Nonlinear problems with singular diffusivity
and inhomogeneous terms

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In this talk we consider a singular diffusion equation associated with total variation with inhomogeneous terms as follows

$$u : [0,1] \times [0, T) \to \mathbb{R}^n (n \geq 1) : \text{unknown function}$$

(P)  

$$\begin{align*}
  u_t - \frac{1}{b(x)} \text{div} \left( a(x) \frac{u_x}{|u_x|} \right) &= 0, \quad (x, t) \in (0,1) \times (0, T), \\
  u(x,0) &= u_0(x), \quad x \in (0,1), \\
  u(0,t) &= g_0, \quad u(1,t) = g_1, \quad t \in (0, T),
\end{align*}$$

where $a(x), b(x)$ are given positive, continuous functions on $[0,1]$ and $u_0$ is an initial data and $g_0, g_1 \in \mathbb{R}^n$ are boundary condition. This equation (1) is written as the gradient system by taking energy

$$E(u) = \int_0^1 a(x)|u_x| dx$$

with respect to the norm $\|f\|^2 = \int_0^1 b(x)|f(x)|^2 dx$. The equation (1) describes the motion of multi-grain problem studied in [3].

In the scalar valued case with boundary condition $u(0) = 0, u(1) = 1$, if $a(x)$ has a unique minimum point $x_0$, then

$$E(u) = \int_0^1 a(x)|u_x| dx \geq a(x_0) \int_0^1 u_x dx = a(x_0)(u(1) - u(0)) = a(x_0).$$

If $u$ is a step function and jumps only at $x_0$, then the equality holds. So global minimizer is unique [2]. In general case, a global minimizer quite naturally has a discontinuity since it makes the energy low by concentrating its variation at the point where $a(x)$ is minimal. It follows that many global minimizers may be piecewise constant functions.

We consider stationary problem of (P) in the vector valued case. Suppose that inhomogeneous term $a(x), b(x)$ satisfy "concave condition" (cf [1]). We characterize stationary piecewise constant solutions.

References

