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ON A FREE BOUNDARY PROBLEM OF THE COUPLED NAVIER-STOKES / MEAN CURVATURE EQUATIONS

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1. Introduction and formulation

We are interested in a free boundary problem of viscous incompressible flows as follows. We shall consider the Navier-Stokes systems:

$$
\text{(NS)} \quad \begin{cases} \partial_t u - \Delta u + (u, \nabla) u + \nabla p = \sigma_1 H \nu \mathcal{H}^{n-1}_{\llcorner \Gamma_t}, \ 0 < t \leq T, \ x \in \mathbb{R}^n, \\ \nabla \cdot u = 0, \ 0 < t \leq T, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^n, \end{cases}
$$

where $u = (u_1, \cdots, u_n)$ and p are unknown velocity field and pressure field, respectively. The symbol Γ_t represents an unknown free interface evolving from the initial interface Γ_0 which is the boundary of a bounded domain Ω_0. The positive constant σ_1 represents the surface tension, and H, ν are the mean curvature, the exterior unit normal vector of Γ_t, respectively. The symbol $\mathcal{H}^{n-1}_{\llcorner \Gamma_t}$ means the $n-1$ dimensional Hausdorff measure restricted on Γ_t.

We assume that the free interface is given by $\Gamma_t = \{x(t, x_0) \in \mathbb{R}^n \ ; \ x_0 \in \Gamma_0\}$ where $x(t, x_0)$ is the solution of the ODE:

$$
\text{(BC)} \quad \begin{cases} \dfrac{dx(t)}{dt} = u(t, x(t)) + \sigma_2 H(t, x(t)) \nu(t, x(t)), \ 0 < t \leq T, \\ x(0) = x_0 \in \Gamma_0, \end{cases}
$$

where σ_2 is a fixed positive constant.

The right hand side of the first equation in (NS) is the free boundary condition taken into account in weak sense. That is, the term $\sigma_1 H \nu \mathcal{H}^{n-1}_{\llcorner \Gamma_t}$ is formally equivalent to the free boundary condition

$$
[(-p\delta_{ij} + \partial_j u_i + \partial_i u_j)_{1 \leq i,j \leq n}]_{\Gamma_t} \nu = \sigma_1 H \nu,
$$

where $[\cdot]_{\Gamma_t}$ expresses the jump across the interface Γ_t.

This report is a continuation of the author's paper [15], in which the fluid motion is assumed to be described by the Stokes equations instead of the Navier-Stokes equations.

Our problem is motivated by the phase transition of materials in a flowing fluid. That is, the motion of the phase is not only governed by its mean curvature but also convected by the fluid velocity. The motion of the fluid is also influenced by the interface, which is represented by the free boundary condition. The coupled equations for fluid motion and the phase transition have lately attracted considerable attention. Gurtin-Polignore-Viñals [9] and Liu-Shen [13] considered the coupled Navier-Stokes/Cahn-Hilliard equations. In [2] Blesgen formulated the coupled compressible Navier-Stokes/Allen-Cahn equations. Feng-He-Liu [6] and Tan-Lim-Khoo [27] discussed the system of the Stokes / Allen-Cahn equations; see also [13]. Especially, [6] is closely related with our problem, since it is formally derived through the singular interface limit of the system considered in [6].

Our model is also related with the following two phase Navier-Stokes flows problem (in weak form)

(TP)

$$\begin{cases} \partial_t u - \nabla \cdot T(\kappa Du, p) + (u, \nabla)u = \sigma_1 H\nu \mathcal{H}^{n-1}_{\llcorner \Gamma_t}, \ 0 < t \leq T, \ x \in \mathbb{R}^n, \\ \nabla \cdot u = 0, \ 0 < t \leq T, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^n, \end{cases}$$

(BC')

$$\begin{cases} \dfrac{dx(t)}{dt} = u(t, x(t)), \ 0 < t \leq T, \ x(t) \in \Gamma_t, \\ x(0) = x_0 \in \Gamma_0, \end{cases}$$

where $T(\kappa Du, p) := 2\kappa_1 \chi_{\Omega_t} Du + 2\kappa_2(1 - \chi_{\Omega_t})Du - pI$ is the stress tensor, $2Du = (\partial_j u_i + \partial_i u_j)_{1 \leq i,j \leq n}$ is the deformation tensor, $\kappa_i > 0$ are viscosity coefficients of fluids, and Ω_t is a bounded domain with $\Gamma_t = \partial\Omega_t$. The function χ_{Ω_t} is the characteristic function of Ω_t.

Then our problem can be regarded as the relaxation of the problem (TP), since the viscosities and the densities of the two fluids are assumed to be the same value and the term $\sigma_2 H\nu$ in the kinematic boundary condition has a regularizing effect for the interface. Such relaxation in the kinematic boundary condition is used as the level set methods in numerical analysis; see Chang-Hou-Merriman-Osher [3]. The advantage of this method (or the phase-field method in [9, 13, 2, 6, 27]) is that one can capture the interface even when it develops singularities such as merging and reconnection.

Since u satisfies the divergence free condition in whole space, we have from (NS),

$$\partial_t u - \Delta u + \mathbf{P}\nabla \cdot u \otimes u = \mathbf{P}\sigma_1 H\nu \mathcal{H}^{n-1}_{\llcorner \Gamma_t}, \tag{1.1}$$

where $\mathbf{P} = (R_i R_j)_{1\le i,j\le n} + I$ is the Helmholtz projection, and $R_j = \partial_j(-\Delta)^{-\frac{1}{2}}$ is the Riesz transformation. One can check that the term $\mathbf{P}\sigma_1 H\nu\mathcal{H}^{n-1}_{\llcorner\Gamma_t}$ is well-defined at least in the class of tempered distributions if the hypersurface Γ_t is a smooth boundary of a bounded domain.

In this paper we shall construct the velocity field as the mild solution of the equation (1.1), that is, the integral equation associated with (1.1). Thus we shall consider the system as follows.

(FBP)

$$\begin{cases} u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla\cdot u\otimes u ds + \int_0^t e^{(t-s)\Delta}\mathbf{P}\sigma_1 H\nu\mathcal{H}^{n-1}_{\llcorner\Gamma_s} ds, \\ (BC). \end{cases}$$

Here, $e^{t\Delta}$ is the heat semigroup. We assume that u_0 belongs to the class of α-Hölder continuous functions ($= C^\alpha(\mathbb{R}^n)$) and Γ_0 is a $C^{2+\alpha}$ hypersurface for some $\alpha\in(0,1)$. Our aim is to construct the pair $(u, \{\Gamma_t\}_{0\le t\le T})$ solving (FBP) with initial data (u_0, Γ_0).

We say that a family of hypersurfaces $\{\Gamma_t\}_{0\le t\le T}$ belongs to $C^{1,2+\alpha}$ when the signed distance function of Γ_t belongs to $C^{1,2+\alpha}$ in a neighborhood of $\{\Gamma_t\}_{0\le t\le T}$. The precise definition will be given in Section 3.

Now the main result of this paper is as follows.

Theorem 1.1 (Existence and uniqueness).
Let $\alpha\in(0,1)$. Assume that $u_0\in C^\alpha(\mathbb{R}^n)$ with $\nabla\cdot u_0 = 0$ and Ω_0 is a bounded domain with $C^{2+\alpha}$ boundary. Let $\Gamma_0 = \partial\Omega_0$. Then there exists a positive T such that there is a unique solution $(u, \{\Gamma_t\}_{0\le t\le T})$ solving (FBP) with initial data (u_0, Γ_0) satisfying that $u\in C^{\frac{\alpha}{2},\alpha}([0,T]\times\mathbb{R}^n)$ and $\{\Gamma_t\}_{0\le t\le T}$ belongs to $C^{1,2+\alpha}$.

As far as the author knows, there are few mathematical results for the free boundary problems in the presence of the term $\sigma_2 H\nu$ in (BC). But under the kinematic boundary condition (BC'), there is much literature for the free boundary problems of viscous incompressible (Navier-Stokes) flows with or without surface tension.

Solonnikov [25] and Shibata-Shimizu [23] proved the local well-posedness in Sobolev spaces for one phase flow problems without surface tension. Mogilevskii-Solonnikov [17] showed the local well-posedness in Hölder spaces for one face flow problems with surface tension; see also Solonnikov [26], Shibata-Shimizu [24]. Denisova [4] and Tanaka [29] studied the two phase flows problems in the Sobolev-Slobodetskii spaces. It is also known that the global solvability holds near the equilibrium states for one or two phase flows problems; see Padula-Solonnikov [25] and Tanaka [28].

In these papers regular solutions are considered and Lagrangian coordinates are used in order to reduce the problem to the case of a fixed domain. But in our problem such reduction is less useful because of the term $\sigma_2 H\nu$ in (BC). So we shall deal with the equation directly as in the formulation (FBP), and the free boundary condition appears in the layer potential term. Although the term $\sigma_2 H\nu$ could lead to more complicated interactions between the interface and the fluid velocity, we have a mathematical advantage such that we do not need the compatibility conditions between the boundary and the initial data. We remark that such compatibility conditions are required in the above papers.

Let us comment on weak solutions of two phase flows problem. Giga-Takahashi [8] studied two phase Stokes flows, and Nouri-Poupaud-Demay [19] studied the multi-phase flows. Both papers deal with the case without surface tension. In Plotnikov [21], Nespoli-Salvi [18], and Abels [1], the case with surface tension is discussed. However, if surface tension is present, the existence of weak solutions is still open even for the Stokes flows, and only measure-valued varifold solutions or varifold solutions are obtained; see [21], [1] for details.

Now let us state the main idea and the outline of the proof for the main theorem. As the first step, for a given u in an appropriate class of functions, we shall construct the family of hypersurfaces evolving by the equation in (BC). Since it is regarded as the mean curvature equation with the perturbation term u, we will follow the arguments of Evans-Spruck [5] (see also A. Lunardi [14] and Giga-Goto [7]), which reduces the equation to the one for the signed distance function of interfaces.

Next, for a given family of hypersurfaces, we estimate the layer potential term in the integral equation in (FBP). The main difficulty is that we cannot expect high regularity for u in whole space (for example, we cannot expect $u(t) \in C^{1+\alpha}(\mathbb{R}^n)$ in general) because of the jump relation of the layer potential. However, in order to obtain a unique regular solution for the perturbed mean curvature equation in (BC), we need the regularity for the perturbation term u such as $u(t) \in C^{1+\alpha}(\mathbb{R}^n)$. To overcome these difficulties, we make use of the regularity for u in tangential directions to the interface. More precisely, if each interface has $C^{2+\alpha}$ regularity (and suitable regularity with respect to time), we have the optimal regularity for the layer potential term such as $C^{1+\alpha}$ in tangential directions. In order to establish this optimal regularity, we use the Hölder-Zygmund spaces. The desired result in the main theorem is obtained by constructing a suitable contraction mapping for velocity fields.

This report is organized as follows. In Section 2 we give the definitions of function spaces. In Section 3 we collect the results in [15], in which the mean curvature equation with a convection term is solved and the estimates for the layer potential term in (FBP) are established. In Section 4 we solve the integral equation for the Navier-Stokes equations with a given layer potential term. In Section 5 we shall construct a suitable contraction mapping and obtain the desired results.

2. Function spaces and embedding properties

First of all, we introduce several function spaces in which we deal with the problems. Let D be either $\mathbb{R}^n$ or an open set in $\mathbb{R}^n$ with uniformly C^2 boundary. Let $C(\overline{D})$ denote the Banach space of all continuous and bounded functions in $\overline{D}$, endowed with the sup norm. Let $C^m(\overline{D})$ denotes the set of all m times continuously differentiable functions in D, with derivatives up to the order m bounded and continuously extendable up to the boundary. The norm of $C^m(\overline{D})$ is defined as

$$||f||_{C^m(\overline{D})} := \sum_{0\le k\le m} ||\partial_x^k f||_{C(\overline{D})}$$

$$||\partial_x^k f||_{C(\overline{D})} := \sum_{|\theta|=k} ||\partial_x^\theta f||_{C(\overline{D})}.$$

Here, $\theta = (\theta_1, \cdots, \theta_n)$ is a multi-index. We recall that $C([a,b]\times\overline{D})$ is the space of all the continuous and bounded functions in $[a,b]\times\overline{D}$, endowed with the norm

$$||f||_{C([a,b]\times\overline{D})}(= ||f||_\infty) := \sup_{(t,x)\in[a,b]\times\overline{D}} |f(t,x)|.$$

For $0<\alpha<1$, we denote by $C^{0,\alpha}([a,b]\times\overline{D})$ (respectively, $C^{\frac{\alpha}{2},0}([a,b]\times\overline{D})$) the space of continuous functions that are α-Hölder continuous with respect to the space variables (respectively, $\frac{\alpha}{2}$-Hölder continuous with respect to time), i.e.,

$$C^{0,\alpha}([a,b]\times\overline{D}) := \{f\in C([a,b]\times\overline{D});\ f(t,\cdot)\in C^\alpha(\overline{D}),\ t\in[a,b]\},$$
$$||f||_{C^{0,\alpha}([a,b]\times\overline{D})}(= ||f||_{C^{0,\alpha}}) := ||f||_\infty + \sup_{t\in[a,b]}[f(t,\cdot)]_{C^\alpha(\overline{D})},$$

where

$$[g]_{C^\alpha(\overline{D})} := \sup_{x,y\in\overline{D},x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\alpha}$$

(respectively,

$$C^{\frac{\alpha}{2},0}([a,b]\times\overline{D}) := \{f\in C([a,b]\times\overline{D});\ f(\cdot,x)\in C^{\frac{\alpha}{2}}([a,b]),\ x\in\overline{D}\},$$
$$||f||_{C^{\frac{\alpha}{2},0}([a,b]\times\overline{D})}(= ||f||_{C^{\frac{\alpha}{2},0}}) := ||f||_\infty + \sup_{x\in\overline{D}}[f(\cdot,x)]_{C^{\frac{\alpha}{2}}([a,b])},$$

where

$$[h]_{C^{\frac{\alpha}{2}}([a,b])} := \sup_{t,s\in[a,b],t>s} \frac{|h(t)-h(s)|}{(t-s)^{\frac{\alpha}{2}}}\ ).$$

Moreover, the function spaces $C^{\frac{\alpha}{2},\alpha}([a,b]\times\overline{D})$, $C^{1,2}([a,b]\times\overline{D})$, $C^{1,2+\alpha}([a,b]\times\overline{D})$, $C^{1+\frac{\alpha}{2},2+\alpha}([a,b]\times\overline{D})$ are defined as follows.

$$C^{\frac{\alpha}{2},\alpha}([a,b]\times\overline{D}) := C^{\frac{\alpha}{2},0}([a,b]\times\overline{D})\cap C^{0,\alpha}([a,b]\times\overline{D}),$$
$$||f||_{C^{\frac{\alpha}{2},\alpha}([a,b]\times\overline{D})}(=||f||_{C^{\frac{\alpha}{2},\alpha}}) := ||f||_{C^{\frac{\alpha}{2},0}([a,b]\times\overline{D})} + ||f||_{C^{0,\alpha}([a,b]\times\overline{D})}.$$

$$C^{1,2}([a,b]\times\overline{D}) := \{f\in C([a,b]\times\overline{D});\ \partial_t f,\ \partial_{ij}f\in C([a,b]\times\overline{D}),\ 1\le i,j\le n\},$$
$$||f||_{C^{1,2}([a,b]\times\overline{D})}(=||f||_{C^{1,2}}) := ||f||_\infty + ||\partial_x f||_\infty + ||\partial_t f||_\infty + ||\partial_x^2 f||_\infty.$$

$$C^{1,2+\alpha}([a,b]\times\overline{D}) := \{f\in C^{1,2}([a,b]\times\overline{D});\ \partial_t f,\partial_{ij}f\in C^{0,\alpha}([a,b]\times\overline{D}),\ 1\le i,j\le n\},$$
$$||f||_{C^{1,2+\alpha}([a,b]\times\overline{D})}(=||f||_{C^{1,2+\alpha}}) := ||f||_\infty + ||\partial_x f||_\infty + ||\partial_t f||_{C^{0,\alpha}} + ||\partial_x^2 f||_{C^{0,\alpha}}.$$

Let X be a Banach space endowed with the norm $||\cdot||_X$. We denote by $C^\alpha([a,b];X)$ the Hölder space such that

$$\begin{aligned} C^\alpha([a,b];X) \ &:=\ \{f\in C([a,b];X);\ [f]_{C^\alpha([a,b];X)} := \sup_{t,s\in[a,b],t>s}\frac{||f(t)-f(s)||_X}{(t-s)^\alpha}, \\ &\qquad ||f||_{C^\alpha([a,b];X)} := \sup_{a\le t\le b}||f(t)||_X + [f]_{C^\alpha([a,b];X)} < \infty\}. \end{aligned}$$

Similarly,

$$\begin{aligned} Lip([a,b];X) \ &:=\ \{f\in C([a,b];X);\ [f]_{Lip([a,b];X)} := \sup_{t,s\in[a,b],t>s}\frac{||f(t)-f(s)||_X}{t-s}, \\ &\qquad ||f||_{Lip([a,b];X)} := \sup_{a\le t\le b}||f(t)||_X + [f]_{Lip([a,b];X)} < \infty\}. \end{aligned}$$

Now we state the embedding properties of the Hölder spaces defined above. The following lemma will be used freely in this paper.

Lemma 2.1. *Let $0<\alpha<1$. Then there exists a positive constant K_α such that for any $f\in C^{1,2+\alpha}([a,b]\times\overline{D})$,*

$$\begin{aligned} (2.1)\quad & ||f||_{C^{\frac{1}{2}}([a,b];C^{1+\alpha}(\overline{D}))} + ||f||_{Lip([a,b];C^\alpha(\overline{D}))} + ||\partial_x f||_{C^{\frac{1+\alpha}{2},0}} + ||\partial_x^2 f||_{C^{\frac{\alpha}{2},0}} \\ \le\ & K_\alpha||f||_{C^{1,2+\alpha}} \end{aligned}$$

holds. Here, the constant K_α is independent of $b-a$ and f.

Proof. See Lunardi [14, Lemma 5.1.1].

3. Several key estimates in [15]

In this section we recall several results obtained by [15].

3.1. **Motion of hypersurfaces by mean curvature with a convection term.** We consider the hypersurfaces evolving in time via mean curvature with a convection term. More precisely, we shall construct a family of hypersurfaces $\{\Gamma_t\}_{0\le t\le T}$ such that for $0 \le t_0 \le t \le T$, $\Gamma_t = \{x(t,x_0); x_0 \in \Gamma_{t_0}\}$ satisfies the ODE

(3.1)
$$\begin{cases} \dfrac{dx(t)}{dt} &= -\dfrac{\sigma_2}{n-1}[\text{div }(\nu(t,x(t))]\nu(t,x(t)) + u(t,x(t)), \ t_0 \le t \le T, \\ x(t_0) &= x_0. \end{cases}$$

Here $\nu(t,x)$ is the exterior unit normal vector of Γ_t, σ_2 is a positive constant, and $u(t,x)$ is a continuous function on $[0,T] \times \mathbb{R}^n$. The mean curvature $H(t,x)$ of the surface Γ_t is given by $H(t,x) = -\frac{1}{n-1}\text{div } \nu(t,x)$. So if $u \equiv 0$, the above equation is the well-known mean curvature flow equation. To construct an evolving hypersurfaces starting from a given smooth initial hypersurfaces, we will follow the arguments of Evans-Spruck [5]; see also Lunardi [14]. Let $\{\Gamma_t\}_{0\le t\le T}$ be the evolving hypersurfaces such that each Γ_t is the boundary of a bounded domain Ω_t. We reduce the equation to an equation for the signed distance function

$$d(t,x) = \begin{cases} \text{dist } (x,\Gamma_t), & x \in \mathbb{R}^n \setminus \overline{\Omega_t}, \\ -\text{dist } (x,\Gamma_t), & x \in \Omega_t. \end{cases} \tag{3.2}$$

If Γ_t is smooth, then the above $d(t,\cdot)$ is also smooth in the set

$$D^+ := \{x \in \mathbb{R}^n \ ; \ 0 \le d(t,x) < \delta_0\}$$

and

$$D^- := \{x \in \mathbb{R}^n \ ; \ -\delta_0 < d(t,x) \le 0\},$$

provided $\delta_0 > 0$ and $T > 0$ is small. Moreover, if δ_0 is sufficiently small, for each $x \in D^+$ there exists a unique $y \in \Gamma_t$ such that $d(t,x) = |y-x|$. The equation (3.1) implies that

$$\begin{aligned} d_t(t,x) &= < \frac{dy}{dt}, \frac{y-x}{|y-x|} > \\ &= < -\frac{\sigma_2}{n-1}[\text{div } \nu(t,y)]\nu(t,y) + u(t,y), \frac{y-x}{|y-x|} > \\ &= \frac{\sigma_2}{n-1}\text{div } \nu(t,y) - u(t,x-d\nabla_x d(t,x)) \cdot \nabla_x d(t,x). \end{aligned}$$

It is well-known that the eigenvalues of the Hessian $\nabla^2 d(t,x)$ are given by

$$\lambda_i = -\frac{\kappa_i(t,y)}{1-\kappa_i(t,y)d(t,x)}, \ i = 1, \cdots, n-1, \ \lambda_n = 0, \tag{3.3}$$

where κ_i are the principal curvatures of the surface Γ_t. Since the mean curvature H is defined as $H = \frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i$, we have

$$d_t = \frac{\sigma_2}{n-1} f(d, \nabla^2 d) - u(t, x - d\nabla d)\cdot \nabla d, \tag{3.4}$$

where

$$f(s,q) = \sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_i s}, \quad s\in \mathbb{R},\ q\in \mathbb{R}_S^{n\times n},\ \lambda_i s \neq 1. \tag{3.5}$$

Here λ_i are the eigenvalues of the symmetric matrix q. The same equation can be deduced for $x \in D^-$. Since $|d|$ is a distance function, the spatial gradient ∇d should have modulus 1 at any point. This provides a nonlinear first order boundary condition for d. So the equation (3.1) is reduced to the following fully nonlinear parabolic problem

$$\begin{cases} \partial_t v = \dfrac{\sigma_2}{n-1} f(v,\nabla^2 v) - u(t, x - v\nabla v)\cdot\nabla v, \ t\geq 0,\ x\in \overline{D}, \\ |\nabla v|^2 = 1,\ t\geq 0,\ x\in\partial D, \\ v(0,x) = d_0(x),\ x\in\overline{D}, \end{cases} \tag{3.6}$$

where $D = D^+\cup D^- = \{x\in\mathbb{R}^n, -\delta_0 < d_0(x) < \delta_0\}$, d_0 is the signed distance function from Γ_0, and f is given as above. We choose δ_0 so small that $\lambda_i(\nabla^2 d_0)\delta_0 \neq 1$ for each i, so f is well-defined near the range of $(d_0(x), \nabla^2 d_0(x))$. Since $f(s,q) = \mathrm{Tr}\,(q(I-sq)^{-1})$, f is analytic. Moreover, since $\mathrm{Tr}\,(\frac{\partial f}{\partial q}(s,q)A) = \mathrm{Tr}\,((I-sq)^{-2}A)$ for $A\in\mathbb{R}^{n\times n}$, we have for $\xi\in\mathbb{R}^n$,

$$\begin{aligned} \sum_{i,j=1}^{n} f_{q_{ij}}(s,q)\xi_i\xi_j &= \mathrm{Tr}(\frac{\partial f}{\partial q}(s,q)\xi\otimes\xi) \\ &= \sum_{i=1}^{n}\frac{1}{(1-\lambda_i s)^2} < \xi, \bar{e}_i >^2, \end{aligned}$$

where $\{\bar{e_1},\cdots,\bar{e_n}\}$ is an orthogonal basis in $\mathbb{R}^n$ such that each $\bar{e}_i$ is an eigenvector of q with eigenvalue λ_i. Thus we have

$$\sum_{i,j=1}^{n} f_{q_{ij}}(s,q)\xi_i\xi_j \geq \iota(s,q)|\xi|^2, \tag{3.7}$$

with $\iota(s,q) = \min_{1\leq i\leq n}(1-\lambda_i s)^{-2}$.

Set $g(p) = p^2 - 1$. In order to solve the equation (3.6), we linearize the principal term $f(v,\nabla^2 v)$ near the initial data $(d_0, \nabla^2 d_0)$ and $g(\nabla d_0)$ near (∇d_0). The existence and uniqueness results of the equation is proved by the general results for the linear parabolic equations and the usual contraction arguments. Let $B(d_0,\nabla^2 d_0)$ be a bounded open neighborhood of the set $\{(d_0(x), \nabla^2 d_0(x)) \in \mathbb{R} \times \mathbb{R}^{n\times n} \ ; \ x \in \overline{D}\}$ such that for each $(s,q) \in B(d_0, \nabla^2 d_0)$, the function $f(s,q)$ is well-defined. Set

$$\begin{aligned}
(3.8) \quad \iota &:= \inf\{\iota(s,q) \ ; \ (s,q) \in B(d_0, \nabla^2 d_0)\} > 0 \\
(3.9) K_f &:= \sup\{|\frac{\partial^\beta f}{\partial s \partial q}(s,q)| \ ; \ (s,q) \in B(d_0, \nabla^2 d_0), \ |\beta| = 0,1,2\}.
\end{aligned}$$

Fix $M > 0$. We assume that the perturbation term $u(t,x)$ belongs to $\mathcal{U}_M$, the closed subset of $C^{0,\alpha}([0,T] \times \mathbb{R}^n)$, defined as

$$\begin{aligned}
(3.10) \mathcal{U}_M := \ & \{u(t,x) \in C^{0,\alpha}([0,T] \times \mathbb{R}^n); u(t,\cdot) \in C^{1+\alpha}(\mathbb{R}^n), \mathit{and} \\
& \sup_{0<t<T} ||u(t,\cdot)||_{C^\alpha(\mathbb{R}^n)} + \sup_{0<t<T} t^{\frac{1-\alpha}{2}} ||\partial_x u(t,\cdot)||_{C(\mathbb{R}^n)} \\
& + \sup_{0<t<T} t^{\frac{1}{2}} [\partial_x u(t,\cdot)]_{C^\alpha(\mathbb{R}^n)} \leq M\}
\end{aligned}$$

The following proposition states the existence and uniqueness of the equation (3.6).

Proposition 3.1 ([15]). *Fix $M > 0$. Let $\alpha \in (0,1)$. Assume that Ω_0 is a bounded domain with uniformly $C^{2+\alpha}$ boundary and let d_0 be the signed distance function from $\Gamma_0 = \partial\Omega_0$. Then there is some $T > 0$ such that for any $u \in \mathcal{U}_M$, there exists a unique $v \in C^{1,2+\alpha}([0,T] \times \overline{D})$, solution of (3.6). Moreover, the solution v satisfies*

$$(3.11) \qquad ||v||_{C^{1,2+\alpha}([0,T]\times\overline{D})} \leq ||d_0||_{C^{2+\alpha}} + 2C(||d_0||_{C^{2+\alpha}}, M),$$

$$(3.12) \qquad ||\partial_x v||_{C^{1,2+\alpha}([t_1,t_2]\times\overline{D'})} \leq C(\frac{(t_2-t_1)^{\frac{1}{2}}}{t_1} + t_1^{-\frac{1}{2}}),$$

for any open set $D' \subset\subset D$ and $0 < t_1 < t_2 \leq T$. Especially, the existence time of the solution does not depend on each $u \in \mathcal{U}_M$, and Γ_t is a $C^{3+\alpha}$ hypersurface for each $0 < t \leq T$. Hence, this $\{\Gamma_t\}_{0\leq t\leq T}$ is a unique family of $C^{2+\alpha}$ hypersurfaces evolving by the perturbed mean curvature equation (3.1) starting from Γ_0.

Proof. See [15, Proposition 3.1].

Let $u(t,x)$ and $\tilde{u}(t,x)$ be two functions in $\mathcal{U}_M$. Let v, $\tilde{v} \in C^{1,2+\alpha}([0,T]\times \overline{D})$ be solutions of the equation (3.6) with initial data $v(0,x) = \tilde{v}(0,x) = d_0(x)$ and with velocity fields u, $\tilde{u}$, respectively. Note that for fixed $M > 0$ and d_0, the above T can be taken uniformly in u belonging to $\mathcal{U}_M$. In order to solve (FBP) we need the following

Proposition 3.2 ([15]). *Fix $M > 0$. Let $\alpha \in (0,1)$. Assume that Ω_0 be a bounded domain with uniformly $C^{2+\alpha}$ boundary and let d_0 be the signed distance function from $\Gamma_0 = \partial\Omega_0$. Let u, $\tilde{u}$, v, $\tilde{v}$ be functions defined* *above. Then it follows that*

$$\|v - \tilde{v}\|_{C^{1,2+\alpha}([0,T]\times\overline{D})} \le C\|u - \tilde{u}\|_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}, \tag{3.13}$$

where C depends only on n, α, ι, M, σ_2, K_f, K_α, and $\|d_0\|_{C^{2+\alpha}}$.

Proof. See [15, Proposition 3.3].

3.2. **Estimates for layer potential.** In this section, we shall recall the estimates of the term

$$F(t,x) := \int_0^t e^{(t-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}ds, \tag{3.14}$$

which reflects the boundary condition on Γ_t when $h = H\nu$. First, we define the class of the evolving hypersurfaces which we deal with. Let Γ_0 be a boundary of a smooth bounded domain Ω_0. Let d_0 be the signed distance function of Γ_0

$$d_0(x) = \begin{cases} \text{dist }(x,\Gamma_0), & x \in \mathbb{R}^n \setminus \overline{\Omega_0}, \\ -\text{dist }(x,\Gamma_0), & x \in \Omega_0. \end{cases} \tag{3.15}$$

We set

$$D := \{x \in \mathbb{R}^n \ ; \ -\delta_0 < d_0(x) < \delta_0\} \tag{3.16}$$

for sufficiently small δ_0. We assume that Γ_0 is uniformly $C^{2+\alpha}$, that is, $d_0 \in C^{2+\alpha}(\overline{D})$. Since d_0 is a distance function, we have $|\partial_x d_0(x)| \equiv 1$ on $x \in \overline{D}$.

Definition 3.1. *Let $R \ge 1$ be a given number and $\alpha \in (0,1)$. We define the set $\mathcal{S}(\alpha, R, T, d_0)$ as the set of families of hypersurfaces $\{\Gamma_t\}_{0\le t\le T}$ such that each Γ_t is a boundary of a bounded domain $\Omega_t \subset \mathbb{R}^n$ and represented as*

$$\Gamma_t = \{x \in D; \ v(t,x) = 0\} \tag{3.17}$$

by the signed distance function $v \in C^{1,2+\alpha}([0,T]\times\overline{D})$ satisfying $\|v\|_{C^{1,2+\alpha}([0,T]\times\overline{D})} \le R$ and $v(0,x) = d_0(x)$.

The following estimates for the layer potential play essential roles.

Proposition 3.3 ([15]). *Let $p \in (1,\infty]$, $\alpha, \beta \in (0,1)$. Assume that $R \ge 1$ is a given number and Γ_0 is a given $C^{2+\alpha}$ hypersurface. Let d_0 be the signed distance function and let $\{\Gamma_t\}_{0\le t\le T}$ be an evolving hypersurface belonging to $\mathcal{S}(\alpha, R, T, d_0)$. Then for sufficiently small $T > 0$ the function $F(t,x)$ given as (3.14) satisfies the following estimates.*

$$\|F\|_{C^{\frac{\beta}{2},\beta}([0,T]\times\mathbb{R}^n)} \le C_1 T^{\frac{1-\beta}{2}} \|h\|_{C([0,T]\times\overline{D})}, \tag{3.18}$$

$$\sup_{0\le t\le T} \|F(t)\|_{L^p(\mathbb{R}^n)} \le C_2 T^{\frac{1}{2}} \|h\|_{C([0,T]\times\overline{D})}, \tag{3.19}$$

$$\sup_{0\le t\le T} \|F(t)\|_{C^{1+\alpha}(\Gamma_t)} \le C_3 \|h\|_{C^{0,\alpha}([0,T]\times\overline{D})}, \tag{3.20}$$

where $C_1 = C_1(n,\beta,r,R)$, $C_2 = C_2(n,p,r,R)$, *and* $C_3 = C_3(n,\alpha,r,R)$.

Proof. See [15, Proposition 4.1].

4. Mild solutions of the Navier-Stokes equations

In this section we shall construct the mild solution of the Navier-Stokes equation with initial velocity $u_0 \in C^\alpha(\mathbb{R}^n)$ and with a term of the layer potential $h\mathcal{H}^{n-1}_{\llcorner\Gamma_t}$ for convenience to reader. Thanks to the estimates for the layer potential term stated in the previous section, we can obtain the appropriate regularity for solutions in tangential directions to Γ_t. We recall that the mild solution of the Navier-Stokes equations is the solution of the integral equation

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla\cdot u\otimes u ds + \int_0^t e^{(t-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s} ds.$$

Let $\alpha \in (0,1)$. Assume that $u_0 \in C^\alpha(\mathbb{R}^n)$ satisfies $\nabla\cdot u_0 = 0$ and d_0 is the distance function of a $C^{2+\alpha}$ hypersurface Γ_0. Let $R \ge 1$ be a given number and let $\{\Gamma_t\}_{0\le t\le T_1}$ be an evolving hypersurfaces belonging to $\mathcal{S}(\alpha, R, T_1, d_0)$. Then we have the following proposition.

Proposition 4.1. *There exists a positive* $T \le T_1$ *such that the mild solution* u *belonging to* $C^{\frac{\alpha}{2},\alpha}([0,T]\times\mathbb{R}^n)$ *uniquely exists. The existence time* T *can be taken uniformly in* $\mathcal{S}(\alpha,R,T_1,d_0)$. *Moreover, this solution satisfies the following estimates.*

$$\|u\|_{C^{\frac{\alpha}{2},\alpha}([0,T\times\mathbb{R}^n)} \le C\|u_0\|_{C^\alpha(\mathbb{R}^n)} + C_1 T^{\frac{1-\alpha}{2}} \|h\|_{C([0,T]\times\overline{D})}, \tag{4.1}$$

$$\begin{aligned}\sup_{0<t<T} t^{\frac{1-\alpha}{2}} \|u(t,\cdot)\|_{C^1(\Gamma_t)} + \sup_{0<t<T} t^{\frac{1}{2}} \|u(t,\cdot)\|_{C^{1+\alpha}(\Gamma_t)} \\ \le C\|u_0\|_{C^\alpha(\mathbb{R}^n)} + C_2 T^{\frac{1-\alpha}{2}} \|h\|_{C^{0,\alpha}([0,T]\times\overline{D})},\end{aligned} \tag{4.2}$$

where $C = C(n,\alpha)$, $C_1(n,\alpha,d_0,R)$, *and* $C_2(n,\alpha,d_0,R)$.

. *Proof.* We will follow the contraction argument by Kato [10]. Since this argument is well-known, we state only the outline of the proof. Set

$$F_0 := e^{t\Delta}u_0,$$
$$B(f,g) := \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla\cdot f\otimes g ds, \quad f,\ g\in C^{0,\alpha}([0,T]\times\mathbb{R}^n),$$
$$F := \int_0^t e^{(t-s)\Delta}\mathbf{P}h\mathcal{H}^{n-1}_{\llcorner\Gamma_s}ds.$$

From the pointwise estimate of the kernel $e^{t\Delta}\mathbf{P}\nabla\cdot$ in [16], it is not difficult to derive the estimate

$$(4.3)\qquad ||B(f,g)||_{C^{\frac{\alpha}{2},\alpha}([0,T]\times\mathbb{R}^n)} \le CT^{\frac{1}{2}}||f||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}||g||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}$$

where C depends only on n and α. Combining the estimates for the heat semigroup and Proposition 3.3, we easily see that

$$(4.4)\qquad G(w) := F_0 - B(w,w) + F$$

is a contraction mapping from the usual closed ball in $C^{\frac{\alpha}{2},\alpha}([0,T]\times\mathbb{R}^n)$ with radius $2(||F_0||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)} + ||F||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)})$ into itself for sufficiently small $T\le T_1$. This implies the time-local existence of the unique solution u. Note that the existence time can be taken uniformly in $\mathcal{S}(\alpha,R,T_1,d_0)$ since the constants in Proposition 3.3 do not depend on each family of hypersurfaces in $\mathcal{S}(\alpha,R,T_1,d_0)$. The estimate (4.1) is obvious. We shall show the estimate (4.2). Note that we have

$$||\partial_x F_0(t,\cdot)||_{C(\mathbb{R}^n)} \le Ct^{-\frac{1-\alpha}{2}}||u_0||_{C^\alpha},$$
$$[\partial_x F_0(t,\cdot)]_{C^\alpha(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}}||u_0||_{C^\alpha}.$$

As for the nonlinear term $B(f,g)$, we recall the estimates

$$||\nabla e^{t\Delta}\mathbb{P}\nabla\cdot f||_{C(\mathbb{R}^n)} \le Ct^{-1}||f||_{C(\mathbb{R}^n)},$$
$$||\nabla e^{t\Delta}\mathbb{P}\nabla\cdot f||_{C(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}}||\nabla f||_{C(\mathbb{R}^n)},$$

for any $f\in (C^1(\mathbb{R}^n))^{n\times n}$; for example, see [16, Corollary 3.1]. By interpolating these estimates, we have

$$(4.5)\qquad ||\nabla e^{t\Delta}\mathbb{P}\nabla\cdot f||_{C(\mathbb{R}^n)} \le Ct^{-1+\frac{\alpha}{2}}||f||_{C^\alpha}.$$

Hence we have

$$\begin{aligned}||\nabla B(f,g)(t,\cdot)||_{C(\mathbb{R}^n)} &\le C\int_0^t (t-s)^{-1+\frac{\alpha}{2}}||f\otimes g(s)||_{C^\alpha(\mathbb{R}^n)}ds\\ &\le Ct^{\frac{\alpha}{2}}||f||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}||g||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)},\end{aligned}$$

thus

$$||\nabla B(f,g)||_{C([0,T]\times\mathbb{R}^n)} \le CT^{\frac{\alpha}{2}}||f||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}||g||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}.$$

Next since the Helmholtz projection $\mathbf{P}$ is bounded in the homogeneous counterpart of $C^{\alpha}(\mathbb{R}^n)$, we see

$$[\nabla B(f,g)(t,\cdot)]_{C^{\alpha}(\mathbb{R}^n)} \leq C[\nabla \int_0^t e^{(t-s)\Delta}\nabla \cdot f \otimes g ds]_{C^{\alpha}(\mathbb{R}^n)}.$$

Then, from the maximal regularity estimates for the heat equation (see [14]), we easily obtain

$$(4.6) \qquad ||\nabla B(f,g)||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)} \leq C||f||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}||g||_{C^{0,\alpha}([0,T]\times\mathbb{R}^n)}.$$

Combining the above estimates with the estimate for F in tangential direction to Γ_t established in Proposition 3.3, we have the desired estimates. This completes the proof.

5. Construction of the solution for free boundary problem

Now we return to the problem (FBP). In this section we shall prove the main theorem. Let u_0 be a function in $C^{\alpha}(\mathbb{R}^n)$ and satisfy $\nabla\cdot u_0 = 0$. Let Γ_0 be a $C^{2+\alpha}$ hypersurface which is a boundary of a bounded domain Ω_0 and let d_0 be the signed distance function of Γ_0. We set $F_0(t,\cdot) = e^{t\Delta}u_0$ and

(5.1)

$$M := 2(||F_0||_{C^{0,\alpha}([0,\infty)\times\mathbb{R}^n)} + \sup_{t>0} t^{\frac{1-\alpha}{2}}||\partial_x F_0(t,\cdot)||_{C(\mathbb{R}^n)} + \sup_{t>0} t^{\frac{1}{2}}[\partial_x F_0(t,\cdot)]_{C^{\alpha}(\mathbb{R}^n)}).$$

Recall that $\mathcal{U}_M$ is the closed subset of $C^{0,\alpha}([0,T]\times\mathbb{R}^n)$ defined as

$$(5.2) \qquad \begin{aligned}\mathcal{U}_M = \{u(t,x) \in C^{0,\alpha}([0,T]\times\mathbb{R}^n);\, u(t,\cdot) \in C^{1+\alpha}(\mathbb{R}^n),\\ L_u := \sup_{0<t<T}||u(t,\cdot)||_{C^{\alpha}(\mathbb{R}^n)} + \sup_{0<t<T} t^{\frac{1-\alpha}{2}}||\partial_x u(t,\cdot)||_{C(\mathbb{R}^n)}\\ + \sup_{0<t<T} t^{\frac{1}{2}}[\partial_x u(t,\cdot)]_{C^{\alpha}(\mathbb{R}^n)} \leq M\}\end{aligned}$$

From Proposition 3.1 there exists a positive T_1 such that for any $u \in \mathcal{U}_M$, there exists a unique family of hypersurfaces $\{\Gamma_t^u\}_{0\leq t\leq T_1}$ evolving via perturbed mean curvature equation (3.1) starting from Γ_0. Moreover, this $\{\Gamma_t^u\}_{0\leq t\leq T_1}$ belongs to $\mathcal{S}(\alpha,R,T_1,d_0)$ with $R = ||d_0||_{C^{2+\alpha}} + 2C(||d_0||_{C^{2+\alpha}(\overline{D})}, K_f, M)$. Let v be the signed distance function of $\{\Gamma_t^u\}_{0\leq t\leq T_1}$. From Proposition 3.1 we also have

$$\sup_{0<t<T_1} t^{\frac{1}{2}}||\partial_x^3 v(t,\cdot)||_{C^{\alpha}(\overline{D'})} < \infty,$$

for any open set $D' \subset\subset D$. Then if $T_2 < T_1$ is sufficiently small, we can set

$$(5.3) \qquad C_4 := \sup_{0<t<T_2} t^{\frac{1}{2}}||\partial_x^3 v(t,\cdot)||_{C^{\alpha}(\cup_{1\leq k\leq m}\overline{O_k})},$$

where $\{O_k\}_{k=1}^m, O_k \subset\subset D$ is a suitable family of open sets covering any hypersurfaces belonging to $\mathcal{S}(\alpha, R, T_2, d_0)$. Note that C_4 is bounded uniformly in each function belonging to $\mathcal{U}_M$. Set

$$F^u(t,\cdot) := \int_0^t e^{(t-s)\Delta}\mathbf{P}\sigma_1 H^u\nu^u\mathcal{H}_{\llcorner\Gamma_s^u}ds, \tag{5.4}$$

where H^u, ν^u are the mean curvature and the exterior unit normal vector of Γ_t^u, respectively. For the signed distance function v, the exterior unit normal vector $\nu^u(t,x)$ and the mean curvature $H^u(t,x)$ of the surface Γ_t are given by

$$\nu^u(t,x) = \nabla_x v(t,x), \tag{5.5}$$

$$H^u(t,x) = -\frac{1}{n-1}\mathrm{div}\ \nu(t,x) = -\frac{1}{n-1}\Delta v(t,x). \tag{5.6}$$

Since v is a function on $[0,T]\times\overline{D}$, ν and H^u can be also regarded as functions on $[0,T]\times\overline{D}$. Especially, if $\{\Gamma_t\}_{0\le t\le T}$ is an evolving hypersurface belonging to $\mathcal{S}(\alpha,R,T,d_0)$, then the mean curvature vector $H^u\nu^u$ belongs to $C^{0,\alpha}([0,T]\times\overline{D})$ as the function on $[0,T]\times\overline{D}$. Moreover, if the above v satisfies $\sup_{0<t<T} t^{\frac{1}{2}}||\partial_x v^3(t,\cdot)||_{C^\alpha(\overline{D'})} < \infty$ for an open set $D' \subset\subset D$, then

$$\sup_{0<t<T} t^{\frac{1}{2}}||\partial_x H^u(t,\cdot)||_{C^\alpha(\overline{D'})} \le C \sup_{0<t<T} t^{\frac{1}{2}}||\partial_x^3 v(t,\cdot)||_{C^\alpha(\overline{D'})}. \tag{5.7}$$

From Proposition 3.3, the function F^u satisfies

$$||F^u||_{C^{\frac{\alpha}{2},\alpha}([0,T_2]\times\overline{D})} \le \sigma_1 C T^{\frac{1-\alpha}{2}}, \tag{5.8}$$

$$\sup_{0\le t\le T_2} ||F^u(t)||_{C^{1+\alpha}(\Gamma_t)} \le \sigma_1 C, \tag{5.9}$$

where C depends only on n, α, R, and Γ_0. Since $F^u(t,\cdot)$ belongs to $C^{1+\alpha}(\Gamma_t^u)$ for each $t\in(0,T_2]$, we can construct the function in $C^{1+\alpha}(\mathbb{R}^n)$ as the extension of $\gamma_{\Gamma_t^u}F^u(t,\cdot)$, where $\gamma_{\Gamma_t^u}$ is the restriction operator on Γ_t^u. We fix the way of the extension as in [15, Section 5]. Then obviously we have $E^v(F^u)(t,x) = F^u(t,x)$ for all $x\in\Gamma_t$ and

$$||E^v(F^u)||_{C^{\frac{\alpha}{2},\alpha}([0,T_2]\times\overline{D})} \le C_5\sigma_1 C T^{\frac{1-\alpha}{2}}, \tag{5.10}$$

$$\sup_{0\le t\le T_2} ||E^v(F^u)(t)||_{C^{1+\alpha}(\mathbb{R}^n)} \le C_5\sigma_1 C, \tag{5.11}$$

where C_5 depends only on n, R, and Γ_0. Note that, although the extension E^v depends on $\{\Gamma_t^u\}_{0\le t\le T_2}$, the constant C_5 is independent of each evolving hypersurface belonging to $\cup_{0<T\le T_2}\mathcal{S}(\alpha,R,T,d_0)$. Let $\Psi_0(u)$ be the unique mild solution to the Navier-Stokes equations with initial velocity u_0 and with the layer potential $\sigma_1 H^u\nu^u\mathcal{H}^{n-1}_{\llcorner\Gamma_t}$, i.e., $\Psi_0(u)$ satisfies

(5.12) $$\Psi_0(u) = W \quad \text{where} \quad W = F_0 - B(W,W) + F^u$$

where F_0 and $B(W,W)$ are defined in Section 4. We define the map $\Psi(u)$ by

(5.13) $$\Psi(u) = F_0 - B(W,W) + E^v(F^u) =: w + E^v(F^u).$$

By the estimates in Proposition 4.1, it is not difficult to see that $\Psi(u)$ maps $\mathcal{U}_M$ into $\mathcal{U}_M$ if we take $T < T_2$ sufficiently small. So we shall show that $\Psi(u)$ is a contraction mapping. Let u, $\tilde{u} \in \mathcal{U}_M$ and let v, $\tilde{v}$ be the signed distance functions of $\{\Gamma_t^u\}_{0\le t\le T}$, $\{\Gamma_t^{\tilde{u}}\}_{0\le t\le T}$, respectively. Then from [15, Proposition 5.1] we have

(5.14) $$||E^v(F^u) - E^{\tilde{v}}(F^{\tilde{u}})||_{C^{0,\alpha}} \le CT^{\frac{1}{2}}||v-\tilde{v}||_{C^{1,2+\alpha}([0,T]\times\cup_{1\le k\le m}\overline{\mathcal{O}_k})}.$$

Next we estimate $w - \tilde{w}$ where $w = W - F^u$ and $\tilde{w} = \tilde{W} - F^{\tilde{u}}$. Since w solves the equation

$$w = F_0 - B(w,w) - B(w,F^u) - B(F^u,w) - B(F^u,F^u),$$

we can show that

$$\begin{aligned}||w-\tilde{w}||_{C^{0,\alpha}} \le\ & C\big(||B(F^u - F^{\tilde{u}}, w)||_{C^{0,\alpha}} + ||B(w, F^u - F^{\tilde{u}})||_{C^{0,\alpha}} \\ & + ||B(F^u - F^{\tilde{u}}, F^u)||_{C^{0,\alpha}} + ||B(F^{\tilde{u}}, F^u - F^{\tilde{u}})||_{C^{0,\alpha}}\big).\end{aligned}$$

Since $F^u(t) - F^{\tilde{u}}(t)$ belongs to $L^p(\mathbb{R}^n)$ for $1 < p \le \infty$ by (3.19), we obtain

$$\begin{aligned}||B(F^u - F^{\tilde{u}}, w)(t)||_{C^\alpha} &\le C\int_0^t (t-s)^{-\frac{1+\alpha}{2}-\frac{n}{2p}}||(F^u(s) - F^{\tilde{u}}(s))\otimes w(s)||_{L^p}ds \\ &\le CT^{\frac{1-\alpha}{2}-\frac{n}{2p}}||w||_{C^{0,\alpha}}\sup_{0<t<T}||F^u(t) - F^{\tilde{u}}(t)||_{L^p}\end{aligned}$$

for sufficiently large $p < \infty$. Here we used the fact that the Helmholtz projection is bounded in $L^p(\mathbb{R}^n)$. Similar estimates for other terms lead to

(5.15) $$||w-\tilde{w}||_{C^{0,\alpha}} \le CMT^{\frac{1-\alpha}{2}-\frac{n}{2p}}\sup_{0<t<T}||F^u(t) - F^{\tilde{u}}(t)||_{L^p}.$$

The calculations using the partition of unity as in [15, Appendix] give the estimate of $\sup_{0<t<T}||F^u(t) - F^{\tilde{u}}(t)||_{L^p(\mathbb{R}^n)}$ such as

(5.16) $$||F^u(t) - F^{\tilde{u}}(t)||_{L^p(\mathbb{R}^n)} \le Ct^{\frac{1}{2p}}||v-\tilde{v}||_{C^{1,\alpha}}$$

where C dependsonly on p, n, α, and the initial interface Γ_0. We omit the details here. Now by Proposition 3.2 we observe that Ψ is a contraction mapping if T is sufficiently small.

Finally, let u^* be the unique fixed point of Ψ in $\mathcal{U}_M$. Then we can check that the pair $(\Psi_0(u^*), \{\Gamma_t^{u^*}\}_{0\le t\le T})$ is the unique solution of the free boundary problem. Indeed, since $u^* = \Psi(u^*)$, we have from the definition of Ψ_0,

$$\begin{aligned}
\Psi_0(u^*)(t,x) &= F_0(t,x) - B(\Psi_0(u^*), \Psi_0(u^*)) + F^{u^*}(t,x) \\
&= F_0(t,x)) - B(\Psi_0(u^*), \Psi_0(u^*)) + E^{v^*}(F^{u^*})(t,x) \\
&= \Psi(u^*)(t,x) \\
&= u^*(t,x)
\end{aligned}$$

for any $(t,x) \in \cup_{0 \le t \le T}\{t\} \times \Gamma_t^{u^*}$. Hence, $\{\Gamma_t^{u^*}\}_{0\le t\le T}$ evolves by the equation

$$\begin{cases}
\dfrac{dx}{dt} & = \sigma_2 H^*(t,x)\nu^*(t,x) + u^*(t,x) \\
& = \sigma_2 H^*(t,x)\nu^*(t,x) + \Psi_0(u^*)(t,x), \\
x(0) & = x_0 \in \Gamma_0,
\end{cases}$$

that is, the pair $(\Psi_0(u^*), \{\Gamma_t^{u^*}\}_{0\le t\le T})$ is a solution of our free boundary problem.

Although the above mapping Ψ depends on the particular way of the extension, we can see that the solution, in fact, does not depend on such extension and is unique in the class stated in the main theorem. To see this, let $(u, \{\Gamma_t\}_{0\le t\le T})$ be another pair of the solution for (FBP). Let v be the signed distance function of $\{\Gamma_t\}_{0\le t\le T}$. Then, v belongs to $C^{1,2+\alpha}([0,T]\times \overline{D})$ where $D = \{x \in \mathbb{R}^n \ ; \ -\delta < d_0(x) < \delta\}$ for sufficiently small $\delta > 0$. Since $\{\Gamma_t\}_{0\le t\le T}$ evolves by the equation in (BC), v satisfies the equation (3.6) in Section 3. The important fact is that for any $x \in \overline{D}$, the point $x - v(t,x)\nabla_x v(t,x)$ must belong to Γ_t by the definition of the signed distance function. This implies that for any $\tilde{u}$ satisfying $\tilde{u} = u$ on $\cup_{0\le t\le T}\{t\} \times \Gamma_t$, the function v is also the solution of the equation (3.6) with $\tilde{u}$ instead of u. This concludes that the solution $(u, \{\Gamma_t\}_{0\le t\le T})$ does not depend on the particular extension, and the above $(\Psi_0(u^*), \{\Gamma_t^{u^*}\}_{0\le t\le T})$ is the unique solution solving (FBP) in the class stated in the theorem. Now the proof of the main theorem is completed.

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