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# Nonlinear problems with singular diffusivity and inhomogeneous terms

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In this talk we consider a singular diffusion equation associated with total variation with inhomogeneous terms as follows

$$u : [0,1] \times [0,T) \to \mathbb{R}^n \ (n \geq 1) \ : \ \text{unknown function}$$

$$(P) \begin{cases} u_t - \dfrac{1}{b(x)} \operatorname{div}\left(a(x)\dfrac{u_x}{|u_x|}\right) = 0, & (x,t) \in (0,1) \times (0,T), & (1) \\ u(x,0) = u_0(x), & x \in (0,1), & (2) \\ u(0,t) = g_0, \quad u(1,t) = g_1, & t \in (0,T), & (3) \end{cases}$$

where $a(x), b(x)$ are given positive, continuous functions on $[0,1]$ and $u_0$ is an initial data and $g_0, g_1 \in \mathbb{R}^n$ are boundary condition. This equation (1) is written as the gradient system by taking energy

$$E(u) = \int_0^1 a(x)|u_x|\,dx$$

with respect to the norm $\|f\|^2 = \int_0^1 b(x)|f(x)|^2\,dx$. The equation (1) describes the motion of multi-grain problem studied in [3].

In the scalar valued case with boundary condition $u(0) = 0, u(1) = 1$, if $a(x)$ has a unique minimum point $x_0$, then

$$E(u) = \int_0^1 a(x)|u_x|\,dx \geq a(x_0)\int_0^1 u_x\,dx = a(x_0)(u(1) - u(0)) = a(x_0).$$

If $u$ is a step function and jumps only at $x_0$, then the equality holds. So global minimizer is unique [2]. In general case, a global minimizer quite naturally has a discontinuity since it makes the energy low by concentrating its variation at the point where $a(x)$ is minimal. It follows that many global minimizers may be piecewise constant functions.

We consider stationary problem of (P) in the vector valued case. Suppose that inhomogeneous term $a(x), b(x)$ satisfy "concave condition" (cf [1]). We characterize stationary piecewise constant solutions.

## References

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[3] R. Kobayashi, J. A, Warren and W. C. Carter, A continuum model of grain boundaries, Physica D, **140**(2000), 141–150.