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Topological Spaces of Discrete Distributions

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Let f be a real-valued function defined on a non-empty set S. If f satisfies the following conditions, then we call that f is a discrete distribution on S:

(1) $0 \leq f(s) \leq 1$ for any $s \in S$.

(2) $\Sigma\{f(s) : s \in S\} \leq 1$.

In case $\Sigma\{f(s) : s \in S\} = 1$ is satisfied in (2), f is called to be a discrete probability distribution on S. Let the support $spt(f)$ of f be the set $\{s \in S : f(s) > 0\}$. Obviously, $spt(f)$ is a countable subset of S for any discrete distribution f.

Let $DD(S)$ and $DPD(S)$ be the set of all discrete distributions and the set of all discrete probability distributions on a set S respectively. $DPD(S)$ is sometimes denoted by Σ_S. We consider the topology of pointwise convergence on $DD(S)$. This topology on $DD(S)$ is the relative topology induced by naturally embedding $DD(S)$ into the power space $I^{|S|}$ of the unit interval $I = [0,1]$.

It is obvious that $DD(S)$ is a closed subset of $I^{|S|}$, and hence $DD(S)$ is compact. $DD(S)$ is Fréchet as a subspace of a Σ-product of the unit intervals. A cardinal κ is considered as the topological space with the usual interval topology.

Let us call a non-decreasing continuous function $f : \kappa \to [0,1]$ with $f(0) = 0$ to be a cumulative, discrete distribution on κ. Let $CDD(\kappa)$ be the set of all cumulative, discrete distributions on κ. On $CDD(\kappa)$, we consider also the topology of pointwise convergence. The space $CDPD(\kappa)$ is the subspace of $CDD(\kappa)$ consisting of functions satisfing $\lim_{\alpha\to\kappa} f(\alpha) = 1$

Fact 1 *(Dydak). A topological space X is metrizable if and only if X is embedded in $DPD(S)$ for some set S. In other words, for a cardinal κ, $DPD(\kappa)$ is universal for metrizable spaces of weight $\leq \kappa$.*

Theorem 1 *The spaces $DPD(\kappa)$ and $CDPD(\kappa)$ are homeomorphic.*

Proof. For $f \in DPD(\kappa)$, let $F : \kappa \to [0,1]$ be the function defined by

$$F(0) = 0, F(\alpha) = \sum\{f(\beta) : \beta < \alpha\} \quad \text{for} \quad 0 < \alpha < \kappa.$$

Then the map $\Psi : DPD(\kappa) \to CDPD(\kappa)$ defined by $\Psi(f) = F$ is obviously one-to-one and onto.

Claim 1. Ψ is continuous.

Let f be an arbitrary point in $DPD(\kappa)$ and let $\Psi(f) = F$. For any $\epsilon > 0$, there is a finite subset A of $spt(f)$ such that $\sum_{a \in A} f(a) > 1 - \frac{\epsilon}{4}$. Assume that A is composed of n elements. Let $\delta = \frac{\epsilon}{4n}$ and $g \in DPD(\kappa)$ be an arbitrary point such that $|g(a) - f(a)| < \delta$ for every $a \in A$. Then

$$1 - \frac{\epsilon}{2} < \sum_{a \in A} g(a) \leq 1,$$

and hence $\sum_{a \notin A} g(a) < \frac{\epsilon}{2}$. Further, for any $B \subset A$

$$|\sum_{b \in B} g(b) - \sum_{b \in B} f(b)| \leq \sum_{b \in B} |g(b) - f(b)| \leq |B| \frac{\epsilon}{4n} < \frac{\epsilon}{4}.$$

Then for any $\alpha < \kappa$,

$$|\sum_{\beta < \alpha} g(\beta) - \sum_{\beta < \alpha} f(\beta)| \leq |\sum_{\beta < \alpha, \beta \notin A} g(\beta)| + |\sum_{\beta < \alpha, \beta \notin A} f(\beta)| + |\sum_{\beta < \alpha, \beta \in A} g(\beta) - \sum_{\beta < \alpha, \beta \in A} f(\beta)|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.$$

This means that for $G = \Psi(g)$, $|G(\alpha) - F(\alpha)| < \epsilon$ for every $\alpha < \kappa$, and hence Ψ is continuous.

Claim 2. Ψ^{-1} is continuous.

Let $F \in CDPD(\kappa)$ and $f = \Psi^{-1}(F)$. Notice that $f(\alpha) = F(\alpha + 1) - F(\alpha)$. For any $\epsilon > 0$, there is a subset A of $spt(f)$ such that $\sum_{a \in A} f(a) > 1 - \frac{\epsilon}{4}$. We can assume that A consists of n elements. Let $\delta = \frac{\epsilon}{4n}$. Suppose that $G \in CDPD(\kappa)$ satisfies that $|G(a) - F(a)| < \delta$ for any $a \in A \cup A+1$, where $A+1 = \{a+1 : a \in A\}$. Then $g = \Psi^{-1}(G)$ satisfies

$$\begin{aligned} |g(a) - f(a)| &= |(G(a+1) - G(a)) - (F(a+1) - F(a))| \\ &\leq |G(a+1) - F(a+1)| + |G(a) - F(a)| < 2\delta = \frac{2\epsilon}{4n} \leq \epsilon \end{aligned}$$

for any $a \in A$. Since

$$\sum_{a \in A} g(a) \geq \sum_{a \in A} f(a) - n\frac{2\epsilon}{4n} > 1 - \frac{3\epsilon}{4},$$

$|g(b) - f(b)| < |g(b)| + |f(b)| < \frac{3\epsilon}{4} + \frac{\epsilon}{4} = \epsilon$ for any $b \notin A$. This shows that Ψ^{-1} is continuous.

We assume that all topological spaces considered here are Tychonoff. What is the class of topological spaces embedded in $DD(S)$ for some set S ? This is the theme of this note. If S is an uncountable set, then the constant zero function 0 is in $DD(S)$ and the pseudo-character at 0 in $DD(S)$ is uncountable. That is, $DD(S)$ is not metrizable. It is obvious that $DPD(S)$ is a dense metrizable subspace of $DD(S)$. A space embedded in $DD(S)$ for some set S is called a DD-space here. That is, X is a DD-space if and only if there exists a family $\{f_\alpha : \alpha \in \kappa\}$ of continuous functions from X to I such that $\Sigma\{f_\alpha(x) : \alpha \in \kappa\} \leq 1$ for each $x \in X$ and the topology of X coincides with the topology induced by $\{f_\alpha : \alpha \in \kappa\}$.

Theorem 2 *(0) Every metrizable space is a DD-space.*

(1) If Y is a subspce of a DD-space X, then Y is a DD-space.

(2) If $\{X_\alpha : \alpha \in A\}$ is a family of DD-spaces, then the topological sum $\bigoplus\{X_\alpha : \alpha \in A\}$ is a DD-space.

(3) If $\{X_n : n \in \omega\}$ is a countable family of DD-spaces, then the product space $\prod\{X_n : n \in \omega\}$ is a DD-space.

(4) Every DD-space has a compactification which is also a DD-space.

Theorem 3 *Let X be a DD-space. Then there is a real-valued function $\phi : X \to I$ such that the topology induced by the topology of X and $\{\phi^{-1}((u,v)) : (u,v)$ is an open interval in $[0,1]\}$ is metrizable. Especially, let $\phi : DD(S) \to I$ be the function defined by $\phi(f) = 1 - \Sigma\{f(s) : s \in S\}$. Then the space with the topology induced by the topology of $DD(S)$ and inverse images of open intervals by ϕ is homeomorphic to $DPD(S)$.*

Let us recall that a compact space K is called uniformly Eberlein compact if it is homeomorphic to a weakly compact subsets of a Hilbert space. The space $c_0(\Gamma)$, for a set $\Gamma \neq \emptyset$, is defined by

$$c_0(\Gamma) = \{x \in \mathbf{R}^\Gamma : |\{\gamma \in \Gamma : |x(\gamma)| > \epsilon\}| < \omega, \forall \epsilon > 0\}.$$

The norm on $c_0(\Gamma)$ is the sup norm. The weak topology on a weakly compact subset of $c_0(\Gamma)$ is exactly the topology of pointwise convergence.

Fact 2 *(Benyamini-Starbird). A compact space K is uniformly Eberlein compact if and only if K is homeomorphic to a subset K' of $c_0(\Gamma)$ for some Γ with the property that for every $\epsilon > 0$ there exists $N(\epsilon) \in \omega$ such that for every $x \in K'$,*

$$|\{\gamma \in \Gamma : |x(\gamma)| > \epsilon\} < N(\epsilon).$$

We say that a family $\mathcal{A}$ of subsets of a set X is boundedly point finite if there exists some $n \in \omega$ such that for every $x \in X$ $\text{ord}(x, \mathcal{A}) \leq n$. A family $\mathcal{A}$ of subsets of X is said to be σ-boundedly point finite if $\mathcal{A} = \bigcup_{k \in \omega} \mathcal{A}_k$ such that each family $\mathcal{A}_k$ is boundedly point finite. A family $\mathcal{A}$ of subsets of a set X is called T_0-separating if whenever $x, y \in X$ are distinct, then some $A \in \mathcal{A}$ contains exactly one of x and y.

Fact 3 *(Benyamini-Rudin-Wage). A compact space K is uniformly Eberlein compact if and only if K has a σ-boundedly point finite T_0-separating family by cozero-sets.*

Theorem 4 *The space $DD(S)$ is uniformly Eberlein compact for any set S.*

In fact, let $\mathbf{Q}'$ be the set of all rational numbers in $[0,1]$. For each $q \in \mathbf{Q}'$ and $s \in S$, let

$$U_s(q) = \{f \in DD(S) : f(s) > q\}.$$

Then

$$\mathcal{A}_q = \{U_s(q) : s \in S\}$$

is a boundedly point finite family by cozero-sets in $DD(S)$. Further, let

$$\mathcal{A} = \bigcup_{q \in \mathbf{Q}'} \mathcal{A}_q.$$

Then $\mathcal{A}$ is a σ-boundedly point finite T_0-separating family by cozero-sets.

Theorem 5 *Every uniformly Eberlein compact space is a DD-space.*

Let $\mathcal{A} = \bigcup_{n \in \omega} \mathcal{A}_n$ be a σ-boundedly point finite T_0-separating family by cozero-sets in a uniformly Eberlein compact space X. For each $n \in \omega$, let k_n be a positive integer such that $\text{ord}(x, \mathcal{A}_n) \leq k_n$ for any $x \in X$. For each $U \in \mathcal{A}$, we take a $[0,1]$-valued continuous function f_U on X with $f_U^{-1}((0,1]) = U$. Further, the function g_U is defined by

$$g_U = \frac{1}{2^n k_n} f_U.$$

Let $\mathcal{F}_n = \{g_U : U \in \mathcal{A}_n\}$ and $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n$. Then the map $\Phi : X \to [0,1]^{\mathcal{A}}$ defined by

$$\Phi(x) = \{g_U(x) : U \in \mathcal{A}\}$$

is a topological embedding of X into $DD(\mathcal{A})$. Note that

$$\sum_{U \in \mathcal{A}} g_U(x) = \sum_{n=1}^{\infty} \frac{1}{2^n k_n} \sum_{U \in \mathcal{A}_n} f_U(x) \leq 1.$$

Corollary 1 *$DD(\kappa)$ is universal for uniformly Eberlein compact spaces of weight $\leq \kappa$.*

The following two theorems may be proved under more general conditions. But, we give direct proofs here.

Theorem 6 *Let X be a DD-space. If X is countably compact, then X is compact.*

Proof. For each $r \leq 1$, let $D_{\leq r}$ (resp. $D_{<r}$) be the subset of of $DD(S)$ consisting of all f with $\Sigma\{f(s) : s \in S\} \leq r$ (resp. $\Sigma\{f(s) : s \in S\} < r$). It suffices to show that X is Lindelöf. Assume that X is not Lindelöf. Then there is an open cover $\mathcal{U}$ with no countable subcover. Let $X_{\leq r} = X \cap D_{\leq r}$ and $X_{<r} = X \cap D_{<r}$ for $0 \leq r \leq 1$. Then there exists

$$r_0 = \sup\{r : X_{\leq r} \text{ is covered by a countable subfamily of } \mathcal{U}\}.$$

It follows that there exists a countable subfamily $\mathcal{U}_0$ of $\mathcal{U}$ which covers $X_{<r_0}$ It is also true that $X_{\leq r_0}$ is covered by $\mathcal{U}_0$, since $X_{=r_0} = X_{\leq r_0} - X_{<r_0}$ is metrizable. Further, let

$$r_1 = \inf\{r : (X - \cup\mathcal{U}_0) \cap X_{\leq r} \neq \emptyset\}.$$

Then $r_0 = r_1$ must be satisfied. Let $F_n = X_{\leq(r_0+1/n)} - \cup\mathcal{U}_0$ for $n = 1, 2, \cdots$. Then this is a decreasing sequence of closed subsets of X such that $\cap\{F_n : n = 1, 2, \cdots\} = \emptyset$. This contradicts the countable compactness of X.

Theorem 7 *For a DD-space X, the cardinalities $c(X), d(X)$ and $w(X)$ are all the same.*

Proof. Let $d(X) = \lambda$ and D be a dense subset of X such that $|D| = \lambda$. Then the cardinality of $A = \cup\{spt(x) : x \in D\}$ is λ. Since D is a subset of the compact set

$$(DD(S) \cap I^A) \times \{0\}^{S-A}$$

in $DD(S)$, X must be a subspace of $I^A \times \{0\}^{S-A}$ whose weight is λ. It follows that $d(X) = w(X)$.

Next, we will show that $d(X) \leq c(X)$. Of course, we can assume that $d(X)$ is uncountable. Let $\kappa \leq d(X)$ be an arbitrary uncountable regular cardinal. then there exists a transfinite sequence $\{x_\alpha : \alpha < \kappa\}$ of points in X such that $spt(x_\alpha) - \cup\{spt(x_\beta) : \beta < \alpha\} \neq \emptyset$. Further, we can fix a positive integer k such that there exists $u_\alpha \in spt(x_\alpha) - \cup\{spt(x_\beta) : \beta < \alpha\}$ with $x_\alpha(u_\alpha) > 1/k$ for any $\alpha < \kappa$. Let $U_\alpha = \{x \in X : x(u_\alpha) > 1/k\}$. Then the family $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ satisfies that each intersection of k members of $\mathcal{U}$ is empty. Hence there must be a disjoint family consisting of κ non-empty open subsets.

As mentioned previously, the spaces $DPD(\kappa), CDPD(\kappa)$ are homeomorphic for any cardinal number κ. However $DD(\kappa)$ and $CDD(\kappa)$ are not homeomorphic for an uncontable cardinal κ. In fact, $DD(\kappa)$ is compact. On the other hand, $CDD(\kappa)$ is not compact.

Moreover, let $IDD_0(\kappa)$ be the space of all non-decreasing $[0,1]$-valued functions f (which need not be continuous) such that $f(0)=0$, with the topology of pointwise convergence. Then $IDD_0(\kappa)$ is a compactification of $CDD(\kappa)$. Further,

Theorem 8 *$CDD(\kappa)$ is not a DD-space for any uncountable cardinal κ.*

For each $\alpha<\kappa$, let $f_\alpha \in CDD(\kappa)$ be the function defined by

$$f_\alpha(\beta)=0 \quad \text{for} \quad \beta\le\alpha, \quad f_\alpha(\beta)=1 \quad \text{for} \quad \beta>\alpha.$$

Then $A=\{f_\alpha:\alpha\in\kappa\}$ is a discrete subset of $CDD(\kappa)$ and the constant zero function $\mathbf{0}$ is in the closure of this set. But there is no sequence in A converging to $\mathbf{0}$, which means that $CDD(\kappa)$ is not Fréchet. Hence $CDD(\kappa)$ is not a DD-space.

Theorem 9 *There is a one-to-one continuous map from $CDD(\kappa)$ onto $DD(\kappa)$.*

In fact, the map $\Psi : CDD(\kappa)\to DD(\kappa)$ defined by $\Psi(F)(\alpha)=F(\alpha+1)-F(\alpha)$ is one-to-one, onto and continuous.

Let us call a topological space X to be a CDD-space if X is homeomorphic to a subspace of $CDD(\kappa)$ for some cardinal κ.

Theorem 10 *(1) Every metrizable space is a CDD-space.*

(1) If a CDD-space X is compact, then X is a DD-space.

(2) If X is a CDD-space, then there is a σ-boundedly point-finite, T_0-separating cozero-family.

Hence, it follows that there is a CDD-space X such that every compactification of X is not a CDD-space.

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