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On asymptotic behaviors of positive solutions for fuzzy difference equations
-ファジィ差分方程式の正値解の漸近挙動について-

同志社大学理工学部 数理システム学科 齋藤誠慈
Faculty of Science and Engineering
Doshisha University
Seiji SAITO

Abstract.

In this paper we introduce results on asymptotic behaviors of Papaschinopoulos et al. [1,2], which are concerning positive equilibrium points of fuzzy difference equations with delays. We discuss sufficient conditions of the stability for positive solutions to real types of difference equations with delays and also we speak about open problems on the existence and stability of positive equilibrium points for fuzzy difference equations with finite delays.

1. Introduction

Consider the following fuzzy difference equation with finite delays

$$x_{n+1} = A + \frac{x_n}{x_{n-k}} \quad (n = 0,1,\ldots), \tag{1}$$

where $\{ x_n : n = 0,1,\ldots\}$ is a sequence of positive numbers, A, x_{-k}, x_{-k+1}, ..., x_0 are positive fuzzy numbers and $k \geq 1$ is an integer. If A is a function from $\mathbf{R}^+ = (0, +\infty)$ into the interval $[0,1]$ with the normality that there exists a real number c such that $A(c) = 1$, with the upper semicontinuity and fuzzy convexity, then A is a called fuzzy number (fuzzy set of one variable). Sequences of positive fuzzy numbers x_n for $n = 0,1,..$ are bounded and persist if the set $\{ \text{supp } x_n : n = 0,1,.. \}$ is uniformly bounded and away from the zero and the infinity. Denote the a-cut sets with $0 < a \leq 1$ of A by $[A]_a = \{ x \in \mathbf{R} : A(x) \geq a \}$. The a-cut set $[A]_a$ denotes by a bounded closed interval $[A_{l,a}, A_{r,a}]$ from the fuzzy convexity, where $A_{l,a}$, $A_{r,a}$ are the left-, right- end point of

$[A]_a$, respectively. In [1,2] Papaschinopoulos et al. proved boundedness, convergence and stability results on positive solutions of fuzzy difference equations with delays(Section 2). A solution $\{x_n\}$ of Eq.(1) is positive if all functions x_n for $n = 0,1,\ldots$ are bounded away from the zero. In Section 3 we discuss asymptotic behaviors for real types of difference equations with delays and also we deal with open problems of real types and fuzzy types of difference equations.

2. Boundedness, convergence and stability results

In [1] Papaschinopoulos et al. proved boundedness and stability theorems on positive solutions of fuzzy difference equations (1) as follows.

Proposition P1. The following statements (i) and (ii) hold truly.

(i) Suppose that $A_{l,a} > 1$ hold for all $0 < a \leq 1$. Then every positive solution to Eq.(1) is bounded and persists.

(ii) Suppose that there exists $0 < b \leq 1$ such that $A_{l,b} > 1$. Then Eq. (1) has at least unbounded solutions.

In [1] they gave the following results on the existence and convergence of positive equilibrium points for Eq.(1) as follows.

Proposition P2. Suppose that there exists a constant $M > 1$ such that $A_{l,a} > M$. Then following statements (i) and (ii) hold truly.

(i) Eq.(1) has a unique positive equilibrium x with $[x]_a = [L_a, R_a]$ such that

$$L_a = \frac{A_{l,a}A_{r,a} - 1}{A_{r,a} - 1}, \quad R_a = \frac{A_{l,a}A_{r,a} - 1}{A_{l,a} - 1}$$

for $0 < a \leq 1$.

(ii) Every positive solution $\{ x_n \}$ is of Eq.(1) tends to the positive equilibrium x_e as $n \to \infty$.

The above propositions are proved by an elementary method. It is possible that the Liapunov's second method (e.g., [3 - 9]) leads to so many fruitful results and weaker conditions for stability results on fuzzy difference equations.

In [2] Papaschinopoulos et al. proved the existence of an infinite number of positive equilibrium and convergence theorems to the following equation.

(2) $$x_{n+1} = \sum_{i=0}^{k} \frac{A_i}{x_{n-i}^{p_i}} \quad (n = 0,1,\ldots).$$

Here k is positive integer, A_i for i = 0,1, ..., k are positive fuzzy numbers and p_i for i = 0,1, ..., k are positive real constants. [2] gave the following boundedness, persistence, the existence and stability of equilibrium points and the unboundedness theorems of positive solutions for Eq.(2).

Proposition P3. The following statements (i) and (iii) hold truly.

(i)Every positive solution for Eq.(2) is bounded and persists if either the following conditions (a), or, (b) holds.

(a) For every i = 0,1,..,k, there exists a λ_i such that $p_i p_{\lambda_i} < 1$;

(b) p_i =1 and every A_i is positive real constant for i=0,1,.., k.

(ii) If the relation $p_i \le 1$ for i = 0,1,..,k and

(3) $$\sum_{i=0}^{k} p_i < k+1$$

holds, Eq.(2) has a unique positive equilibrium x_e. Moreover, every positive solution $\{x_n\}$ of Eq.(2) nearly converges to x_e with respect to D and D_1 as $n \to +\infty$.

(iii) Under the same condition (b) of (i), Eq.(2) has an infinite number of positive solutions and every positive solution $\{x_n\}$ of Eq.(2) nearly converges to a positive equilibrium with respect to D and D_1 as $n \to +\infty$.

Here the metrics D and D_1 between fuzzy numbers y, z are as follows. Denote $[y]_a = [y_{l,a}, y_{r,a}]$ and $[z]_a = [z_{l,a}, z_{r,a}]$ for $0 < a \le 1$.

$$D(y,z) = \max_{0<a\le 1}(|y_{l,a} - z_{l,a}|, |y_{r,a} - z_{r,a}|),$$

$$D_1(y,z) = \int_0^1 (|y_{l,a} - z_{l,a}| + |y_{r,a} - z_{r,a}|)da,$$

$$D_S(y,z) = \sup_{a\in(0,1]-S} \max(|y_{l,a} - z_{l,a}|, |y_{r,a} - z_{r,a}|),$$

where S $\subset (0,1]$ is a measurable set. An equilibrium x of fuzzy Eq.(2) is said to be nearly asymptotically stable if x is stable with respect to D and for each $\delta > 0$ and each solution $\{x_n\}$ of Eq.(2) there exists a measurable set S $\subset (0,1]$ with the measure meas(S) $< \delta$ such that $\lim_{n \to +\infty} D_S(x_n, x) = 0$. The following theorem is concerning the above asymptotic stability of the equilibrium of Eq.(2).

Proposition P4. Suppose that each A_i and p_i for i = 0,1,..,k satisfy

$$h = \sum_{i=0}^{k} p_i < 1, \qquad \left(\frac{E}{H}\right)^{\frac{1}{1-h}} > p_i \ (i = 0,1,\dots,k),$$

where E = min{ K_i : i = 0,1,..,k }, H = max{H_i : i = 0,1,..,k }, K_i, H_i are defined as supp(Ai) $\subset$ [Ki, Hi] for i = 0,1,..,k. Then (2) has a unique positive equilibrium x which is nearly asymptotically stable.

The authors[2] proved an unboundedness theorem as follows.

Proposition P5. If there exists non-real number A_j at $0 \le j \le k$ and each $p_i = 1$ for i = 0,1,..,k, then every solution of Eq.(2) neither is bounded nor persists.

3. Discussion and open problems

In this section we discuss asymptotic behaviors of positive solutions for real types of various difference equations (see [8]). Moreover we give open problems to real types and fuzzy types of difference equations.

Consider the following real type of difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), \tag{4}$$

for n = 0,1,..., and k is a positive integer. Assume that F=F(x_0, x_1, ...,x_k) is a C^1-class function defined on $\mathbf{R}^{k+1}$ to $\mathbf{R}^1$ and let an $x_e \in \mathbf{R}^1$ be an equilibrium of Eq.(4). If all the roots of the polynomial equation

$$\lambda^{k+1} = \sum_{i=0}^{k} \frac{\partial F}{\partial x_i}(x_e, x_e, \dots, x_e)\lambda^{k-i} \tag{5}$$

lie in the open unit disk $\{z \in C : |z| < 1\}$, then the equilibrium x_e of Eq.(4) is asymptotically stable. If at least one root of (5) has absolute value greater than one, then the equilibrium e of (4) is unstable. See [8].

In case where the real-valued function $F(x_n, x_{n-1}) = A + \dfrac{x_n}{x_{n-1}}$ (A, x_n, x_{n-1} are real numbers) with k=1 Eq.(4) has the equilibrium $x_e = A + 1$ with $A > -1$, because we consider positive solutions for Eq.(4). The characteristic Eq. $(A+1)\lambda^2 - \lambda + 1 = 0$ has the roots $\lambda = \dfrac{1 \pm \sqrt{-(4A+3)}}{2(A+1)}$. The relation $|\lambda| < 1$ leads to an inequality

(6) $$A > 0.$$

Then we have the following open problems Q1 – Q3.

Q1: Prop.P2 shows that the equilibrium for fuzzy types of (1) with $A_{l,a} > M > 0$ is the globally property. In real types of Eq.(1) with k = 1 and $A > 0$, does the local stability mean the global stability ?

Q2: Find a condition for stability for real types of (1) with $k > 2$.

Q3: Is fuzzy types of Eq.(1) stable with $0 < M \leq 1$?

We consider positive solutions for the following real type of difference equation

(7) $$x_{n+1} = \frac{a + bx_n}{c + x_{n-1}} \quad (n = 0, 1, \ldots),$$

where a, b, c are positive real. Eq.(7) has the positive equilibrium $x_e = \dfrac{b - c + \sqrt{(b-c)^2 + 4a}}{2}$. In the special case where (8) , or, (9) hold, the equilibrium of Eq.(7) is globally asymptotically stable (see[8]):

(8) $b < c$,

(9) $b \geq c$ with either $a \leq bc$, or, $bc < a \leq 2c(b + c)$.

Ref.[8] gives the following open problem Q4.

Q4: Is the positive equilibrium of Eq.(7) globally asymptotically stable under that $a,b,c > 0$ hold ?

Q5: Find the local, global stability conditions for the positive equilibrium of fuzzy types of Eq.(7).

Every positive solution of Eq. $x_{n+1} = \frac{1}{x_n}$ is 2-periodic. Every positive solution of Eq. $x_{n+1} = \frac{1+x_n}{x_{n-1}}$ is 5-periodic. Every positive solution of Eq. $x_{n+1} = \frac{1+x_n+x_{n-1}}{x_{n-2}}$ is 8-periodic. See [8]. Every positive solution of equation

(10) $$x_{n+1} = \frac{1+x_n+x_{n-1}+\ldots+x_{n-k+1}}{x_{n-k}}$$

is (3k+2) for k = 0,1,2. Ref.[8] gives the following open problem.

Q6: Computer work indicates that positive solutions for real types of Eq.(10) are not all of the same period for any k ≥ 4. Is every positive solution of Eq.(10) (3k+2)-periodic for k ≥ 4.

An positive equilibrium for the following real type of equation

(11) $$x_{n+1} = \frac{A}{x_n^2} + \frac{1}{\sqrt{x_{n-1}}} \quad (n = 0,1,\ldots; A > 0)$$

with 0 < A < 15/4 is locally stable and also Eq.(11) has a 2-periodic solution with A > 15/4. See [8], which gives an open problem to Eq.(11). Ref.[8] gives the following open problems to Eq.(11).

Q7: Show that when 0 < A < 15/4 the equilibrium is global stable.

Q8: Show that when A > 15/4 there exists a positive 2-periodic solution which is global stable.

The following question is concerning fuzzy types of Eqs.(2) and (11).

Q9: The positive equilibrium for real types of Eq.(11) with positive A < 15/4 is stable, which is a weaker stability condition for fuzzy types of Eq.(2) with $p_i p_{\lambda_i} < 1$ of Condition (a) in Prop.P3. Is fuzzy types of Eq.(11) stable?

Real types of Eq. (4) is said to be permanent if there exist positive numbers C < D such that for any positive initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 > 0$ there exists a positive N such that $C < x_n < D$ for n ≥ N. For example, the following real types of equation is permanent under some conditions. See [8].

$$x_{n+1} = \frac{a+bx_n}{1+\sum_{i=0}^{m} b_i x_{n-i}} \quad (n = 0,1,\ldots; a,b,b_i \geq 0)$$

Condition (i) in Prop.P3 to fuzzy types of Eq.(2) shows that $A_{l,a} > 1$ for all $0 < a \le 1$ leads to the boundedness and persistence of positive solutions. The permanence means the boundedness and persistence.

Q10: Find a condition for the permanence to fuzzy types of Eq.(2).

Consider the following real type of

$$x_{n+1} = \frac{a}{x_n^2} + \frac{1}{x_{n-1}} \quad (n = 0,1,\dots; a > 0) \tag{12}$$

If $0 < a < 2\sqrt{3}$, then the equilibrium to Eq.(12) is locally stable and unstable if $a > 2\sqrt{3}$, under which there exists the following 2-periodic solution p,q of Eq.(12) such that

$$p = \frac{a + \sqrt{a^2 + 2 - 2\sqrt{1 + 4a^2}}}{2}, q = \frac{a - \sqrt{a^2 + 2 - 2\sqrt{1 + 4a^2}}}{2}.$$

In [8] the following open problems are given:

Q11: For what values of a is the positive equilibrium of Eq.(12) globally asymptotically stable?

Q12: For what values of a is the 2-periodic solutions p,q of Eq.(12) asymptotically stable?

Moreover it is possible to solve similar problems to fuzzy types of Eq.(12).

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