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A complement to monotonicity of generalized Furuta-type operator functions

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Abstract

Recently, Furuta obtained the results on monotonicity of a generalized Furuta-type operator function $F(\lambda,\mu)=A^{-\lambda}\ \natural_{\frac{1-t+\lambda}{(p-t)\mu+\lambda}}\ (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\mu}$.

In this report, we shall show the result which considers a domain not considered in Furuta's one as follows: Let $A\geq B\geq 0$ with $A>0$, $t\in[0,1]$ and $p\geq 1$. Then $F(\lambda,\mu)$ satisfies

$$F(q,w)\geq F(t,1)\geq F(r,s)\geq F(r',s')$$

for any $s'\geq s\geq 1$, $r'\geq r\geq t$, $\frac{1-t}{p-t}\leq w\leq 1$ and $t-1\leq q\leq t$.

We shall also discuss an equivalence relation related to Ando-Hiai inequality.

1 Introduction

This report is based on our recent paper [21] and preprint [4].

In this report, a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator T is said to be positive (denoted by $T\geq 0$) if $(Tx,x)\geq 0$ for all $x\in\mathcal{H}$, and also an operator T is said to be strictly positive (denoted by $T>0$) if T is positive and invertible.

The following Löwner-Heinz theorem is a famous order preserving operator inequality.

$$A\geq B\geq 0 \quad \text{implies} \quad A^{\alpha}\geq B^{\alpha} \text{ for any } \alpha\in[0,1].$$

In 1987, Furuta inequality [11] is established as an extension of Löwner-Heinz theorem.

Theorem 1.A (Furuta inequality [11]).

If $A\geq B\geq 0$, then for each $r\geq 0$,

(i) $\quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}}\geq(B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\quad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}}\geq(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p\geq 0$ and $q\geq 1$ with $(1+r)q\geq p+r$.

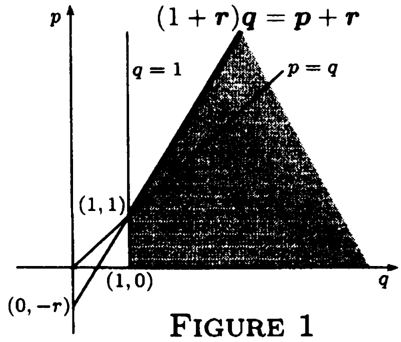

FIGURE 1

By putting $r=0$ in Theorem 1.A, we can get Löwner-Heinz theorem. Alternative proofs of Theorem 1.A are given in [2, 22] and also an elementary one page proof in [12]. Tanahashi [25] showed that the domain drawn for p, q and r in the Figure 1 is the best possible one for Theorem 1.A.

As stated in [22], when $A>0$ and $B\geq 0$, (ii) of Theorem 1.A can be arranged in terms of α-power mean $\sharp_{\alpha}$ for $\alpha\in[0,1]$ introduced by Kubo-Ando [24] as $A \sharp_{\alpha} B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\alpha}A^{\frac{1}{2}}$:

$$A\geq B\geq 0 \text{ with } A>0 \quad \text{implies} \quad A^{-r}\sharp_{\frac{1+r}{p+r}}B^{p}\leq B\leq A \text{ for } p\geq 1 \text{ and } r\geq 0. \tag{F}$$

Next we shall discuss weaker order than usual one $A\geq B$. For $A, B>0$, the order $\log A\geq\log B$ is called chaotic order. It is well known that chaotic order is weaker than usual one since $\log t$ is an operator monotone function for $t>0$.

As a characterization of chaotic order, in [3] and [13] (see also [5, 27]), they showed the following: For $A, B>0$,

$$\log A\geq\log B \quad \text{if and only if} \quad A^{-r}\sharp_{\frac{r}{p+r}}B^{p}\leq I \text{ for all } p\geq 0 \text{ and } r\geq 0, \tag{1.1}$$

and also

$$\log A\geq\log B \quad \text{implies} \quad A^{-r}\sharp_{\frac{\delta+r}{p+r}}B^{p}\leq B^{\delta} \text{ for } p\geq\delta\geq 0 \text{ and } r\geq 0.$$

We remark that an excellent proof of (1.1) which used only Theorem 1.A was shown in [27]. We can summarize above results as follows: For $A, B>0$,

$$\begin{array}{ccl}
A\geq B & \Longrightarrow & A^{-r}\sharp_{\frac{1+r}{p+r}}B^{p}\leq B\leq A \ \text{ for } p\geq 1 \text{ and } r\geq 0.\\
\Downarrow & & \\
A^{q}\geq B^{q}\ (q\in(0,1)) & \Longrightarrow & A^{-r}\sharp_{\frac{q+r}{p+r}}B^{p}\leq B^{q}\leq A^{q} \text{ for } p\geq q \text{ and } r\geq 0.\\
\Downarrow & & \\
\log A\geq\log B & \Longleftrightarrow & (1.1)\text{: } A^{-r}\sharp_{\frac{r}{p+r}}B^{p}\leq I \ \text{ for all } p\geq 0 \text{ and } r\geq 0.\\
\Downarrow & & \\
A^{-r}\sharp_{\frac{\delta+r}{p+r}}B^{p}\leq B^{\delta} & & \\
\text{for } p\geq\delta\geq 0 \text{ and } r\geq 0. & &
\end{array}$$

2 Equivalence relation related to Ando-Hiai inequality

In 1994, Ando and Hiai [1] have shown the following inequality.

Theorem 2.A (Ando-Hiai inequality [1]). *For $A, B > 0$,*

$$A \natural_\alpha B \le I \text{ for } \alpha \in (0,1) \quad \text{implies} \quad A^r \natural_\alpha B^r \le I \text{ for } r \ge 1. \tag{AH}$$

By (AH), they obtained that for $A, B > 0$,

$$A^{-1} \natural_{\frac{1}{p}} A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}} \le I \text{ implies } A^{-r} \natural_{\frac{1}{p}} (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^r \le I \text{ for } p \ge 1 \text{ and } r \ge 1, \tag{AH'}$$

that is,

$$A \ge B > 0 \text{ implies } A^r \ge \{A^{\frac{r}{2}} (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^r A^{\frac{r}{2}}\}^{\frac{1}{p}} \text{ for } p \ge 1 \text{ and } r \ge 1. \tag{AH''}$$

We remark that (AH") is equivalent to the main result of log majorization.

In [8], it was pointed out that the following (C) is the essence of (F).

$$A \ge B > 0 \quad \text{implies} \quad A^{-r} \natural_{\frac{r}{p+r}} B^p \le I \text{ for } p \ge 0 \text{ and } r \ge 0. \tag{C}$$

We remark that (F) implies (C) immediately by Löwner-Heinz theorem. It was shown in [7] that an equivalence relation holds between (AH) and (F) via (C). Here we can obtain an equivalence relation between (AH) and (C) without using (F).

Theorem 2.1 ([4]). (AH) *is equivalent to* (C).

Proof of Theorem 2.1. Suppose that (C) holds and that $A \natural_\alpha B \le I$. We put $p = \frac{1}{\alpha} > 1$. Then the assumption $A \natural_\alpha B \le I$ says that

$$B_1 = (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha \le A^{-1} = A_1.$$

Applying (C) to $A_1 \ge B_1$, we have

$$A_1^{-r} \natural_{\frac{r}{p+r}} B_1^p \le I \quad \text{for } r \ge 0.$$

Moreover it follows that for $p \ge 1$ and $r \ge 0$,

$$\begin{aligned} A_1^{-r} \natural_{\frac{1+r}{p+r}} B_1^p &= B_1^p \natural_{\frac{p-1}{p+r}} A_1^{-r} = B_1^p \natural_{\frac{p-1}{p}} (B_1^p \natural_{\frac{p}{p+r}} A_1^{-r}) \\ &= B_1^p \natural_{\frac{p-1}{p}} (A_1^{-r} \natural_{\frac{r}{p+r}} B_1^p) \le B_1^p \natural_{\frac{p-1}{p}} I = B_1 \le A_1. \end{aligned}$$

Summing up the above discussion, for each $p > 1$,

$$A \natural_{\frac{1}{p}} B \le I \text{ implies } A^r \natural_{\frac{1+r}{p+r}} A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \le A^{-1}, \text{ or } A^{r+1} \natural_{\frac{1+r}{p+r}} B \le I \text{ for } r \ge 0.$$

Noting that

$$B \natural_{\frac{p-1}{p+r}} A^{r+1} = A^{r+1} \natural_{\frac{1+r}{p+r}} B \le I,$$

we apply it for $p_1 = \frac{p+r}{p-1}$ in the following way;

$$I \geq B^{r+1} \natural_{\frac{1+r}{p_1+r}} A^{r+1} = A^{r+1} \natural_{\frac{1}{p}} B^{r+1}$$

by $1 - \frac{1+r}{p_1+r} = \frac{1}{p}$. Namely we obtain (AH).

(AH) $\Rightarrow$ (C) has been already shown in [7]. But we cite it for the sake of convenience: It suffices to show that (C) holds for $p, r > 1$ under the assumption $A \geq B > 0$ because it holds for $0 \leq p, r \leq 1$ by Löwner-Heinz theorem. So we take arbitrary $p, r > 1$, and put $\alpha = \frac{r}{p+r}$ and $q = \max\{p, r\}$. Then, as noted in above, if $A \geq B > 0$, then (C) holds for $p_1 = \frac{p}{q}$ and $r_1 = \frac{r}{q}$, i.e.,

$$A^{-r_1} \natural_{\frac{r_1}{p_1+r_1}} B^{p_1} \leq I.$$

We here apply (AH) to this, that is, we have

$$I \geq A^{-r_1 q} \natural_{\frac{r_1 q}{p_1 q + r_1 q}} B^{p_1 q} = A^{-r} \natural_{\frac{r}{p+r}} B^p,$$

as desired. □

3 A complement to monotonicity of generalized Furuta-type operator functions

In 1995, Furuta [14] obtained the following theorem.

Theorem 3.A (Grand Furuta inequality [14]). *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,*

$$F(r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}} \tag{3.1}$$

is decreasing for $r \geq t$ and $s \geq 1$, and

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{3.2}$$

holds for $r \geq t$ and $s \geq 1$.

Theorem 3.A is established as a generalization of both Furuta inequality (F) and Ando-Hiai inequality (AH"). In fact, Theorem 3.A leads (F) by putting $t = 0$ and $s = 1$, and also leads (AH") by putting $t = 1$ and $s = r$. An alternative proof of Theorem 3.A is given in [6] and an elementary one-page proof of (3.2) is in [15]. Related results to Theorem 3.A are shown in [16, 18, 19, 20, 29] and so on. It is shown in [26] (see also [10, 28]) that the outside exponents of (3.2) are the best possible. We remark that (3.1) can be rewritten by using α-power mean as follows:

$$F(\lambda, \mu) = A^{-\lambda} \natural_{\frac{1-t+\lambda}{(p-t)\mu+\lambda}} (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\mu}. \tag{3.1'}$$

Related to Theorem 3.A, the following result was shown in [23, 9].

Theorem 3.B ([23, 9]). *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p \geq 1$. Then*

$$A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^t \sharp_{\frac{1-t}{p-t}} B^p$$

for $s \geq 1$ and $r \geq t$, where $A \natural_s B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^s A^{\frac{1}{2}}$ for $s \in \mathbb{R}$.

Very recently, as a generalization of Theorem 3.B, the following theorem was shown on monotonicity of a generalized Furuta-type operator function (3.1').

Theorem 3.C ([17]). *Define $F(\lambda, \mu)$ as (3.1'). Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p \geq 1$. Then $F(\lambda, \mu)$ satisfies the following properties:*

(i) $$F(r,w) \geq F(r,1) \geq F(r,s) \geq F(r,s')$$
holds for any $s' \geq s \geq 1$, $r \geq t$ and $\frac{1-t}{p-t} \leq w \leq 1$.

(ii) $$F(q,s) \geq F(t,s) \geq F(r,s) \geq F(r',s)$$
holds for any $r' \geq r \geq t$, $s \geq 1$ and $t-1 \leq q \leq t$.

$F(\lambda, \mu)$ is not always decreasing for $\frac{1-t}{p-t} \leq \lambda \leq 1$ and $t-1 \leq \mu \leq t$ (see [17]). But Theorem 3.C says that we can compare $F(r,w)$ with $F(r,1)$ for $\frac{1-t}{p-t} \leq w \leq 1$, and $F(q,s)$ with $F(t,s)$ for $t-1 \leq q \leq t$. We remark that Theorem 3.C leads Theorem 3.B by putting $w = \frac{1-t}{p-t}$ in (i) or $q = 0$ in (ii).

Here, we shall consider a domain not considered in Theorem 3.C, that is, we shall show that we can also compare $F(q,w)$ with $F(t,1)$ for $\frac{1-t}{p-t} \leq w \leq 1$ and $t-1 \leq q \leq t$.

Theorem 3.1 ([21]). *Define $F(\lambda, \mu)$ as (3.1'). Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p \geq 1$. Then $F(\lambda, \mu)$ satisfies*

$$F(q,w) \geq F(t,1) \geq F(r,s) \geq F(r',s')$$

for any $s' \geq s \geq 1$, $r' \geq r \geq t$, $\frac{1-t}{p-t} \leq w \leq 1$ and $t-1 \leq q \leq t$.

Proof of Theorem 3.1. We have only to show $F(q,w) \geq F(t,1)$ since $F(t,1) \geq F(r,s) \geq F(r',s')$ is just Theorem 3.A.

By Löwner-Heinz theorem, $A^{t-q} \geq B^{t-q}$ since $t-q \in [0,1]$ and $A^t \geq B^t$ since $t \in [0,1]$, so that we have

$$\begin{aligned} F(q,w) &= A^{-q} \sharp_{\frac{1-t+q}{(p-t)w+q}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^w = A^{\frac{-t}{2}} \{A^{t-q} \sharp_{\frac{1-t+q}{(p-t)w+q}} (A^t \sharp_w B^p)\} A^{\frac{-t}{2}} \\ &\geq A^{\frac{-t}{2}} \{B^{t-q} \sharp_{\frac{1-t+q}{(p-t)w+q}} (B^t \sharp_w B^p)\} A^{\frac{-t}{2}} = A^{\frac{-t}{2}} B A^{\frac{-t}{2}} = A^{-t} \sharp_{\frac{1}{p}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}}) \\ &= F(t,1). \end{aligned}$$

Hence the proof is complete. □

Figure 2 expresses the domain of λ and μ in which Theorem 3.A, Theorem 3.C and Theorem 3.1 hold.

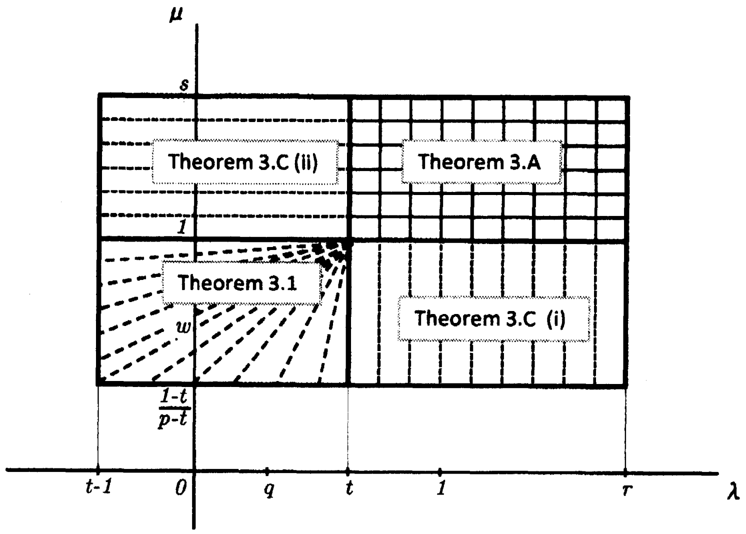

FIGURE 2

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