数理解析研究所講究録
第 1633 巻 2009 年 138-150

On Game Interpretations for the Curvature Flow Equation and Its Boundary Problems

Qing Liu*
Graduate School of Mathematical Sciences, The University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914 Japan

Abstract

In this paper, we start with studying a family of deterministic discrete-time optimal control problems whose value functions converge to the oblique boundary problems of first-order Hamilton-Jacobi equations. Our method is based on the billiard semiflow. We finally apply this method to the case of second-order geometric flow equations.

1 Introduction

This paper investigates several generalizations of our previous work in [4], which provides a discrete game interpretation for the Neumann boundary problem of motion by curvature. We recall that such kind of optimal control approach is first proposed by Kohn and Serfaty (see [8, 9]), who drew a connection between two-person games and second-order PDEs. It turns out that by the convergence argument, a time-optimal problem is related to the Dirichlet problem of an elliptic equation and a time-dependent game corresponds to the Cauchy problem of a parabolic equation; see [3, 7] for generalizations in distinct directions. Our goal here is different, mainly resting on the general boundary problems of evolutionary equations. To simplify our proofs and emphasize our idea, we mainly discuss first-order Hamilton-Jacobi equations on the ground of deterministic optimal control theory (see, e.g., [1]). The well-posedness of these oblique boundary problems in the viscosity sense is due to [10] for first-order cases and [2, 5, 12, 13] for second-order ones.

A billiard semiflow is studied in [4]. Based on it, discrete deterministic games are constructed so that their value functions converge to the unique solution of the Neumann

*The author is supported by the Japanese Government Scholarship. E-mail: liuqing@ms.u-tokyo.ac.jp

boundary problem of curve shortening flow equation. In this paper, we apply the same method to the first-order Hamilton-Jacobi equations with Neumann type boundary:

$$\text{(E1)} \qquad \begin{cases} u_t(x,t) + \sup_{a\in A}\{-f(x,a)\cdot \nabla u(x,t) - l(x,a)\} = 0 & \text{in } \Omega \times [0,\infty), \\ \nabla u(x,t)\cdot \nu(x) = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a C^2 and convex domain, $\nu(x)$ denotes the outward unit normal to $\partial\Omega$ at x, and f and l are given functions. Refer to Section 3 for details. It is worthwhile to mention that our approach can be distinguished from that in [10], which pioneers the study of the boundary conditions in the viscosity sense and their applications in optimal control based on the Skorokhod problem; see [11, 14] for the topic on the Skorokhod problem. We use the very simple billiard law: the angle of incidence equals to the angle of reflection, in place of the Skorokhod map or any of its discrete versions. However, it is interesting to find that billiard and Skorokhod reflections are analogous in form [4, Lemma 2.3], which makes our arguments more understandable.

Another way of generalizing [4] is to devise a more general billiard law, as we call the oblique billiard, so as to get differential games for oblique boundary conditions. When creating the obliqueness, we do not imitate the usual billiard law via angles at hitting points, but instead follow the idea of decomposing each incident ray along the normal and tangent and then simply switching the direction of its normal component. Such an operation certainly gives a generalization of the classical billiard but its properties, especially those about its singular phenomena, turn out to be obscure. In this paper, without touching too complicated situations, we conduct our game interpretation for the oblique boundary problem of Hamilton-Jacobi equations only in the half plane, where any billiard move hits the boundary at most once. A typical equation is like

$$\text{(E2)} \qquad \begin{cases} u_t(x,t) + \sup_{a\in A}\{-f(x,a)\cdot \nabla u(x,t) - l(x,a)\} = 0 & \text{in } \Omega \times [0,\infty), \\ \nabla u(x,t)\cdot \gamma(x) = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where $\gamma(x)$ is the unit oblique normal, satisfying for every $x \in \partial\Omega$,

$$\langle \gamma(x), \nu(x)\rangle = \theta, \quad 0 < \theta \leq 1.$$

In order to avoid redundancy, we slightly modify our proof for Neumann boundary problem to adapt it to this oblique boundary case. As a matter of fact, in this case, the similarity between billiard and Skorokhod reflections still holds. We shall explain in detail in Section 4. Another application of our oblique billiards to the curve shortening equation is quite natural and a formal derivation is included as well in the section. A question remains unsolved how to get any extension of our results or find another type of oblique billiards for more general domains.

We remark that both first and second-order equations can be derived from discrete game settings but their difference is spectacular. In contrast to the simple way of deducing first-order Hamilton-Jacobi equations, for second-order time-dependent case, we usually need to eliminate the first-order space term by adding a null condition and arrange the coexistence of the first-order time derivative and second-order space derivatives by using an irregular clock. Our previous work [3] contains a discussion on it.

2 Planar Billiard Dynamics

Let us start with a review of the results about the normal billiard semiflow. (We specify the word "normal" because a more general and complicated oblique billiard will be discussed later.) All of the proofs, omitted in this paper, are given in [4]. We first recall the billiard flow. Suppose that there is a domain Ω, said to be a *billiard table*, satisfying the following assumption:

(A1) Ω is a bounded and convex domain in $\mathbb{R}^2$ with C^2 boundary.

The billiard flow in Ω, denoted by $T^t : \overline{\Omega} \times \mathbf{S}^1 \to \overline{\Omega}$ $(t \in \mathbb{R})$, describes the billiard motion in the table. By billiard motion, we mean that a mass point is moving along straight-lines in the interior of the domain and following the optic law on the boundary, namely, the angle of incidence equals the angle of reflection. For a fixed pair (x, v), $T^t(x, v)$ represents the ball's position at time t. The set $\{T^t(x, v) \in \overline{\Omega} : t \geq 0\}$ is called a *billiard trajectory* starting from (x, v) and the hitting points on the boundary are called *vertices* of the trajectory. It is obvious that T^t satisfies the group property restricted in $\Omega \times \mathbf{S}^1$ with the identity T^0 and $T^{-t}(x, v) = T^t(x, -v)$ for any $x \in \Omega$ and $v \in \mathbf{S}^1$.

We stress here that such a billiard motion is not always proper. Indeed, a so-called terminating phenomenon may occur even in this C^2 domain, or in other words, the sequence of vertices $\{p_n\}_{n \geq 1}$ may converge to a point on $\partial\Omega$. For further explanation, we refer the readers to [6], from which an important property is drawn to be stated in Lemma 2.1 below.

We hereafter utilize the arc-length parametrization $\Gamma(\cdot) : \mathbb{R} \to \mathbb{R}^2$, a function of class C^2, to represent $\partial\Omega$. Its derivative with respect to s is denoted by Γ_s.

Lemma 2.1. *Suppose that Ω satisfies* (A1). *If a trajectory terminates at a point $\Gamma(s_\infty) \in \partial\Omega$, with a sequence of vertices $\{\Gamma(s_n)\}_{n \geq 1}$ arranged in order, then there exists $N > 0$ such that for $n \geq N$, s_n monotonically converges to s_∞ and $(\Gamma(s_\infty) - \Gamma(s_n))/|s_\infty - s_n|$ converges to a unit tangent, denoted by v_∞, to the boundary at $\Gamma(s_\infty)$.*

We next present a modified billiard dynamics as follows.

Definition 2.1. Let Ω satisfy (A1).

(i) If $x \in \partial\Omega$, and v equals to the tangent of $\partial\Omega$, then

$$S^t(x, v) := \Gamma(t), \quad \text{for any } t \geq 0,$$

where $\Gamma(\cdot)$ is the arc-length parametrization of $\partial\Omega$ such that $\Gamma(0) = x$ and $\Gamma_s(0) = v$;

(ii) If $x \in \Omega$ and v is such that $T^t(x, v)$ terminates on $\partial\Omega$ at time t_0, then

$$S^t(x, v) := \begin{cases} T^t(x, v) & \text{if } 0 \leq t < t_0, \\ S^{t-t_0}(T^{t_0}(x, v), v_\infty) & \text{if } t \geq t_0, \end{cases}$$

where v_∞ is obtained from Lemma 2.1;

(iii) If $x \in \partial\Omega$ and v points inside Ω, then

$$S^t(x,v) := \begin{cases} x & \text{if } t = 0, \\ S^{t-\varepsilon}(x+\varepsilon v, v) & \text{if } t > 0, \end{cases}$$

where $\varepsilon > 0$ is such that $x + \delta v \in \Omega$ for all $\delta \in (0, \varepsilon)$.

It is easily seen that S^t is a semiflow. For $t \geq 0$, $x \in \overline{\Omega}$ and $v \in \mathbf{S}^1$, we set

$$\alpha^t(x,v) = x + tv - S^t(x,v) \tag{2.1}$$

and call it the *boundary adjustor*. An important property of our semiflow is given in the following lemma.

Lemma 2.2 ([4, Lemma 2.3]). *Assume that Ω satisfies* (A1). *For any fixed $t \geq 0$, $x \in \overline{\Omega}$ and $v \in \mathbf{S}^1$, let $\alpha^t(x,v)$ be the boundary adjustor of $S^t(x,v)$. Then there exist $d_l \geq 0$ and $y_l \in \partial\Omega \cap B_t(x)$, $l = 1, 2, \ldots$ such that*

$$\alpha^t(x,v) = \sum_{l=0}^{\infty} d_l \nu(y_l), \tag{2.2}$$

where the convergence on the right hand side is in $\mathbb{R}^2$. In addition, the following estimates hold:

$$\left|\alpha^t(x,v)\right| \leq 2t. \tag{2.3}$$

$$\left|\sum_{l=k}^{\infty} d_l \nu(y_l)\right| \leq 4t, \quad \text{for all } k = 1, 2, \ldots \tag{2.4}$$

$$\sum_{l=1}^{\infty} |y_{l+1} - y_l| \leq 2t. \tag{2.5}$$

This lemma tells us that the effect of billiard reflection is nothing but a series of inward normal impacts. Such an observation, resembling the Skorokhod problem, turns out to play a significant role in our game setting.

We conclude this section with another property, which is a direct consequence of the separation theorem for convex sets in $\mathbb{R}^2$.

Lemma 2.3 ([4, Lemma 2.4]). *Assume that Ω satisfies* (A1). *Then*

$$|x_0 - S^t(x,v)| \leq |x_0 - (x+tv)| \text{ for any } x, x_0 \in \overline{\Omega},\ v \in \mathbf{S}^1 \text{ and } t \geq 0. \tag{2.6}$$

We discuss in this paper only a convex domain. For more general domains, we need a few additional techniques since the above lemma no longer holds. See [4] for further study.

3 Neumann Boundary of HJ Equations

We establish a discrete system on the basis of the billiard semiflow investigated in Section 2. At first assume

(A2) $$A \text{ is a compact topological space,}$$

(A3) $$f : \overline{\Omega} \times A \to \mathbb{R}^2 \text{ satisfies } \sup_{x \in \overline{\Omega},\ a \in A} |f(x,a)| \le M_1,$$

and

(A4) $$|f(x_1, a) - f(x_2, a)| \le L_1 |x_1 - x_2| \text{ for } L_1 > 0 \text{ independent of } a \in A.$$

Notice that there exists a function $v_f : \overline{\Omega} \times A \to S^1 = \{v \in \mathbb{R}^2 : |v| = 1\}$ such that for any $(x, a) \in \overline{\Omega} \times A$,

$$f(x,a) = |f(x,a)| v_f(x,a).$$

To formulate our control system, we take the step size $\varepsilon > 0$ and a sequence $y_k, k = 0, 1, 2, \ldots$, which satisfies the following:

(3.1) $$\begin{cases} y_{k+1} = S^{|f_k|\varepsilon}\left(y_k, v_f(y_k, a_{k+1})\right); \\ y_0 = x, \end{cases}$$

where the control variable $a_k \in A$ and $|f_k| = |f(y_k, a_{k+1})|$ for all $k = 0, 1, 2, \ldots$ It seems to be at question whether our definition above is valid since v_f is not uniquely determined when $|f| = 0$. However, there is essentially no problem in the system (3.1) thanks to our billiard structure, which yields a temporary stop whenever $f_k = 0$.

For every $t \ge 0$, let N be the largest integer less than t/ε. Given $x \in \mathbb{R}^2$, $t \ge 0$ and $a = (a_1, \ldots, a_N) \in A^N$, we define a control objective as

(3.2) $$\begin{aligned} J^\varepsilon(x,t,a) &:= \sum_{k=1}^{N} \varepsilon l(y_{k-1}, a_k) + u_0(y_N), \text{ if } t \ge \varepsilon \text{ and} \\ J^\varepsilon(x,t,a) &:= u_0(x), \text{ if } 0 \le t < \varepsilon, \end{aligned}$$

where $l : \overline{\Omega} \times A \to \mathbb{R}$ stands for the running cost fulfilling

(A5) $$\sup_{x \in \overline{\Omega},\ a \in A} |l(x,a)| \le M_2; \text{ and}$$

(A6) $$|l(x_1, a) - l(x_2, a)| \le L_2 |x_1 - x_2| \text{ for } L_2 > 0 \text{ independent of } a \in A.$$

and the function $u_0 : \overline{\Omega} \to \mathbb{R}$ is a terminal cost. We next define a value function for every $x \in \overline{\Omega}$ and $t \ge 0$

(3.3) $$u^\varepsilon(x,t) := \inf_{a \in A^N} J^\varepsilon(x,t,a)$$

and it clearly satisfies the dynamic programming equation

(DPP) $$u^\varepsilon(x,t) := \inf_{a \in A} \{u^\varepsilon(S^{|f(x,a)|\varepsilon}(x, v_f(x,a)),\ t - \varepsilon) + \varepsilon l(x,a)\}$$
$$\text{for all } x \in \overline{\Omega} \text{ and } t \ge \epsilon.$$

The first theorem we get is

Theorem 3.1. *Assume* (A1)-(A6). *Let* u^ε *be the game value in* (3.3) *and* u_0 *be a continuous function in* $\overline{\Omega}$, *then* u^ε *converges, as* $\varepsilon \to 0$, *uniformly on every compact set of* $\overline{\Omega} \times [0, \infty)$ *to the unique solution of the Neumann boundary problem of Hamilton-Jacobi equation* (E1).

We below present the definition of viscosity solutions of (E1).

Definition 3.1. An upper semicontinuous (resp., lower semicontinuous) function u on $\overline{\Omega} \times [0, \infty)$ is a *viscosity subsolution* (resp., *viscosity supersolution*) of (E1) if

$$u(x, 0) \le u_0(x) \quad (\text{resp., } u(x, 0) \ge u_0(x))$$

and whenever there are $(\hat{x}, \hat{t}) \in \overline{\Omega} \times (0, \infty)$, a neighborhood $\mathcal{O}$ relative to $\overline{\Omega} \times (0, \infty)$ of $(\hat{x}, \hat{t})$ and a smooth function $\varphi : \mathcal{O} \to \mathbb{R}$ such that

$$\max_{\mathcal{O}}(u - \varphi) = (u - \varphi)(\hat{x}, \hat{t})$$
$$\left(\text{resp., } \min_{\mathcal{O}}(u - \varphi) = (u - \varphi)(\hat{x}, \hat{t})\right),$$

the following holds:

(i) If $\hat{x} \in \Omega$, then

$$\partial_t \varphi(\hat{x}, \hat{t}) + \sup_{a \in A}\{-f(\hat{x}, a) \cdot \nabla\varphi(\hat{x}, \hat{t}) - l(\hat{x}, a)\} \le 0$$
$$\left(\text{resp., } \partial_t \varphi(\hat{x}, \hat{t}) + \sup_{a \in A}\{-f(\hat{x}, a) \cdot \nabla\varphi(\hat{x}, \hat{t}) - l(\hat{x}, a)\} \ge 0\right).$$

(ii) If $\hat{x} \in \partial\Omega$, then

$$\partial_t \varphi(\hat{x}, \hat{t}) + \sup_{a \in A}\{-f(\hat{x}, a) \cdot \nabla\varphi(\hat{x}, \hat{t}) - l(\hat{x}, a)\} \le 0$$
$$\left(\text{resp., } \partial_t \varphi(\hat{x}, \hat{t}) + \sup_{a \in A}\{-f(\hat{x}, a) \cdot \nabla\varphi(\hat{x}, \hat{t}) - l(\hat{x}, a)\} \ge 0\right)$$

or

$$\langle \nabla\varphi(\hat{x}, \hat{t}), \nu(\hat{x}) \rangle \le 0 \quad \left(\text{resp., } \langle \nabla\varphi(\hat{x}, \hat{t}), \nu(\hat{x}) \rangle \ge 0\right).$$

Definition 3.2. A function u on $\overline{\Omega} \times [0, \infty)$ is called a *viscosity solution* of (E1) if it is both a viscosity subsolution and a viscosity supersolution.

Before we prove Theorem 3.1, let us first introduce the *upper* and *lower relaxed limits* of u^ε as

$$\overline{u}(x, t) := \limsup_{\varepsilon \to 0}{}^* u^\varepsilon(x, t) = \lim_{\delta \to 0} \sup\{u^\varepsilon(y, s) : \varepsilon < \delta,\ |x - y| + |t - s| < \delta\}$$

and

$$\underline{u}(x, t) := \liminf_{\varepsilon \to 0}{}_* u^\varepsilon(x, t) = \lim_{\delta \to 0} \inf\{u^\varepsilon(y, s) : \varepsilon < \delta,\ |x - y| + |t - s| < \delta\}.$$

In what follows, we give our proof of Theorem 3.1 by showing $\overline{u} = \underline{u}$, which consists of three propositions.

Proposition 3.2. $\underline{u}(x,0) = \overline{u}(x,0) = u_0(x)$.

Proof. We only show $\overline{u}(x,0) \le u_0(x)$ for every fixed $x \in \overline{\Omega}$. A symmetric argument gives $\underline{u}(x,0) \ge u_0(x)$ and our conclusion is thus reached in virtue of a basic fact that $\underline{u} \le \overline{u}$. To this end, we adopt a barrier argument since our initial datum u_0 is only continuous and needs regularizing. More precisely, for any $\lambda > 0$, there exists a constant C_λ such that

$$u_0(y) \le \lambda + u_0(x_0) + C_\lambda |y - x| \text{ for all } y \in \overline{\Omega}.$$

We denote by $\overline{V}_\lambda$ the right hand side of the above inequality and now observe the same discrete-time optimal control problem but only with the terminal cost changed from u_0 to $\overline{V}_\lambda$. Let the value function of this new game be $\tilde{u}^\varepsilon$. Then, in view of the definition (3.3), the boundedness of functions f and l (i.e., (A3) and (A5)) and (2.6) with $x_0 = x$, we directly evaluate for $y \in \overline{\Omega}$ and $t \ge 0$

$$\begin{aligned} \tilde{u}^\varepsilon(y,t) &\le M_2 N\varepsilon + \lambda + u_0(x) + C_\lambda(|y-x| + M_1 N\varepsilon) \\ &\le \lambda + u_0(x) + C_\lambda|y-x| + (C_\lambda M_1 + M_2)t. \end{aligned}$$

Noting that game values preserve the order of objectives, that is $u^\varepsilon \le \tilde{u}^\varepsilon$ in our special case, we are thus led, by the definition of relaxed limits as $y \to x$ and $t \to 0$, to

$$\overline{u}(x,0) \le \lambda + u_0(x).$$

Sending $\lambda \downarrow 0$, we are done. □

Proposition 3.3. $\overline{u}$ *is a subsolution of* (E1).

Proof. We argue by contradiction. Since $\overline{u}$ fulfills the initial data by Proposition 3.2, assume there exist $(x_0, t_0) \in \partial\Omega \times (0,\infty)$ (our argument actually works for the case $x_0 \in \Omega$ as well), a δ-neighborhood B_δ of (x_0,t_0) relative to $\overline{\Omega} \times (0,\infty)$ and a smooth function ϕ on $\overline{\Omega} \times (0,\infty)$ such that

(i) $\overline{u}(x_0,t_0) - \phi(x_0,t_0) > \overline{u}(x,t) - \phi(x,t)$ for all $(x,t) \in B_\delta$;

(ii) $\partial_t\phi(x_0,t_0) + \sup_{a\in A}\{-f(x_0,a)\cdot\nabla\phi(x_0,t_0) - l(x_0,a)\} \ge \eta_0 > 0$; and

(iii) $\nabla\phi(x_0,t_0)\cdot\nu(x_0) \ge \eta_0 > 0$.

Assumption (ii), together with the continuity of f and l in x (i.e., (A4) and (A6)), implies the existence of $\overline{a} \in A$ satisfying

$$\partial_t\phi(x,t) - f(x,\overline{a})\cdot\nabla\phi(x,t) - l(x,\overline{a}) \ge \eta_0/2 \text{ for all } (x,t) \in B_\delta \tag{3.4}$$

and (iii) gives rise to

$$\nabla\phi(x,t)\cdot\nu(x) \ge \eta_0/2 \text{ for all } (x,t) \in B_\delta \text{ and } x \in \partial\Omega. \tag{3.5}$$

By the definition of $\overline{u}$, we can take a sequence $(x_0^\varepsilon, t_0^\varepsilon) \to (x_0,t_0)$ with

$$u^\varepsilon(x_0^\varepsilon, t_0^\varepsilon) \to \overline{u}(x_0,t_0).$$

Let us use the constant control $\overline{a}$ to get a sequence of states

$$X_1 \equiv (x_1, t_1) = (x_0^\varepsilon, t_0^\varepsilon);$$

$$X_{k+1} \equiv (x_{k+1}, t_{k+1}) = \left(S^{|f_k|\varepsilon}(x_k, v_f(x_k, \overline{a})), t_k - \varepsilon\right), \quad k \geq 1.$$

We assume for the moment that any X_k does not exceed B_δ, which requires that k should not be too large. In terms of the dynamic programming principle (DPP), we have

$$u^\varepsilon(X_k) \leq u^\varepsilon(X_{k+1}) + \varepsilon l(x_k, \overline{a}). \tag{3.6}$$

On the other hand, applying Taylor's formula and the notion of boundary adjustor (2.1), we get

$$\phi(X_{k+1}) = \phi(X_k) - \varepsilon\phi_t(X_k) + \nabla\phi(X_k) \cdot (f(x_k, \overline{a})\varepsilon - \alpha_k^\varepsilon). \tag{3.7}$$

(If x_0 is originally an interior point, taking a sufficient small δ makes all $\alpha_k^\varepsilon = 0$.) Due to the special structure of α as in (2.2) in Lemma 2.2 and an application of (3.5), the above equality yields

$$\phi(X_{k+1}) - \phi(X_k) \leq \varepsilon\left(-\phi_t(X_k) + \nabla\phi(X_k) \cdot f(x_k, \overline{a})\right). \tag{3.8}$$

Combining (3.4), (3.6) and (3.8), we are led to

$$(u^\varepsilon - \phi)(X_{k+1}) - (u^\varepsilon - \phi)(X_k) \geq \frac{\eta_0}{2}\varepsilon$$

and furthermore

$$(u^\varepsilon - \phi)(X_k) - (u^\varepsilon - \phi)(X_1) \geq \frac{(k-1)\eta_0\varepsilon}{2} \text{ for all } k = 1, 2, \ldots.$$

It means that we can take a subsequence of X_k, still indexed by k, in B_δ but converging to $(x', t') \neq (x_0, t_0)$. Hence, by letting $\varepsilon \to 0$, we see

$$(\overline{u} - \phi)(x', t') \geq (\overline{u} - \phi)(x_0, t_0),$$

which is a contradiction to our assumption (i).

□

Proposition 3.4. *$\underline{u}$ is a supersolution of* (E1).

Showing Proposition 3.4 is not largely different from what has been done for Proposition 3.3. We omit the whole process here.

The proof of Theorem 3.1 is actually completed. The last step in our proof is only a comparison principle to obtain $\overline{u} \leq \underline{u}$ based on our three propositions above. This part of work is classical and elaborated well in [10].

4 An Extension to Oblique Type Boundary

We mentioned before that our discrete-time optimal control setting also provides an approach of characterizing the general oblique boundary problem. In this section, our intention is to deal with the Hamilton-Jacobi equation (E2) and curve shortening flow equation but only in the domain of a half plane, that is, we assume

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}. \tag{A7}$$

4.1 Hamilton-Jacobi Equation

We first need to seek an oblique billiard, whose existence is suggested by the succeeding simple example.

Under the assumption (A7), let us denote respectively by $\vec{i}$ and $\vec{j}$ the unit vectors in a pair of coordinates, and then the outward normal of $\partial\Omega$ reduces to a constant vector $-\vec{j}$. Furthermore, for any hitting point x, we suppose,

$$\gamma(x) = -\sqrt{1-\theta^2}\vec{i} - \theta\vec{j}. \tag{4.1}$$

In this domain, a planar oblique billiard trajectory $S_o^t(x, v_1)$ can be set as a straight line with initial data (x, v_1) until it touches $\partial\Omega$. At the hitting moment, varying from the familiar optic reflection law, we linearly decompose the unit vector v_1 according to the vectors γ and $\vec{i}$, only oppose the sign of its component in the γ direction, and in this way get the reflected-off vector v_2 which will lead another straight line move. More precisely, taking a linear transformation

$$R = \begin{pmatrix} 1 & -2\frac{\sqrt{1-\theta^2}}{\theta} \\ 0 & -1 \end{pmatrix}, \tag{4.2}$$

we can express the new type of reflection by

$$v_2 = Rv_1, \tag{4.3}$$

where v_1 satisfies $\left\langle v_1, \vec{j} \right\rangle \le 0$.

It merits mentioning that v_1 and v_2 here should be viewed as speed vectors in stead of directions, because $|v_2| \ne 1$ in general. In other words, the speed shifts at vertices, whose total number is however evidently not more than 1 in this half plane case. For a general domain, the number of vertices could be very large and we know little about the singular phenomena especially the termination. This problem actually obstructs us to handle more complicated domains. It is of great interest if one can generalize Lemma 2.1 for our application in this new circumstance.

In our special domain, the definition of S_o^t is

Definition 4.1. Assume (A7) and let t_0 be the hitting time, and then define $S_o^t(\cdot,\cdot)$: $\overline{\Omega} \times S^1 \to \overline{\Omega}$ as

$$S_o^t(x, v) = \begin{cases} x + tv & \text{if } t \le t_0 \\ x + t_0 v + (t - t_0)Rv & \text{if } t > t_0, \end{cases} \tag{4.4}$$

where R is given in (4.2).

Generalized from normal billiards, this oblique billiard dynamic has an analogue of (2.2) in Lemma 2.2 as

$$\beta^t(x, v) := x + tv - S_o^t(x, v) = C(t)\gamma \text{ and } |\beta^t| = C(t) \le \frac{2t}{\theta}, \tag{4.5}$$

where $C(t)$ is a constant depending on t.

With all the preparation above, the game corresponding to (E2) can be established almost the same as we have done in Section 3. Merely substituting S^t with S^t_o, one confirms that the new value function u^ε satisfies

$$u^\varepsilon(x,t) := \inf_{a\in A}\{u^\varepsilon(S_o^{|f(x,a)|\varepsilon}(x, v_f(x,a)),\ t-\varepsilon) + \varepsilon l(x,a)\} \tag{4.6}$$
$$\text{for all } x \in \overline{\Omega} \text{ and } t \geq \epsilon.$$

Then we get

Theorem 4.1. *Assume* (A2)-(A7). *Let u^ϵ be the value function of the game based on oblique billiards above and u_0 be a continuous function in $\overline{\Omega}$. Then u^ϵ converges, as $\varepsilon \to 0$, uniformly on every compact subset of $\overline{\Omega} \times [0,\infty)$ to the unique viscosity solution of* (E2).

Remark 4.1. The definition of solutions of (E2) appears the same as that of (E1) if one replaces ν by γ in Definition 3.1 and 3.2.

Proof. We almost repeat the proof in Section 3. Indeed, the proof of Proposition 3.2 needs little modification. A crucial point is that, in light of (4.5), for oblique billiard games the distance of each move is still bounded in spite of a necessary alteration of the bound from ε to $(1+2/\theta)\varepsilon$. Variants of Propositions 3.3 and 3.4 work well too and probably even simpler in that our boundary adjustor now is not a series of normals but instead a single oblique one. □

4.2 Curve Shortening Flow Equation

Another example of oblique billiards related PDEs is the Neumann boundary problem of two-dimensional curvature flow equation:

$$\text{(E3)} \qquad \begin{cases} \partial_t u - \Delta u + \left(\nabla^2 u \frac{\nabla u}{|\nabla u|}\right) \cdot \frac{\nabla u}{|\nabla u|} = 0 & \text{in } \Omega \times (0,T), \\ u(x,T) = u_0(x) & \text{in } \overline{\Omega}, \\ \nabla u(x,t) \cdot \gamma(x) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where $\gamma(x)$ is the outward oblique normal, defined in the same way as in the Section 4.1. We again consider the problem in the half plane, i.e., Ω is assumed to satisfy (A7).

For a second-order equation like (E3), optimal control theory is generally inadequate to give any explicit representation for its solution. Instead, we resort to the two-person games, in which both players adopt measures adverse to their opponents. Although the game theory is known to connect again with first-order Hamilton-Jacobi equations ([1]), it sometimes behaves more interesting than just optimal control by two controllers. In fact, the conflict of players could cause singularity. To be more precise, we recall in the proof of Proposition 3.3 that Taylor expansion (equation (3)) of the test function ϕ only involves first derivatives. For second-order games, the expansion is conducted up to second-order while the first-order term come to vanish when ε tends to 0. Such heuristics being carried out and combined with a scaled clock, deterministic game values may approximate the solution of a second-order equation. See [8] or [3, 4, 7] for a better understanding.

We next pose the game setting in a concise manner. Start the game at $x \in \overline{\Omega}$ and $t = 0$. We keep in mind that for any $\varepsilon > 0$ each step brings about a movement of length

$\sqrt{2}\varepsilon$ but costs time ε^2, which certainly implies that for any time t, the total of game steps N equals the largest integer less than or equal to t/ε^2.

The discrete system now writes

$$\begin{aligned} &y_k = S_o^{\sqrt{2}\varepsilon}(y_{k-1}, b_k v_k),\ b_k = \pm 1 \text{ and } v_k \in \mathbf{S}^1 \text{ for all } k = 1, 2, \ldots, N; \\ &y_0 = x \in \overline{\Omega}, \end{aligned}$$

where v and b are control variables of two players who are adverse to each other on the quantity $u_0(y_N)$. Giving the information advantage to the player in charge of control b, we define the value function as

$$u^\varepsilon(x,t) = \inf_{|v_1|=1} \sup_{b_1=\pm 1} \ldots \inf_{|v_N|=1} \sup_{b_N=\pm 1} u_0(y(N)), \tag{4.7}$$

which, in particular, implies $u^\varepsilon(x,t) = u_0(x)$ when $t \in [0, \varepsilon^2)$. As usual, we can show that u^ε satisfies the dynamic programming principle

$$u^\varepsilon(x,t) = \inf_{|v|=1} \sup_{b=\pm 1} u^\varepsilon\left(S_o^{\sqrt{2}\varepsilon}(x, bv), t - \varepsilon^2\right) \text{ for all } t \in [\varepsilon^2, \infty). \tag{4.8}$$

It follows formally by Taylor's formula and the billiard representation (4.5) that at (x,t)

$$\begin{aligned} 0 \approx -\varepsilon^2 u_t^\varepsilon + \inf_{|v|=1} \sup_{b=\pm 1} &\left\{ \nabla u^\varepsilon \cdot \left(\sqrt{2}\varepsilon bv - \beta^{\sqrt{2}\varepsilon}\right) \right. \\ &\left. + \frac{1}{2}\nabla^2 u^\varepsilon \left(\sqrt{2}\varepsilon bv - \beta^{\sqrt{2}\varepsilon}\right) \cdot \left(\sqrt{2}\varepsilon bv - \beta^{\sqrt{2}\varepsilon}\right) \right\}. \end{aligned} \tag{4.9}$$

Moreover, we assume $x \in \partial\Omega$ since the case $x \in \Omega$ is comparatively easy, as already seen in the proof of Theorem 3.1. Viewing for the moment that $u^\varepsilon(x,t)$ has bounded derivatives and converges in some sense to a function $u(x,t)$, we discuss two cases for every subsequence, still indexed by ε:

1. Boundary condition dominant case: There exists $C > 0$ such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |\beta^{\sqrt{2}\varepsilon}| = C.$$

We then divide both sides of (4.9) by ε, pass to the limit $\varepsilon \longrightarrow 0$ and get via (4.5) that

$$0 = \sqrt{2} \inf_{|v|=1} \sup_{b=\pm 1} |\nabla u(x,t) \cdot bv| - C \nabla u(x,t) \cdot \gamma(x).$$

Since the first term on the right hand side is zero, the classical oblique boundary condition remains.

2. Mixed type case: Assume on the contrary to the former case

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |\beta^{\sqrt{2}\varepsilon}| = 0.$$

Then the same first-order operation as above yields that the "inf sup" is attained at $v = \frac{\nabla^\perp u}{|\nabla u|}$, where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. To avoid directly realizing the oblique boundary again, we additionally assume that $\nabla u(x,t) \neq 0$. If $\frac{1}{\varepsilon^2}|\beta^{\sqrt{2}\varepsilon}| \to \infty$ as $\varepsilon \to 0$, we divide both sides of (4.9) by $|\beta^{\sqrt{2}\varepsilon}|$ and send $\varepsilon \to 0$ to get

$$\nabla u(x,t) \cdot \gamma(x) = 0.$$

If $|\beta^{\sqrt{2}\varepsilon}|$ is of order $o(\varepsilon^2)$, we in turn use ε^2 as the divisor and obtain the limit equation

$$u_t - \nabla^2 u \frac{\nabla^\perp u}{|\nabla u|} \cdot \frac{\nabla^\perp u}{|\nabla u|} = 0 \text{ at } (x,t).$$

Even when $|\alpha^{\sqrt{2}\varepsilon}|$ exactly has the order ε^2, the limit of the divided equation then is

$$u_t(x,t) - \nabla^2 u(x,t) \frac{\nabla^\perp u(x,t)}{|\nabla u(x,t)|} \cdot \frac{\nabla^\perp u(x,t)}{|\nabla u(x,t)|} + M \nabla u(x,t) \cdot \gamma(x) = 0,$$

where M is a positive constant. It follows that either

$$u_t - \nabla^2 u \frac{\nabla^\perp u}{|\nabla u|} \cdot \frac{\nabla^\perp u}{|\nabla u|} \geq 0 \text{ and } \nabla u \cdot \gamma(x) \leq 0 \text{ at } (x,t)$$

or

$$u_t - \nabla^2 u \frac{\nabla^\perp u}{|\nabla u|} \cdot \frac{\nabla^\perp u}{|\nabla u|} \leq 0 \text{ and } \nabla u \cdot \gamma(x) \geq 0 \text{ at } (x,t),$$

which reveals that u fulfills the boundary condition in the viscosity sense.

The preceding mechanism gives rise to the following result.

Theorem 4.2. *Assume that Ω satisfies* (A7) *and u_0 is a continuous function in $\overline{\Omega}$. Let u^ε be the value function of the game defined by* (4.7). *Then u^ε converges, as $\varepsilon \to 0$, to the unique viscosity solution of* (E3) *uniformly on compact subsets of $\overline{\Omega} \times [0,\infty)$.*

As the formal deduction is well developed above, a rigorous proof is skipped. One may find such a proof for the Neumann boundary problem of curve shortening flow equation in [4], which is closely related to the problem here. The existence and uniqueness of the solution of (E3) is clarified in [13] and the comparison theorem we shall rely on is included there as well.

We conclude finally that a deterministic game interpretation is given for our oblique boundary problem but the domain is too special. It is of further interest to tackle more general domains. The central part is an appropriate definition of oblique billiard dynamics. There may be several ways to implement it and we are looking for the most sensible one.

References

[1] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Systems and Control: Foundations and Applications, Birkhäuser Boston, Boston, 1997.

[2] Y. Giga, *Surface evolution equations, a level set approach*, Monographs in Mathematics 99, Birkhäuser Verlag, Basel, 2006.

[3] Y. Giga and Q. Liu, *A remark on the discrete deterministic game approach for curvature flow equations*, preprint, Hokkaido University Preprint Series in Mathematics #901.
http://eprints.math.sci.hokudai.ac.jp/archive/00001842/01/pre901.pdf

[4] Y. Giga and Q. Liu, *A billiard-based game interpretation of the Neumann problem for the curve shortening equation*, to appear in Adv. Differential equations.

[5] Y. Giga and M.-H. Sato, *Neumann problem for singular degenerate parabolic equations*, Differential Integral Equations, 6 (1993), 1217-1230.

[6] B. Halpern, *Strange billiard tables*, Tran. Amer. Math. Soc., 232 (1977), 297-305.

[7] K. Kasai, *Representation of solutions for nonlinear parabolic equations via two-person game with interest rate*, preprint.

[8] R. V. Kohn and S. Serfaty, *A deterministic-control-based approach to motion by curvature*, Comm. Pure Appl. Math., 59 (2006), 344-407.

[9] R. V. Kohn and S. Serfaty, *Second-order PDE's and deterministic games*, preprint.

[10] P. L. Lions, *Neumann type boundary conditions for Hamilton-Jacobi equations*, Duke Math. J., 52 (1985), 793-820.

[11] P. L. Lions and A. S. Sznitman, *Stochastic differential equations with reflecting boundary conditions*, Comm. Pure Appl. Math., 37 (1984), 511-537.

[12] M.-H. Sato, *Interface evolution with Neumann boundary condition*, Adv. Math. Sci. Appl., 4 (1994), 249-264.

[13] M.-H. Sato, *Capillary problem for singular degenerate parabolic equations on a half space*, Differential Integral Equations, 9 (1996), 1213-1224.

[14] H. Tanaka, *Stochastic differential equations with reflecting boundary conditions in convex regions*, Hiroshima Math. J., 9 (1979), 163-177.