sigma_p(T)$ for the point spectrum of T. We first take note of some cases treated in [6].

Proposition 2.1 ([6]). *If $T = D_\Lambda + u\otimes v \in \mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ and there exists $n_0\in\mathbb{N}$ such that $\alpha_{n_0}\beta_{n_0} = 0$, then either $\lambda_{n_0}\in\sigma_p(T)$ or $\bar{\lambda}_{n_0}\in\sigma_p(T^*)$. Moreover, if there exist $m_0, n_0\in\mathbb{N}$ with $m_0\neq n_0$ such that $\lambda_{m_0} = \lambda_{n_0}$, then $\lambda_{n_0}\in\sigma_p(T)$. Finally, if the derived set Λ' of Λ is a singleton, then $\{T\}'$ contains a nonzero compact operator. Consequently, in all cases T has a n.h.s.*

Thus in what follows we restrict our attention to the class $(\mathcal{RO})$ consisting of all operators $T = D_\Lambda + u\otimes v$ in $\mathcal{L}(\mathcal{H})$ for which all coefficients α_n and β_n are nonzero, $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ is a one-to-one map of $\mathbb{N}$ into $\mathbb{C}$, and Λ' is not a singleton. We remark that it follows easily that if $T_1 = D_{\Lambda_1} + u_1\otimes v_1$ and $T_2 = D_{\Lambda_2} + u_2\otimes v_2$ belong to $(\mathcal{RO})$ with $T_1 = T_2$, then the sequences Λ_1 and Λ_2 coincide and $u_1\otimes v_1 = u_2\otimes v_2$ ([6, Prop. 1.1]). It is also clear that for all $T = D_\Lambda + u\otimes v\in(\mathcal{RO})$, we have $\sigma_e(T) = \sigma_{lre}(T) = \sigma_{lre}(D_\Lambda) = \Lambda'$.

The following proposition gives very useful necessary and sufficient conditions that a number $\lambda\in\mathbb{C}$ belong to $\sigma_p(T)$.

Proposition 2.2 ([6]). *Let $T = D_\Lambda + u\otimes v\in(\mathcal{RO})$. Then a point $\mu\in\mathbb{C}$ is an eigenvalue of T if and only if*

a) $\mu \notin \Lambda$,

b) $\sum_{n\in\mathbb{N}} \frac{|\alpha_n|^2}{|\mu-\lambda_n|^2} < +\infty$ (*which implies by the Schwarz inequality that* $\sum_{n\in\mathbb{N}} \frac{|\alpha_n\bar{\beta}_n|}{|\mu-\lambda_n|} < +\infty$), *and*

c) $f_T(\mu) := \sum_{n\in\mathbb{N}} \frac{\alpha_n\bar{\beta}_n}{\mu-\lambda_n} = +1$.

Moreover, if $\mu \in \sigma_p(T)$ [*resp.*, $\bar{\mu} \in \sigma_p(T^*)$], *then the eigenspace associated with* μ [*resp.* $\bar{\mu}$] *is spanned by the single vector* $\sum_{n\in\mathbb{N}}(\frac{\alpha_n}{\mu-\lambda_n})e_n$ [*resp.*, $\sum_{n\in\mathbb{N}}(\frac{\beta_n}{\bar{\mu}-\bar{\lambda}_n})e_n$], *and so is one-dimensional. Finally,* $(\Lambda\backslash\Lambda') \cap \sigma(T) = \varnothing$ (*i.e., all isolated points* λ_n *of the set* Λ *lie outside of* $\sigma(T)$).

We observe that the last statement of Proposition 2.2 can be proved in two lines by noting that if λ_n is isolated in Λ, then $(D_\Lambda - \lambda_n)$ (and thus $(T - \lambda_n)$) is a Fredholm operator of index zero, and hence necessarily either $\lambda_n \in \sigma_p(T)$ or $\lambda_n \in \mathbb{C}\backslash\sigma(T)$.

One might expect that an arbitrary T in $(\mathcal{RO})$ would satisfy $\sigma_p(T) \cup \sigma_p(T^*) \neq \varnothing$ (and thus trivially have a n.h.s.), but that this is false has been known (in the case $D_\Lambda = D_\Lambda^*$) for at least fifty years (cf., e.g., [1]).

Example 2.3 ([9]). Let $\{D_n\}_{n\in\mathbb{N}}$ be the (non-tangential) disjoint open disks centered at λ_n and has radius r_n with $D_n \subset \mathbb{D}$ such that $m(\mathbb{D} \setminus \cup_{n\in\mathbb{N}}\bar{D}_n) = 0$ and $\sum_{n\in\mathbb{N}} r_n < \infty$. In fact, $\{D_n\}$ can be constructed using an introduction argument, covering at each step a closed set of whose measure is a fixed nonzero fraction of the measure of the open set uncovered by the disk constructed at previous steps. Now consider $u = \sum r_n e_n$. If $z \notin \bar{\mathbb{D}}$, then $f_T(z) = 1/z$. By Proposition 2.2, obviously z is not eigenvalue of T for any $z \notin \bar{\mathbb{D}}$. For $z \in \bar{\mathbb{D}} \setminus \cup_{n\in\mathbb{N}}\bar{D}_n$, we have $f_T(z) = \bar{z}$, and so such z is eigenvalue of T if and only if $z = 1$. Finally, suppose $z \in \bar{D}_n \setminus \{\lambda_n\}$. Then $f_T(z) = \infty$. Hence by Proposition 2.2, $\sigma_p(T) = \{1\}$.

It looks that the first example of an operator $T \in (\mathcal{RO})$ such that Λ' has positive planar Lebesgue measure and $\sigma_p(T) = \varnothing$ was given by Stampfli [8]. Hence it is important to construct other methods different from finding point spectra to find invariant subspaces for an operator.

3. HYPERINVARIANT SUBSPACES

The following is a partial solution of Problem 1.1 and the results in this section come from [3].

Theorem 3.1. *Let* $T = D_\Lambda + u \otimes v$ *be any rank-one perturbation of a diagonal normal operator such that* $T \notin \mathbb{C}1_{\mathcal{H}}$ *and* $\sum_{n\in\mathbb{N}}(|\alpha_n|^{2/3} + |\beta_n|^{2/3}) < +\infty$. *Then* T *has a n.h.s.*

The following theorem is technical result to represent Theorem 3.1.

Theorem 3.2. *Suppose* $T = D_\Lambda + u \otimes v$ *is such that*

i) *the map* $n \longrightarrow \lambda_n$ *of* $\mathbb{N}$ *onto* Λ *is injective and* Λ' *is not a singleton,*

ii) *for every* $n \in \mathbb{N}$, $\alpha_n\beta_n \neq 0$, *and*

iii) $\sum_{n\in\mathbb{N}}(|\alpha_n|^{2/3} + |\beta_n|^{2/3}) < +\infty$ (*the nontrivial assumption*).

Then either

I) *there exists an idempotent F with $0 \neq F \neq 1_{\mathcal{H}}$ such that $F \in \{T\}''$, and consequently, T has a complemented n.h.s. (i.e., there exist n.h.s. $\mathcal{M}$ and $\mathcal{N}$ of T with $\mathcal{M} \cap \mathcal{N} = (0)$ and $\mathcal{M} + \mathcal{N} = \mathcal{H}$), or*

II) *there exists an uncountable set $\{\mu : \mu \in P\}$ of eigenvalues of T and an associated family $\{u_\mu\}_{\mu \in P}$ of linearly independent eigenvectors (with $Tu_\mu = \mu u_\mu$) such that $\mathcal{M} = \vee_{\mu \in P}\{u_\mu\}$ is a n.h.s. for T and $\mathcal{H} \ominus \mathcal{M}$ is infinite dimensional.*

In [4] it is established that the commutants of such rank-one perturbation operators are abelian, paralleling thereby the properties of the commutants of normal operators of multiplicity one. Also it is shown by example that this behavior does not extend to the commutants of rank-one perturbations of all normal operators of multiplicity one, and we discuss similarity and quasisimilarity questions associated with this class of operators below.

For operators $T = D_\Lambda + u \otimes v \in (\mathcal{RO})$, we now turn to a certain property of the commutant $\{T\}'$ of T.

Theorem 3.3. *Suppose $T = D_\Lambda + u \otimes v \in (\mathcal{RO})$, where $\Lambda = \{\lambda_n\}_{n \in \mathbb{N}}$, D_Λ, and u and v are as in* (1.1) *and* (1.2) *(and the definition of the class $(\mathcal{RO})$). Then the map $\varphi : \{T\}' \to \{T\}'u$ defined by $\varphi(A) = Au$ for $A \in \{T\}'$ is a one-to-one, bounded linear transformation from $\{T\}'$ onto the linear manifold $\{T\}'u$.*

The following is one of main results in [4].

Theorem 3.4. *Suppose $T = D_\Lambda + u \oplus v \in (\mathcal{RO})$, where the notation is as established in* (1.1) *and* (1.2). *Then (the unital,* WOT*-closed algebra) $\{T\}'$ is abelian.*

The following corollary comes from [4], which should be compared with Theorem 3.1.

Proposition 3.5. *Suppose $T = D_\Lambda + u \otimes v \in (\mathcal{RO})$, $0 \in \Lambda' \backslash (\Lambda \cup \sigma_p(T) \cup \sigma_p(T^*))$, and $D_\Lambda^{1/2}$ is any fixed square root of the (normal) operator D_Λ. If*

$$\sum_{n \in \mathbb{N}} (|\alpha_n|\,|\lambda_n^{-1/2}|)^{2/3} < \infty, \quad \text{and} \quad \sum_{n \in \mathbb{N}} (|\beta_n|\,|\lambda_n^{1/2}|)^{2/3} < \infty,$$

then T has a n.h.s.

Remark 3.6. It is worthwhile to study the normality, hyponormality, and weak hyponormality of operators $T = D_\Lambda + u \otimes v$. In [8], the characterization for rank-one perturbation of isometries was developed by finding operator matrix structures. This technique will be applied to p-hyponormality for rank-one perturbation of weighted shifts. In [2], they studied a special rank-one perturbation of weighted shifts and operator gaps.

Remark 3.7. The idea of this article will be important in the sequel [5] in proving the decomposability of the operators in the class $(\mathcal{RO})$.

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