数理解析研究所講究録
第 1634 巻 2009 年 9-12

Homology groups of box complexes of chordal graphs

筑波大学 数理物質科学研究科 上別府陽 (Akira Kamibeppu)

Graduate School of Pure and Applied Sciences, University of Tsukuba

筑波大学 数学系 川村一宏 (Kazuhiro Kawamura)

Institute of Mathematics, University of Tsukuba

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a finite set and $E(G)$ is a family of 2-element subsets of $V(G)$. Throughout this paper, all graphs are undirected and connected. We follow [5] with respect to the standard notation in graph theory. For a graph G, J. Matoušek and G. M. Ziegler [11] introduced an abstract free simplicial $\mathbb{Z}_2$-complex $\mathsf{B}(G)$, called the *box complex* of G, to obtain a lower bound for the chromatic number $\chi(G)$ of G;

$$\chi(G) \geq \mathrm{ind}_{\mathbb{Z}_2}(\mathsf{B}(G)) + 2,$$

where $\mathrm{ind}_{\mathbb{Z}_2}(\mathsf{B}(G))$ is the $\mathbb{Z}_2$-*index* of $\mathsf{B}(G)$ (see [11] for the definition). P. Csorba [3], D. N. Kozlov [10] and R. T. Živaljević [12] proved that for any graph G, the box complex $\mathsf{B}(G)$, the neighborhood complex $\mathsf{N}(G)$, the Lovász complex $\mathsf{L}(G)$ and the homomorphism complex $\mathrm{Hom}(K_2, G)$ have the same simple homotopy type.

In this paper, we focus on $\mathsf{B}(G)$ and are interested in the relation between topology of $||\mathsf{B}(G)||$ and combinatorics of G. In [7], it is proved that $\mathsf{B}(G)$ is disconnected if and only if G is bipartite. Also in [6], it is shown that a graph G contains no 4-cycle if and only if $||\overline{G}||$ is a strong $\mathbb{Z}_2$-deformation retract of $||\mathsf{B}(G)||$, where $\overline{G}$ is a 1-dimensional subcomplex of $\mathsf{B}(G)$ introduced in [6].

Our next step is to understand topology of $||\mathsf{B}(G)||$ when G contains many 4-cycles. As a typical class of such graphs, we introduce chordal graphs. Let G be a graph and C a cycle of length at least 4 contained in G. An edge of G is a chord of C if the endvertices of the edge belong to C and the edge is not an edge of C. A graph G is *chordal* if every cycle of length at least 4 contained in G has a chord. It is known that for any chordal graph G, $\chi(G)$ is equal to its clique number which is the greatest integer m such that $K_m \subseteq G$, where K_m is a complete graph with m vertices (see [5], p128, Proposition 5.5.2). Due to the perfectness of chordal graphs, it is not difficult to see $\chi(G) = \mathrm{ind}_{\mathbb{Z}_2}(\mathsf{B}(G)) + 2$ for any chordal graph G. Some examples suggest that the homology of $\mathsf{B}(G)$ is related to maximal complete subgraphs of a chordal graph G. The purpose of this paper is to make these connections specific for chordal graphs.

We define the box complex of a graph following [11]. Let G be a graph and U a subset of $V(G)$. A vertex $v \in V(G)$ which is adjacent to each $u \in U$ is called a *common neighbor* of U in G. The set of all common neighbors of U in G is denoted by $\mathrm{CN}_G(U)$. For convenience, we define $\mathrm{CN}_G(\phi) = V(G)$. For $U_1, U_2 \subseteq V(G)$ such that $U_1 \cap U_2 = \phi$, we define $G[U_1, U_2]$ as the bipartite subgraph of G with

$$V(G[U_1, U_2]) = U_1 \cup U_2 \text{ and } E(G[U_1, U_2]) = \{u_1u_2 \mid u_1 \in U_1,\ u_2 \in U_2,\ u_1u_2 \in E(G)\}.$$

The graph $G[U_1, U_2]$ is said to be *complete* if $u_1u_2 \in E(G)$ for all $u_1 \in U_1$ and $u_2 \in U_2$. For convenience, $G[\phi, U_2]$ and $G[U_1, \phi]$ are also said to be complete.

Let U_1, U_2 be subsets of $V(G)$. The subset $U_1 \uplus U_2$ of $V(G) \times \{1, 2\}$ is defined as

$$U_1 \uplus U_2 := (U_1 \times \{1\}) \cup (U_2 \times \{2\}).$$

For vertices $u_1, u_2 \in V(G)$, $\{u_1\} \uplus \phi$, $\phi \uplus \{u_2\}$, and $\{u_1\} \uplus \{u_2\}$ are simply denoted by $u_1 \uplus \phi$, $\phi \uplus u_2$ and $u_1 \uplus u_2$ respectively.

The *box complex* of a graph G is an abstract simplicial complex with the vertex set $V(G) \times \{1,2\}$ and the family of simplices

$$\begin{aligned} \mathsf{B}(G) = \{U_1 \uplus U_2 \mid U_1, U_2 \subseteq V(G),\ U_1 \cap U_2 = \phi, \\ G[U_1, U_2] \text{ is complete, } \mathrm{CN}_G(U_1) \neq \phi \neq \mathrm{CN}_G(U_2)\,\}. \end{aligned}$$

The box complex $\|\mathsf{B}(K_m)\|$ of a complete graph K_m with m vertices have same homotopy type as the $(m-2)$-dimensional sphere S^{m-2}. For a chordal graph G, the computation of the homology group of $\mathsf{B}(G)$ relies on the following decompositions, called a *simplicial decomposition* of G: every chordal graph has a sequence $(V_1, \cdots, V_n)$ of its maximal complete subgraphs such that $G = \cup_{i=1}^{n} V_i$ and the intersection $(\cup_{i=1}^{k-1} V_i) \cap V_k$ is a complete subgraph of G for each $k = 2, \cdots, n$ (see [1]). This allows us to make an inductive argument which examines the difference between $\mathsf{B}(\cup_{i=1}^{k-1} V_i) \cup \mathsf{B}(V_k)$ and $\mathsf{B}(\cup_{i=1}^{k} V_i)$. We may consider a more general situation: let G be the union of two induced subgraphs G_1 and G_2 of G, and we study the homomorphism $H_*(\mathsf{B}(G_1) \cup \mathsf{B}(G_2)) \to H_*(\mathsf{B}(G))$ on homology groups induced by the inclusion $\mathsf{B}(G_1) \cup \mathsf{B}(G_2) \hookrightarrow \mathsf{B}(G)$.

Theorem 1 ([8]). If $G_1 \cap G_2$ is complete and $\mathrm{CN}_{G_1}(V(G_1 \cap G_2)) \neq \phi \neq \mathrm{CN}_{G_2}(V(G_1 \cap G_2))$, then the inclusion $\mathsf{B}(G_1) \cup \mathsf{B}(G_2) \hookrightarrow \mathsf{B}(G)$ induces isomorphisms

$$\tilde{H}_*(\mathsf{B}(G_1) \cup \mathsf{B}(G_2)) \to \tilde{H}_*(\mathsf{B}(G))$$

on homology groups.

Remark 2. Under the same condition in Theorem 1, we can say more: the inclusion $\mathsf{B}(G_1) \cup \mathsf{B}(G_2) \hookrightarrow \mathsf{B}(G)$ is a homotopy equivalence (see [8], Remark 3.14).

We apply Theorem 1 to $(G_1, G_2) = (H, K_m)$, where $H \cap K_m = K_l$ and H is a graph with $\mathrm{CN}_H(V(K_l)) \neq \phi$ and $m > l$. The following is one of our main theorems in this paper.

Theorem 3 ([8]). For every chordal graph G, the homology group $H_*(\mathsf{B}(G))$ is torsion free.

Let $\beta_q = \beta_q(\mathsf{B}(H))$ be the q-dimensional Betti number of $\mathsf{B}(H)$. We need the following proposition to prove Theorem 3.

Proposition 4 ([8]). Let G be the union of a graph H and a complete graph K_m such that $H \cap K_m = K_l$, where $m > l$ and $\mathrm{CN}_H(V(K_l)) \neq \phi$. Assume that H satisfies the condition

> $(*)$ for any q, there exists a basis $\{e_1^q, \cdots, e_{\beta_q}^q\}$ of $H_q(\mathsf{B}(H))$ such that for any $(q+2)$-complete subgraph Q of H with the inclusion $j_Q : Q \hookrightarrow H$, $j_{Q*}([\mathsf{B}(Q)]) = 0$ or $\pm e_{k_Q}^q$ for some $k_Q \in \{1, \cdots, \beta_q\}$.

Then, G satisfies the condition $(*)$ as well.

Proof of Theorem 3. Let $(V_1, \cdots, V_n)$ be a simplicial decomposition of G and $G_k := \cup_{i=1}^{k} V_i$, the union of complete graphs $V_1, \cdots, V_k$. We use the induction on k. For $k = 1$, the conclusion holds since G_1 is a complete graph V_1. Note that $G_k = G_{k-1} \cup V_k$ and $G_{k-1} \cap V_k$ is complete. Moreover, for each k, we see

$$\mathrm{CN}_{G_{k-1}}(V(G_{k-1} \cap V_k)) \neq \phi \neq \mathrm{CN}_{V_k}(V(G_{k-1} \cap V_k))$$

by the maximality of $V_1, \cdots, V_n$. Now the theorem follows from Proposition 4. □

From the proof of Proposition 4, we can obtain two more results with respect to the box complex of chordal graphs. We extract some consequences of the proof of Proposition 4. We note that (i) in Remark 5 is also a consequence of the Folding theorem (see [9], Theorem 3.3 and also [2], [4]). Let $[\mathsf{B}(K_m)]$ be a generator of $H_{m-2}(\mathsf{B}(K_m))$.

Remark 5. Let G be a chordal graph with a simplicial decomposition $(V_1, \cdots, V_n)$ and $G_{k-1} = \cup_{i=1}^{k-1} V_i$. We consider the graph $G_k = G_{k-1} \cup V_k$. Put $G_{k-1} \cap V_k = K_{l_{k-1}}$.

(i) If $|V_k| = l_{k-1} + 1$, then we obtain $[\mathsf{B}(V_k)] = [\mathsf{B}(K_{l_{k-1}+1})]$ in $H_{|V_k|-2}(\mathsf{B}(G_k))$ (hence, in $H_{|V_k|-2}(\mathsf{B}(G))$ as well), where $K_{l_{k-1}+1}$ is a complete subgraph of G_{k-1} containing $G_{k-1} \cap V_k$. Also we see that the inclusion $G_{k-1} \hookrightarrow G_k$ induces isomorphisms on homology groups: $H_*(\mathsf{B}(G_k)) \cong H_*(\mathsf{B}(G_{k-1}))$.

(ii) If $|V_k| \geq l_{k-1} + 2$, $[\mathsf{B}(V_k)]$ is one of generators of $H_{|V_k|-2}(\mathsf{B}(G_k))$ for each $k \in \{1, \cdots, n\}$.

(iii) $H_q(\mathsf{B}(G)) = 0$ for each $q \geq \omega(G) - 1$ (note that there exists no clique of G with at least $(\omega(G) + 1)$ vertices).

Let χ_{K} be the Euler characteristic of a complex K. The proof of Proposition 4 reveals the relation between $\chi_{\mathsf{B}(G)}$ and $\chi_{\mathsf{B}(H)}$.

Corollary 6 ([8]). Let G be the union of a graph H and a complete graph K_m such that $H \cap K_m = K_l$, where $m > l$ and $\mathrm{CN}_H(V(K_l)) \neq \phi$. We assume that H satisfies the condition $(*)$ of Proposition 4. Then, we have

$$\chi_{\mathsf{B}(G)} = \chi_{\mathsf{B}(H)} + (1 - \delta_{m,l+1})((-1)^{m-2} + (-1)^l).$$

For a chordal graph G, we can describe the Euler characteristic $\chi_{\mathsf{B}(G)}$ of $\mathsf{B}(G)$ in terms of a simplicial decomposition of G by Corollary 6.

Theorem 7 ([8]). Let G be a chordal graph with a simplicial decomposition $(V_1, \cdots, V_n)$. Moreover, let $m_k := |V_k|$, $l_0 := 0$ and $l_{k-1} := |(\cup_{i \leq k-1} V_i) \cap V_k|$. We have the following equality:

$$\chi_{\mathsf{B}(G)} = \sum_{m_k > l_{k-1}+1} ((-1)^{m_k - 2} + (-1)^{l_{k-1}}).$$

For any complete subgraph V of a chordal graph G, by noting Remark 5, we can determine whether a generator $[\mathsf{B}(V)]$ of $H_{|V|-2}(\mathsf{B}(V))$ is zero or not in $H_{|V|-2}(\mathsf{B}(G))$.

Theorem 8 ([8]). Let V be a maximal complete subgraph of a chordal graph G. The inclusion $i_V : V \hookrightarrow G$ induces the zero homomorphism $i_{V*} : H_{|V|-2}(\mathsf{B}(V)) \to H_{|V|-2}(\mathsf{B}(G))$ if and only if there exists a sequence $\{k_j\}_{j=1}^p \subset \{1, \cdots, n\}$ such that

(1) $|V_{k_j}| = |V|$ and $|V| - |V \cap V_{k_1}| = |V_{k_j}| - |V_{k_j} \cap V_{k_{j+1}}| = 1$ for $j = 1, \cdots, p-1$ and
(2) $|V_{k_p}| > |V_{k_{p-1}}|$.

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory, Graduate Texts in Mathematics 244, Springer-Verlag, 2008.

[2] E. Babson and D. N. Kozlov, Complexes of graph homomorphisms, Israel J. Math. 152 (2006), 285-312.

[3] P. Csorba, On the simple $\mathbb{Z}_2$-homotopy type of graph complexes and their simple $\mathbb{Z}_2$-universality, Canad. Math. Bull. 51 (2008), no.4, 535-544.

[4] S. Cukic and D. N. Kozlov, The homotopy type of complexes of graph homomorphisms between cycles, Discrete Computational Geometry 36 (2006), no.2, 313-329.

[5] R. Diestel, Graph Theory. 3rd ed. Graduate Texts in Mathematics 173, Springer-Verlag, 2005.

[6] A. Kamibeppu, Homotopy type of the box complexes of graphs without 4-cycles, Tsukuba J. Math. 32 (2008), no.2, 307-314.

[7] A. Kamibeppu, Subcomplexes of box complexes of graphs, to appear in Tsukuba J. Math.

[8] A. Kamibeppu and K. Kawamura, Homology groups of box complexes of chordal graphs, submitted.

[9] D. N. Kozlov, A simple proof for folds on both sides in complexes of graph homomorphisms. Proc. Amer. Math. Soc. 134 (2006), no.5, 1265-1270.

[10] D. N. Kozlov, Simple homotopy types of Hom-complexes, neighborhood complexes, Lovász complexes, and atom crosscut complexes. Topology Appl. 153 (2006), no.14, 2445-2454.

[11] J. Matoušek and G. M. Ziegler, Topological lower bounds for the chromatic number: A hierarchy. Jahresbericht der Deutschen Mathematiker-Vereinigung, 106 (2004), no.2, 71-90.

[12] R. T. Živaljević, WI-posets, graph complexes and $\mathbb{Z}_2$-equivalences, J. Combin. Theory Ser. A 111 (2005), no.2, 204-223.