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Upper triangular operators, SVEP and Browder, Weyl theorems

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Abstract

We show the important role played by SVEP, *the single-valued extension property*, and the *polaroid property* in relating the spectrum, and certain distinguished parts thereof, of the operators $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and M_0 for some Banach space operators A, B and $C \in B(\mathcal{X})$.

1. Results

Let $B(\mathcal{X})$ denote the algebra of operators (equivalently, bounded linear transformations) on a Banach space $\mathcal{X}$. For $A, B, C \in B(\mathcal{X})$, let M_C denote the upper triangular operator $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and let $M_0 = A \oplus B$. The spectrum, and certain distinguished parts thereof, of the operators M_C and M_0 has been studied by a number of authors in the recent past; see references. Of particular interest to us is the relationship between the spectral, the Fredholm, the Browder and the Weyl properties.

Most of our notation is standard. For an operator $T \in B(\mathcal{X})$:

$$\begin{aligned}
\sigma(T) &= \text{ spectrum of } T, \\
\sigma_a(T) &= \text{ approximate point spectrum of } T, \\
\Phi_+(\mathcal{X}) &= \{T \in B(\mathcal{X}) : T \text{ is upper semi--Fredholm}\}, \text{ and} \\
\Phi_-(\mathcal{X}) &= \{T \in B(\mathcal{X}) : T \text{ is lower semi--Fredholm}\}.
\end{aligned}$$

The Browder, the Weyl, the upper semi–Fredholm, the lower semi–Fredholm spectrum of T are the sets

$$\begin{aligned}
\sigma_b(T) &= \{\lambda \in \sigma(T) : T - \lambda \text{ is not Fredholm or } \operatorname{asc}(T-\lambda) \neq \operatorname{dsc}(T-\lambda)\}, \\
\sigma_w(T) &= \{\lambda \in \sigma(T) : T - \lambda \text{ is not Fredholm or } \operatorname{ind}(T-\lambda) \not\leq 0\}, \\
\sigma_{SF_+}(T) &= \{\lambda \in \sigma(T) : T - \lambda \notin \Phi_+(\mathcal{X})\}, \text{ and} \\
\sigma_{SF_-}(T) &= \{\lambda \in \mathbf{C} : T - \lambda \notin \Phi_-(\mathcal{X})\},
\end{aligned}$$

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respectively. The Browder essential approximate point spectrum $\sigma_{ab}(T)$, and the Weyl essential approximate point spectrum $\sigma_{aw}(T)$, of T are the sets

$$\begin{aligned}\sigma_{ab}(T) &= \{\lambda \in \sigma_a(T) : T-\lambda \notin \Phi_+(\mathcal{X}) \text{ or } \operatorname{asc}(T-\lambda)=\infty\}, \text{ and}\\ \sigma_{aw}(T) &= \{\lambda \in \sigma_a(T) : T-\lambda \notin \Phi_+(\mathcal{X}) \text{ or } \operatorname{ind}(T-\lambda \not\leq 0\}.\end{aligned}$$

Let

$$\begin{aligned}\Xi(T) &= \{\lambda \in \mathbf{C} : T \text{ does not have SVEP at } \lambda\},\\ \Xi_+(T) &= \{T-\lambda \in \Phi_+(\mathcal{X}) : \lambda \in \Xi(T)\}, \text{ and}\\ \Xi_+^*(T) &= \{T-\lambda \in \Phi_+(\mathcal{X}) : \lambda \in \Xi(T^*)\}.\end{aligned}$$

The following inclusions/equalities are either well known or are easily proved:

$$\begin{aligned}&\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B) = \sigma(M_0) = \sigma(M_C) \cup \{\Xi(A^*) \cup \Xi(B)\},\\ &\sigma_b(M_C) \subseteq \sigma_b(A) \cup \sigma_b(B) = \sigma_b(M_0), \text{ and}\\ &\sigma_w(M_C) \subseteq \sigma_w(M_0) \subseteq \sigma_w(A) \cup \sigma_w(B).\end{aligned}$$

Furthermore, if we let $P = A$ and $Q = B$ or $P = A^*$ and $Q = B^*$, then:

$$\begin{aligned}&\sigma_b(M_0) = \sigma_b(M_C) \cup \{\Xi(A^*) \cup \Xi(B)\}, \text{ and}\\ &\sigma_w(A) \cup \sigma_w(B) \subseteq \sigma_w(M_C) \cup \{\Xi(P) \cup \Xi(Q)\}.\end{aligned}$$

Consequently, *if* $\Xi(P) \cup \Xi(Q) = \emptyset$, *then*

$$\sigma_b(M_0) = \sigma_w(M_0) = \sigma_b(M_C) = \sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B).$$

For the spectra σ_{ab} and σ_{aw}, one has:

$$\begin{aligned}&\sigma_{ab}(M_C) \subseteq \sigma_{ab}(M_0) \subseteq \sigma_{ab}(M_C) \cup \{\Xi_+^*(A) \cup \Xi_+(B)\}, \text{ and}\\ &\sigma_{aw}(M_0) \subseteq \sigma_{aw}(A) \cup \sigma_{aw}(B) \subseteq \sigma_{ab}(A) \cup \sigma_{ab}(B)\\ = \; &\sigma_{ab}(M_0) \subseteq \sigma_{aw}(A) \cup \sigma_{aw}(B) \cup \{\Xi_+(A) \cup \Xi_+(B)\}.\end{aligned}$$

Recall that a necessary and sufficient condition for M_0 to satisfy Browder's theorem (or, *Bt*), $acc\sigma(M_0) \subseteq \sigma_w(M_0) \iff \sigma_w(M_0) = \sigma_b(M_0)$ (resp., a-Browder's theorem (or, $a - Bt$), $acc\sigma_a(M_0) \subseteq \sigma_{aw}(M_0) \iff \sigma_{aw}(M_0) = \sigma_{ab}(M_0)$), is that A and B have SVEP at points $\lambda \in \sigma(M_0)$ such that $A-\lambda$, $B-\lambda$ are Fredholm and $\operatorname{ind}(A-\lambda)+\operatorname{ind}(B-\lambda)=0$ (resp., A and B have SVEP at points $\lambda \in \sigma_a(M_0)$ such that $A-\lambda$, $B-\lambda$ are are upper semi–Fredholm and $\operatorname{ind}(A-\lambda)+\operatorname{ind}(B-\lambda) \leq 0$). Similarly, a necessary and sufficient condition for M_C to satisfy *Bt*, $acc\sigma(M_C) \subseteq \sigma_w(M_C) \iff \sigma_w(M_C) = \sigma_b(M_C)$ (resp., $a - Bt$, $acc\sigma_a(M_0) \subseteq \sigma_{aw}(M_0) \iff \sigma_{ab}(M_C) = \sigma_{aw}(M_C)$), is that M_C has SVEP at points $\lambda \in \sigma(M_0) \setminus \sigma_{aw}(M_C)$ (resp., at points $\lambda \in \sigma_a(M_C) \setminus \sigma_{aw}(M_C)$). It is easily verified that if M_0 satisfies *Bt* then $\sigma_w(M_0) = \sigma_w(A) \cup \sigma_w(B)$, and that if M_0 satisfies $a - Bt$ then $\sigma_{aw}(M_0) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. A similar result, however, fails for the operator M_C, as follows from a consideration of the operator $\begin{pmatrix} U & 1-UU^* \\ 0 & U^* \end{pmatrix}$, where U is the forward unilateral shift on a Hilbert space. Evidently, the complement of $\sigma_w(M_C)$ in the complex plane $\mathbf{C}$ is the union of the complement of $\sigma_w(M_0)$ in $\mathbf{C}$ with $\sigma_w(M_0) \setminus \sigma_w(M_C)$, and the complement of $\sigma_{aw}(M_C)$ in the complex plane $\mathbf{C}$ is the union of the complement of $\sigma_{aw}(M_0)$ in $\mathbf{C}$ with $\sigma_{aw}(M_0) \setminus \sigma_{aw}(M_C)$; thus, if M_0 satisfies *Bt* (resp., $a - Bt$) and M_C has SVEP on $\sigma_w(M_0) \setminus \sigma_w(M_C)$ (resp., $\sigma_{aw}(M_0) \setminus \sigma_{aw}(M_C)$), then M_C satisfies *Bt* (resp., $a - Bt$). This may be achieved in a number of ways.

Theorem 1.1 *(a). If either (i) A has SVEP at points $\lambda \in \sigma_w(M_0) \setminus \sigma_{SF_-}(A)$ and B has SVEP at points $\mu \in \sigma_w(M_0) \setminus \sigma_{SF_-}(B)$, or (ii) both A and A^* have SVEP at points $\lambda \in \sigma_w(M_0) \setminus \sigma_{SF_+}(A)$, or (iii) A^* has SVEP at points $\lambda \in \sigma_w(M_0) \setminus \sigma_{SF_+}(A)$ and B^* has SVEP at points $\mu \in \sigma_w(M_0) \setminus \sigma_{SF_-}(B)$, then M_0 satisfies Bt implies M_C satisfies Bt.*

(b). If (i) A has SVEP on $\lambda \in \sigma_{aw}(M_0) \setminus \sigma_{SF_+}(A)$ and A^* has SVEP on $\mu \in \sigma_w(M_0) \setminus \sigma_{SF_+}(A)$, or (ii) A^* has SVEP on $\lambda \in \sigma_w(M_0) \setminus \sigma_{SF_+}(A)$ and B^* has SVEP on $\mu \in \sigma_w(M_0) \setminus \sigma_{SF_+}(B)$, then M_0 satisfies $a - Bt$ implies M_C satisfies $a - Bt$.

As a consequence one has:

Corollary 1.2 *(a)[4, Proposition 4.1] If $\{\Xi(A) \cap \Xi(B^*)\} \cup \Xi(A^*) = \emptyset$, then M_0 satisfies Bt (resp., $a - Bt$) implies M_C satisfies Bt (resp., $a - Bt$).*

(b) [2, Theorem 3.2] If either $\sigma_{aw}(A) = \sigma_{SF_+}(B)$ or $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B) = \emptyset$, then M_0 satisfies Bt (resp., $a - Bt$) implies M_C satisfies Bt (resp., $a - Bt$).

Both Theorem 1.1 and Corollary 1.2 are a particular case of the following theorem.

Theorem 1.3 *(i) If $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then M_0 satisfies Bt if and only if M_C satisfies Bt.*
(ii) If $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, then M_0 satisfies $a - Bt$ implies M_C satisfies $a - Bt$. If in addition either A^* or B has SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$, then M_C satisfies $a - Bt$ if and only if M_0 satisfies $a - Bt$.

Here, we observe that if $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then $\sigma_w(M_0) = \sigma_w(M_C)$ and $\sigma(M_C) = \sigma(M_0)$; the hypothesis $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ implies that $\sigma_{aw}(M_C) = \sigma_{aw}(M_0)$ (and $\sigma(M_C) = \sigma(M_0)$). Observe also that if B has SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$ and M_C satisfies $a-Bt$, then A and B have SVEP on $\sigma_a(M_C)\setminus\sigma_{aw}(M_C)$; if A^* has SVEP on $\sigma_a(M_C)\setminus\sigma_{aw}(M_C)$, then $\sigma_a(M_C) = \sigma_a(A) \cup \sigma_a(B)$.

Let $\pi_0(M_C) = \{\lambda \in \mathrm{iso}\sigma(M_C) : 0 < \dim\ker(M_C - \lambda)^{-1}(0) < \infty\}$, and let $\pi_0^a(M_C) = \{\lambda \in \mathrm{iso}\sigma_a(M_C) : 0 < \dim\ker(M_C - \lambda)^{-1}(0) < \infty\}$. An operator T is said to be *polaroid* (resp., *a-polaroid*) at a points $\lambda \in \mathrm{iso}\sigma(T)$ (resp., $\lambda \in \mathrm{iso}\sigma_a(T)$) if λ is a pole of the resolvent of T (resp., $(T-\lambda)\mathcal{X}$ is closed and $\mathrm{asc}(T-\lambda) < \infty$). Let $\mathcal{R}_0(T) = \{\lambda \in \mathrm{iso}\sigma(T) : \lambda$ is a finite rank pole of the resolvent of $T\}$ and $\mathcal{R}_0^a(T) = \{\lambda \in \mathrm{iso}\sigma_a(T) : T - \lambda \in \Phi_+(\mathcal{X}), \mathrm{asc}(T-\lambda) < \infty\}$.

In common with current terminology, we say that T satisfies *Weyl's theorem, or Wt* (resp., *a-Weyl's theorem, or $a - Wt$*) if $\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$ (resp., $\sigma_a(T) \setminus \sigma_{aw}(T) = \pi_0^a(T)$).

The following result is not difficult to prove:

Theorem 1.4 *(a). M_C satisfies Wt if and only if M_C has SVEP at $\lambda \notin \sigma_w(M_C)$ and M_C is polaroid at points $\mu \in \pi_0(M_C)$.*

(b). M_C satisfies $a - Wt$ if and only if M_C has SVEP at $\lambda \notin \sigma_{aw}(M_C)$ and M_C is a-polaroid at points $\mu \in \pi_0^a(M_C)$.

Combining Theorems 1.1 and 1.4, an additional well known argument, see [5] and [6], implies the following:

Theorem 1.5 *[5, Theorem 3.7] If either of the SVEP hypotheses (i), (ii) and (iii) of Theorem 1.1(a) is satisfied, then M_C satisfies Wt for every $C \in B(\mathcal{X})$ if and only if M_0 satisfies Wt and A is polaroid at $\lambda \in \pi_0(M_C)$.*

Theorem 1.5 implies the following:

Corollary 1.6 *(a) [4, Theorem 4.2] If $\{\Xi(A) \cap \Xi(B^*)) \cup \Xi(A^*) = \emptyset$, A is polaroid at $\lambda \in \pi_0(M_C)$ (or A is isoloid and satisfies Wt) and M_0 satisfies Wt, then M_C satisfies Wt.*

(b) [2, Theorem 3.3] If $\sigma_{aw}(A) = \sigma_{SF_+}(B)$ or $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B) = \emptyset$, A is polaroid at $\lambda \in \pi_0(M_C)$ (or A is isoloid and satisfies Wt) and M_0 satisfies Wt, then M_C satisfies Wt.

Both Theorem 1.5 and Corollary 1.6 are subsumed by the following general result.

Theorem 1.7 *If $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then M_0 satisfies $Wt \Longleftrightarrow M_C$ satisfies Wt if and only if $\mathcal{R}_0(M_0) = \pi_0(M_C)$.*

The proof of the theorem is a straightforward consequence of the facts that $Wt \Longrightarrow Bt$, and $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B) \Longrightarrow \sigma_w(M_0) = \sigma_w(M_C)$ and $\sigma(M_C) = \sigma(M_0)$.

The result corresponding to Theorem 1.7 for $a - Wt$ is the following:

Theorem 1.8 *(i). If $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$, then M_0 satisfies $a - Wt$ implies M_C satisfies $a - Wt$ if and only if $\pi_0^a(M_C) \subseteq \pi_0^a(M_0)$.*
(ii). Conversely, if $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$ and A^* has SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$, then M_C satisfies $a - Wt$ implies M_0 satisfies $a - Wt$ if and only if $\pi_0^a(M_0) \subseteq \pi_0^a(M_C)$.

Theorem 1.8 implies in particular that:

Corollary 1.9 *[5, Theorem 3.11] If $\Xi(A^*) \cup \Xi(B^*) = \emptyset$, A is polaroid at $\lambda \in \pi_0^a(M_C)$ (or, A is isoloid and satisfies Wt) and B is polaroid at $\mu \in \pi_0^a(B)$, then M_C satisfies $a - Wt$.*

We note here that the hypothesis $\Xi(A^*) \cup \Xi(B^*) = \emptyset$ implies that a-poles of A and B are indeed poles of their respective resolvents.

It is well known that if a Banach space operator T is such that T^* has SVEP, then T satisfies Wt if and only if T satisfies $a - Wt$. Observe that a sufficient condition for ${M_0}^*$ and ${M_C}^*$ to have SVEP is that both A^* and B^* have SVEP. More generally:

Theorem 1.10 *Let $M_X = M_0$ or M_C. If A^* has SVEP on $\sigma(A) \setminus \sigma_{SF_+}(A)$ and B^* has SVEP on $\sigma(B) \setminus \sigma_{SF_+}(B)$, then M_X satisfies $Wt \Longleftrightarrow M_X$ satisfies $a - Wt$.*

Although T has SVEP and satisfies Wt does not guarantee T^* satisfies $a - Wt$, we do have that if T has SVEP and is polaroid, then T satisfies Wt and T^* satisfies $a - Wt$. This leads us to: *if A and B have SVEP and are polaroid, then M_C satisfies Wt and M_C^* satisfies $a - Wt$.*

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