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Generalized parallelogram law for operators and its application

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Abstract. The calssical Bohr inequality says that $|a+b|^2 \leq p|a|^2 + q|b|^2$ for all scalars a, b and p, $q > 0$ with $1/p + 1/q = 1$. In this note, we improve the accuracy of the estimate given by the original Bohr inequality. Actually, we present:

If A and B are operators on a Hilbert space and $t \neq 0$, then

$$|A+B|^2 + \frac{1}{t}|tA - B|^2 = (1+t)|A|^2 + (1+\frac{1}{t})|B|^2.$$

We discuss applications and further generalizations of it.

§1 Introduction

Let H be a complex separable Hilbert space and $\mathbb{B}(H)$ the algebra of all bounded operators on H. Denote by $|A|$ the absolute value operator of $A \in \mathbb{B}(H)$: $|A| = (A^*A)^{1/2}$, where A^* is the adjoint operator of A.

We say that A is a positive operator, if $(Ax, x) \geq 0$ for all $x \in H$, denoted by $A \geq 0$, and $A \geq B$ if A and B are self-adjoint and $A - B \geq 0$.

The classical Bohr inequality for scalar asserts that for any $a, b \in \mathbb{C}$ and all positive conjugate exponents $p, q \in \mathbb{R}$,

$$|a+b|^2 \leq p|a|^2 + q|b|^2 \tag{0}$$

with equality if and only if $(1-p)a = b$ (See [1]).

In 2003, O. Hirzallah[4] proposed an operator version of Bohr inequality as follows:

If $A, B \in \mathbf{B}(H)$ and p, q are both positive real conjugate exponents with $q \geq p$, then

$$|A-B|^2 + |(p-1)A + B|^2 \leq p|A|^2 + q|B|^2. \tag{1}$$

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It is well known that the absolute value operator plays an important role in the polar decomposition $A = U|A|$. Recently, various generalizations of Bohr inequalities have been obtained in [2] and [6].

In this paper, we improve the accuracy of the estimate given by the original Bohr inequality. As a matter of fact, the parallelogram law for absolute value of operators:

$$|A+B|^2 + |A-B|^2 = 2|A|^2 + 2|B|^2 \tag{2}$$

is our viewpoint. An operator version of the Bohr inequality (0) is obtained by a generalization of (2) as follows:

$$|A+B|^2 + |\sqrt{p-1}A - \sqrt{q-1}B|^2 = p|A|^2 + q|B|^2 \tag{3}$$

for operators A, B, and $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, because of $(p-1)(q-1) = 1$.

Furthermore, we extend the Bohr inequality to a three variable case:

If $\frac{1}{p} + \frac{1}{q} + \frac{1}{w} = 1$ for $p, q, w > 0$, then for operators A, B, C, we have

$$|A+B+C|^2 \leq p|A|^2 + q|B|^2 + w|C|^2.$$

§2 Bohr equality for 2 operators

The operator parallelogram law (2) has also the following generalization, which is different from (3) a bit:

Theorem 2.1 If $A, B \in \mathbb{B}(H)$, then $|A+B|^2 + \frac{1}{t}|tA-B|^2 = (1+t)|A|^2 + (1+\frac{1}{t})|B|^2$, for $t \neq 0$.

Proof. It follows that

$$\begin{aligned} & |A+B|^2 + \frac{1}{t}|tA-B|^2 \\ = \; & |A|^2 + |B|^2 + A^*B + B^*A + t|A|^2 + \frac{1}{t}|B|^2 - A^*B - B^*A \\ = \; & (1+t)|A|^2 + (1+\frac{1}{t})|B|^2. \end{aligned}$$

It is immediately obtained from the condition of t.

Corollary 2.2 (i) If $0 < t \leq 1$, then $|A+B|^2 + |tA-B|^2 \leq (1+t)|A|^2 + (1+\frac{1}{t})|B|^2$;

(ii) If $t \geq 1$ or $t < 0$, then $|A+B|^2 + |tA-B|^2 \geq (1+t)|A|^2 + (1+\frac{1}{t})|B|^2$.

As an easy consequence, we have Bohr type inequalities obtained in [2] and [4].

Corollary 2.3 [4, Theorem 1] If $A, B \in \mathbb{B}(H)$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq 2$, then

(i) $|A-B|^2 + |(p-1)A+B|^2 \leq p|A|^2 + q|B|^2$.

(ii) $|A-B|^2 + |A+(q-1)B|^2 \geq p|A|^2 + q|B|^2$.

Corollary 2.4 [2, Theorem 2] If $A, B \in \mathbb{B}(H)$, $\frac{1}{p} + \frac{1}{q} = 1$, $p < 1$, then

(iii) $|A-B|^2 + |(p-1)A+B|^2 \geq p|A|^2 + q|B|^2$.

Corollary 2.5 [2, Theorem 3] If $A, B \in \mathbb{B}(H)$, $|\alpha| \geq |\beta|$, then

$$|A-B|^2 + \frac{1}{|\alpha|^2}||\beta|A + |\alpha|B|^2 \leq (1+\frac{|\beta|}{|\alpha|})|A|^2 + (1+\frac{|\alpha|}{|\beta|})|B|^2$$

with equality if and only if $|\alpha| = |\beta|$ or $|\beta|A + |\alpha|B = 0$;

§3 Bohr-type inequalities for 3 operators

Observe that

$$|A+B+C|^2 = \begin{pmatrix} I & I & I \end{pmatrix} \begin{pmatrix} |A|^2 & A^*B & A^*C \\ B^*A & |B|^2 & B^*C \\ C^*A & C^*B & |C|^2 \end{pmatrix} \begin{pmatrix} I \\ I \\ I \end{pmatrix} \geq 0.$$

Then, due to the idea of [6], we may convert a problem of absolute operators to a problem of 3×3 block operator matrices, while the later approach maybe easier to handle.

For the sake of convenience, we cite the following well-known fact:

Lemma 3.1 If $x, y, z \geq 0$, and $a, b, c \in \mathbb{R}$ with

$$\begin{cases} xy \geq a^2, yz \geq c^2, xz \geq b^2; \\ xyz + 2abc \geq xc^2 + yb^2 + za^2. \end{cases}$$

Then

$$\begin{pmatrix} x & a & b \\ a & y & c \\ b & c & z \end{pmatrix} \geq 0.$$

Lemma 3.2 Let $A_i \in \mathbb{B}(H)$, $\alpha_i, \beta_i \in \mathbb{R}$ with $i = 1, 2, 3$. Then positive operator-valued function

$$F(\alpha_1, \alpha_2, \alpha_3) = |\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3|^2$$

is order preserving if the order $\prec$ among $\mathbb{R}$ is defined by

$$(\alpha_1, \alpha_2, \alpha_3) \prec (\beta_1, \beta_2, \beta_3) \Leftrightarrow |\alpha_i| \leq |\beta_i| \quad \text{for all } i \text{ and } \alpha_i\beta_j = \alpha_j\beta_i \quad \text{for} \quad i \neq j.$$

Proof. Since $|A_1 + A_2 + A_3|^2 = \begin{pmatrix} I & I & I \end{pmatrix} \begin{pmatrix} A_1^* \\ A_2^* \\ A_3^* \end{pmatrix} \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} \begin{pmatrix} I \\ I \\ I \end{pmatrix}$,
it suffice to show that $(\alpha_1, \alpha_2, \alpha_3) \prec (\beta_1, \beta_2, \beta_3)$ implies that

$$\begin{pmatrix} \alpha_1 A_1^* \\ \alpha_2 A_2^* \\ \alpha_3 A_3^* \end{pmatrix} \begin{pmatrix} \alpha_1 A_1 & \alpha_2 A_2 & \alpha_3 A_3 \end{pmatrix} \leq \begin{pmatrix} \beta_1 A_1^* \\ \beta_2 A_2^* \\ \beta_3 A_3^* \end{pmatrix} \begin{pmatrix} \beta_1 A_1 & \beta_2 A_2 & \beta_3 A_3 \end{pmatrix},$$

that is,

$$\begin{pmatrix} \alpha_1^2|A_1|^2 & \alpha_1\alpha_2 A_1^*A_2 & \alpha_1\alpha_3 A_1^*A_3 \\ \alpha_1\alpha_2 A_2^*A_1 & \alpha_2^2|A_2|^2 & \alpha_2\alpha_3 A_2^*A_3 \\ \alpha_1\alpha_3 A_3^*A_1 & \alpha_2\alpha_3 A_3^*A_2 & \alpha_3^2|A_3|^2 \end{pmatrix} \leq \begin{pmatrix} \beta_1^2|A_1|^2 & \beta_1\beta_2 A_1^*A_2 & \beta_1\beta_3 A_1^*A_3 \\ \beta_1\beta_2 A_2^*A_1 & \beta_2^2|A_2|^2 & \beta_2\beta_3 A_2^*A_3 \\ \beta_1\beta_3 A_3^*A_1 & \beta_2\beta_3 A_3^*A_2 & \beta_3^2|A_3|^2 \end{pmatrix}.$$

By the definition and Lemma 3.1, we have

$$\begin{pmatrix} \beta_1^2-\alpha_1^2 & \beta_1\beta_2-\alpha_1\alpha_2 & \beta_1\beta_3-\alpha_1\alpha_3 \\ \beta_1\beta_2-\alpha_1\alpha_2 & \beta_2^2-\alpha_2^2 & \beta_2\beta_3-\alpha_2\alpha_3 \\ \beta_1\beta_3-\alpha_1\alpha_3 & \beta_2\beta_3-\alpha_2\alpha_3 & \beta_3^2-\alpha_3^2 \end{pmatrix} \geq 0,$$

which implies the conclusion.

Theorem 3.3 Let $A, B, C \in \mathbb{B}(H)$, $\alpha_i \in \mathbb{R}$, $p, q, w > 0$ with $i = 1, 2, 3$. If

$$\begin{cases} p \geq \alpha^2; \\ q \geq \beta^2; \\ w \geq \gamma^2. \end{cases} \quad \begin{cases} (p-\alpha^2)(q-\beta^2) \geq (\alpha\beta)^2; \\ (q-\beta^2)(w-\gamma^2) \geq (\beta\gamma)^2; \\ (w-\gamma^2)(p-\alpha^2) \geq (\gamma\alpha)^2; \\ pqw \geq \alpha^2 qw + \beta^2 pw + \gamma^2 pq. \end{cases}$$

Then

$$|\alpha A + \beta B + \gamma C|^2 \leq p|A|^2 + q|B|^2 + w|C|^2.$$

Proof. As in above, we have to show that

$$\begin{pmatrix} \alpha^2|A|^2 & \alpha\beta A^*B & \alpha\gamma A^*C \\ \alpha\beta B^*A & \beta^2|B|^2 & \beta\gamma B^*C \\ \alpha\gamma C^*A & \beta\gamma C^*B & \gamma^2|C|^2 \end{pmatrix} \leq \begin{pmatrix} p|A|^2 & 0 & 0 \\ 0 & q|B|^2 & 0 \\ 0 & 0 & w|C|^2 \end{pmatrix}.$$

Therefore, what we should do is just to prove that

$$\begin{pmatrix} p-\alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & q-\beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & w-\gamma^2 \end{pmatrix} \geq 0.$$

which is obtained by the assumption and Lemma 3.1.

The following corollary is symmetric to Theorem 3.3.

Corollary 3.4 Let $A, B, C \in \mathbb{B}(H)$, $\alpha_i \in \mathbb{R}$, $p, q, w > 0$ with $i = 1, 2, 3$. If

$$\begin{cases} p \leq \alpha^2; \\ q \leq \beta^2; \\ w \leq \gamma^2. \end{cases} \quad \begin{cases} (p-\alpha^2)(q-\beta^2) \geq (\alpha\beta)^2; \\ (q-\beta^2)(w-\gamma^2) \geq (\beta\gamma)^2; \\ (w-\gamma^2)(p-\alpha^2) \geq (\gamma\alpha)^2; \\ pqw \geq \alpha^2 qw + \beta^2 pw + \gamma^2 pq. \end{cases}$$

Then

$$|\alpha A + \beta B + \gamma C|^2 \geq p|A|^2 + q|B|^2 + w|C|^2.$$

Now we have Bohr inequality for 3 operators.

Corollary 3.5 If $p, q, w > 0$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{w} = 1$, then $|A + B + C|^2 \leq p|A|^2 + q|B|^2 + w|C|^2$.

Proof. Given $p, q, w > 0$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{w} = 1$, the following is

$$\begin{cases} p \geq 1; \\ q \geq 1; \\ w \geq 1. \end{cases} \quad \begin{cases} (p-1)(q-1) \geq 1; \\ (q-1)(w-1) \geq 1; \\ (w-1)(p-1) \geq 1; \\ pqw = qw + pw + pq. \end{cases}$$

Therefore, Theorem 3.3 implies that

$$|A+B+C|^2 \le p|A|^2 + q|B|^2 + w|C|^2.$$

§4 Bohr equality for multiple operators

For this, we begin with the reformulation of (3). As a matter of fact, it is just the first step in this section:

Lemma 4.1 Let $A_1, A_2 \in \mathbb{B}(H)$. $\frac{1}{r_1}+\frac{1}{r_2}=1$ with $r_1, r_2 \ge 1$.

$$r_1|A_1|^2 + r_2|A_2|^2 - |A_1+A_2|^2 = |\sqrt{\frac{r_1}{r_2}}A_1 - \sqrt{\frac{r_2}{r_1}}A_2|^2.$$

Theorem 4.2 Suppose that $A_i \in \mathbb{B}(H)$. and $r_i \ge 1$ with $\sum\limits_{i=1}^{n}\frac{1}{r_i}=1$ for $i=1,2,...,n, n\in \mathbb{N}$. Then

$$\sum_{i=1}^{n} r_i|A_i|^2 - |\sum_{i=1}^{n} A_i|^2 = \sum_{1\le i<j\le n}\left|\sqrt{\frac{r_i}{r_j}}A_i - \sqrt{\frac{r_j}{r_i}}A_j\right|^2. \tag{4}$$

Proof. We show it by the induction on n. Note that it is true for $n=2$ by Lemma 4.1. Now suppose that it is true for $n=k$, then we take $A_1,\cdots,A_{k+1} \in \mathbb{B}(H)$ and $r_1,\cdots,r_{k+1} > 1$ satisfying $\sum\limits_{i=1}^{k+1}\frac{1}{r_i}=1$. If we put $r_i' = r_i(1-\frac{1}{r_{k+1}})$ for $i=1,\cdots,k$ for convenience, then $r_i' > 1$ and $\sum\limits_{i=1}^{k}\frac{1}{r_i'}=1$. Hence we have

$$\begin{aligned}
&\sum_{i=1}^{k+1} r_i|A_i|^2 - |\sum_{i=1}^{k+1} A_i|^2 = \sum_{i=1}^{k} r_i|A_i|^2 + r_{k+1}|A_{k+1}|^2 - |\sum_{i=1}^{k} A_i + A_{k+1}|^2 \\
= \;& (1-\frac{1}{r_{k+1}})\sum_{i=1}^{k} r_i|A_i|^2 - |\sum_{i=1}^{k} A_i|^2 \\
&+(r_{k+1}-1)|A_{k+1}|^2 + \frac{1}{r_{k+1}}\sum_{i=1}^{k} r_i|A_i|^2 - (\sum_{i=1}^{k} A_i)^* A_{k+1} - A_{k+1}^*\sum_{i=1}^{k} A_i \\
= \;& \left(\sum_{i=1}^{k} r_i'|A_i|^2 - |\sum_{i=1}^{k} A_i|^2\right) + \sum_{i=1}^{k}\frac{r_i}{r_{k+1}}|A_i|^2 - (\sum_{i=1}^{k} A_i)^* A_{k+1} - A_{k+1}^*\sum_{i=1}^{k} A_i + (r_{k+1}-1)|A_{k+1}|^2 \\
= \;& \sum_{1\le i<j\le k}\left|\sqrt{\frac{r_i}{r_j}}A_i - \sqrt{\frac{r_j}{r_i}}A_j\right|^2 + \sum_{i=1}^{k}\frac{r_i}{r_{k+1}}|A_i|^2 - (\sum_{i=1}^{k} A_i)^* A_{k+1} - A_{k+1}^*\sum_{i=1}^{k} A_i + \sum_{i=1}^{k}\frac{r_{k+1}}{r_i}|A_{k+1}|^2 \\
= \;& \sum_{1\le i<j\le k+1}\left|\sqrt{\frac{r_i}{r_j}}A_i - \sqrt{\frac{r_j}{r_i}}A_j\right|^2.
\end{aligned}$$

Therefore, the equality (4) holds for all $n\in\mathbb{N}$.

Corollary 4.3 [6, Theorem 7] Suppose that $A_i \in \mathbb{B}(H)$, and $r_i \ge 1$ with $\sum\limits_{i=1}^{n}\frac{1}{r_i}=1$ for $i=1,2,...,n$. Then

$$|\sum_{i=1}^{n} A_i|^2 \le \sum_{i=1}^{n} r_i|A_i|^2.$$

Equivalently, we can say that $K(z)=|z|^2$ satisfies (operator) Jensen inequality, in the sense that

$$K(\sum_{i=1}^{n} t_iA_i) \le \sum_{i=1}^{n} t_iK(A_i)$$

for $t_1, \cdots, t_n \geq 0$ with $\sum_{i=1}^{n} t_i = 1$.

Corollary 4.4 Let $A_i \in \mathbb{B}(H)$, $\sum_{i=1}^{n} \frac{1}{r_i} = 1$, and $r_i \neq 0$ with $\sum_{i=1}^{n} \frac{1}{r_i} = 1$ for $i = 1, 2, ..., n$, $n \in \mathbb{N}$. Then

$$\sum_{i=1}^{n} r_i |A_i|^2 - |\sum_{i=1}^{n} A_i|^2 = \sum_{1 \leq i \leq j \leq n} \frac{r_j}{r_i} \left| \frac{r_i}{r_j} A_i - A_j \right|^2.$$

§5 Further generalization of Bohr inequality

Theorem 5.1 Let $A, B, C \in \mathbb{B}(H)$, $\alpha_i \in \mathbb{R}$, $\gamma_i > 0$ with $i = 1, 2, 3$. If

$$\begin{cases} \gamma_1 \geq 1 + \alpha_1^2; \\ \gamma_2 \geq 1 + \alpha_2^2; \\ \gamma_3 \geq 1 + \alpha_3^2. \end{cases} \quad \begin{cases} [\gamma_1 - (1 + \alpha_1^2)][\gamma_2 - (1 + \alpha_2^2)] \geq (1 + \alpha_1\alpha_2)^2; \\ [\gamma_2 - (1 + \alpha_2^2)][\gamma_3 - (1 + \alpha_3^2)] \geq (1 + \alpha_2\alpha_3)^2; \\ [\gamma_1 - (1 + \alpha_1^2)][\gamma_3 - (1 + \alpha_3^2)] \geq (1 + \alpha_1\alpha_3)^2. \end{cases}$$

with $[\gamma_1 - (1 + \alpha_1^2)][\gamma_2 - (1 + \alpha_2^2)][\gamma_3 - (1 + \alpha_3^2)] - 2(1 + \alpha_1\alpha_2)(1 + \alpha_1\alpha_3)(1 + \alpha_2\alpha_3) \geq - [\gamma_3 - (1 + \alpha_3^2)](1 + \alpha_1\alpha_2)^2 - [\gamma_2 - (1 + \alpha_2^2)](1 + \alpha_1\alpha_3)^2 - [\gamma_1 - (1 + \alpha_1^2)](1 + \alpha_2\alpha_3)^2$.

Then

$$|A_1 + A_2 + A_3|^2 + |\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3|^2 \leq \gamma_1 |A_1|^2 + \gamma_2 |A_2|^2 + \gamma_3 |A_3|^2. \tag{5}$$

Proof. Notice that both sides of the inequality (5) correspond to

$$\begin{pmatrix} (1 + \alpha_1^2)|A_1|^2 & (1 + \alpha_1\alpha_2)A_1^* A_2 & (1 + \alpha_1\alpha_3)A_1^* A_3 \\ (1 + \alpha_1\alpha_2)A_2^* A_1 & (1 + \alpha_2^2)|A_2|^2 & (1 + \alpha_2\alpha_3)A_2^* A_3 \\ (1 + \alpha_1\alpha_3)A_3^* A_1 & (1 + \alpha_2\alpha_3)A_3^* A_2 & (1 + \alpha_3^2)|A_3|^2 \end{pmatrix}$$

and

$$\begin{pmatrix} \gamma_1 |A_1|^2 & 0 & 0 \\ 0 & \gamma_2 |A_2|^2 & 0 \\ 0 & 0 & \gamma_3 |A_3|^2 \end{pmatrix}$$

respectively. Hence, it is suffice to show that

$$\begin{pmatrix} \gamma_1 - (1 + \alpha_1^2) & -1 - \alpha_1\alpha_2 & -1 - \alpha_1\alpha_3 \\ -1 - \alpha_1\alpha_2 & \gamma_2 - (1 + \alpha_2^2) & -1 - \alpha_2\alpha_3 \\ -1 - \alpha_1\alpha_3 & -1 - \alpha_2\alpha_3 & \gamma_3 - (1 + \alpha_3^2) \end{pmatrix} \geq 0,$$

which is implied by the assumption and Lemma 3.1.

Corollary 5.2 Let $A, B, C \in \mathbb{B}(H)$, $\alpha_1, \alpha_2, \beta_i \in \mathbb{R}$, $\gamma_i > 0$ with $i = 1, 2, 3$. If

$$\begin{cases} \gamma_1 \geq 1 + \alpha_1^2; \\ \gamma_2 \geq 1 + \alpha_2^2; \\ \gamma_3 \geq 1. \end{cases} \quad \begin{cases} [\gamma_2 - (\alpha_2^2 + 1)](\gamma_3 - 1) \geq 1; \\ [\gamma_1 - (\alpha_1^2 + 1)](\gamma_3 - 1) \geq 1; \\ [\gamma_1 - (\alpha_1^2 + 1)][\gamma_2 - (\alpha_2^2 + 1)] \geq (1 + \alpha_1\alpha_2)^2; \\ 1 + \alpha_1\alpha_2 \leq 0. \end{cases}$$

Then

$$|A_1 + A_2 + A_3|^2 + |\alpha_1 A_1 + \alpha_2 A_2|^2 \leq \gamma_1 |A_1|^2 + \gamma_2 |A_2|^2 + \gamma_3 |A_3|^2.$$

Another extension of Bohr inequality is presented:

Corollary 5.3 If $A, B, C \in \mathbb{B}(H)$, and $\frac{1}{p}+\frac{1}{q}+\frac{1}{w}=1, p, q, w \geq 0$, then

$$|A+B+C|^2+|\frac{p}{\sqrt{p+q}}A-\frac{q}{\sqrt{p+q}}B|^2 \leq p|A|^2+q|B|^2+w|C|^2.$$

Theorem 5.4 Let $A_i \in \mathbb{B}(H)$, $\alpha_i, \beta_i \in \mathbb{R}$, $\gamma_i > 0$ with $i = 1, 2, 3$. If

$$\begin{cases} \gamma_1 \geq 1+\alpha_1^2; \\ \gamma_2 \geq 1+\alpha_2^2; \\ \gamma_3 \geq 1+\alpha_3^2. \end{cases} \quad \begin{cases} [\gamma_1-(\alpha_1^2+1)][\gamma_2-(\alpha_2^2+1)] \geq (\alpha_1\alpha_2+1)^2; \\ [\gamma_2-(\alpha_2^2+1)][\gamma_3-(\alpha_3^2+1)] \geq (\alpha_2\alpha_3+1)^2; \\ \alpha_1\alpha_3+1=0. \end{cases}$$

Then

$$|A_1+A_2+A_3|^2+|\alpha_1A_1+\alpha_2A_2+\alpha_3A_3|^2 \leq \gamma_1|A_1|^2+\gamma_2|A_2|^2+\gamma_3|A_3|^2.$$

Corollary 5.5 If $A, B, C \in \mathbb{B}(H)$, and $\frac{1}{p}+\frac{1}{q}+\frac{1}{w}=1, p, q, w \geq 0$, then

$$|A+B+C|^2+|\frac{1}{\sqrt{w-1}}A+\frac{1}{\sqrt{w-1}}B-\sqrt{w-1}C|^2 \leq p|A|^2+q|B|^2+w|C|^2.$$

Related to [3], we have the following inequalities. As a matter of fact, the right-hand sides are regarded as the weighted arithmetic mean of $|A|^2$, $|B|^2$ and $|C|^2$ in [3, Lemma 1].

Corollary 5.6 If $A, B, C \in \mathbb{B}(H)$, and $t \in (0, 1)$, then we have

$$|A+B+C|^2+|\sqrt{t}A+\sqrt{t}B-\frac{1}{\sqrt{t}}C|^2 \leq \frac{2-t}{1-t}\frac{1+t}{t}|A|^2+\frac{2-t}{1-t}|B|^2+\frac{1+t}{t}|C|^2;$$

$$|A+B+C|^2+|\sqrt{1-t}(A+B)-\frac{1}{\sqrt{1-t}}C|^2 \leq \frac{1+t}{t}|A|^2+\frac{2-t}{1-t}\frac{1+t}{t}|B|^2+\frac{2-t}{1-t}|C|^2.$$

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References

[1] Zur theorie der fastperiodischen funktionmen I, Acta Math., 45(1927), 29-127.

[2] W.-S. Cheung and J.Pecaric, Bohr's inequalities for Hilbert sapce operators, J. Math. Anal. Appl., 323(2006), 403-412.

[3] J. I. Fujii, M. Fujii. M. Nakamura, J. Pecaric and Y. Seo, A reverse inequality for the weighted geometric mean due to Lawson-Lim, Linear Algebra Appl., 427(2007), 272-284.

[4] O. Hirzallah, Non-commutative operator Bohr inequalities, J. Math. Anal. Appl., 282(2003), 578-583.

[5] N.Young, An Introduction to Hilbert Space, Cambridge University Press, 1988.

[6] F. Zhang, On the Bohr inequality of operators, J. Math. Anal. Appl., 333(2007), 1264-1271.