Four-term leaping recurrence relations

弘前大学大学院理工学研究科 小松 尚夫 (Takao Komatsu) 1
Graduate School of Science and Technology
Hirosaki University

1 Introduction

Given a three-term linear recurrence relation \( Z_n = T(n)Z_{n-1} + U(n)Z_{n-2} \) (\( n \geq 2 \)), where the initial values \( Z_0, Z_1 \) are arbitrary integral values, and \((T(n))_{n \geq 0}, (U(n))_{n \geq 0}\) are integer sequences with \( U(n) \neq 0 \) for all \( n \geq 0 \).

In 2008 Elsner and the author constructed a leaping three-term recurrence relation from the original relation. Namely, for fixed positive integers \( k \) and \( 0 \leq i < k \), they obtained a three-term relation concerning \( z_n = Z_{kn+i} \).

For integers \( a, l \) with \( l \geq 1 \) we define the determinant

\[
K_l(a) = \begin{vmatrix} T(a) & 1 & 0 \\ -U(a+1) & T(a+1) & 1 \\ 0 & -U(a+2) & T(a+2) \\ \vdots & \vdots & \vdots \\ T(a+l-2) & 1 & \quad \\ -U(a+l-1) & T(a+l-1) \end{vmatrix},
\]

with \( K_0(a) = 1 \). Let

\[
\Omega(M) = U(M-r)U(M-r+1)\ldots U(M-1)
\]

with \( M = (n-1)r+i+2 \). Then we have the following ([3, Theorem 2]).

Theorem 1 Given a three-term recurrence formula

\[
Z_n = T(n)Z_{n-1} + U(n)Z_{n-2} \quad (n \geq 2)
\]

with arbitrary initial values \( Z_0, Z_1 \) and two sequences of integers,

\[
(T(n))_{n \geq 0} = \left( a_0, a_1, a_2, \ldots, a_\rho, T_1(k), T_2(k), \ldots, T_w(k) \right)_{k=1}^\infty,
\]

\[
(U(n))_{n \geq 0} = \left( b_0, b_1, b_2, \ldots, b_\rho, U_1(k), U_2(k), \ldots, U_w(k) \right)_{k=1}^\infty,
\]

where \( U(n) \neq 0 \) for all \( n \geq 0 \), and \( a_\rho \geq 0 \), \( w \geq 1 \) are fixed integers. Then, for any integers \( r \) and \( i \) with \( r \geq 2, 0 \leq \rho \leq i < \rho + r \) and \( n \geq 2 \),

\[
K_{r-1}(M-r) \cdot Z_n - \left( K_{r-1}(M)K_r(M-r) + U(M)K_{r-1}(M-r)K_{r-2}(M+1) \right) \cdot Z_{n-1}
\]

\[
+(-1)^r\Omega(M)K_{r-1}(M) \cdot Z_{n-2} = 0
\]

holds for \( Z_n = Z_{rn+i} \). For \( T(a) > 0 \) and \( U(a) > 0 \) for all \( a > \rho \) one has \( K_{r-1}(M) \neq 0 \).

In particular, with \( Z_n = q_n \), \( q_0 = 1 \), \( q_1 = a_1 \) or \( Z_n = p_n \), \( p_0 = a_0 \), \( p_1 = a_0a_1 + b_1 \), this recurrence formula for \( Z_n \) is satisfied by the denominators \( q_{rn+i} \) and numerators \( p_{rn+i} \), respectively, of the convergents of a non-regular continued fraction

\[
[\frac{a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \cdots + b_\rho}}}{a_0 + \frac{U_1(k)}{T_1(k) + \frac{U_2(k)}{T_2(k) + \cdots + U_w(k)}}}]_{k=1}^\infty.
\]

1 This research was partially supported by the Grant-in-Aid for Scientific Research (C) (No. 18540006), the Japan Society for the Promotion of Science.
In the case of regular continued fractions this result is reduced as follows.

**Corollary 1** Given a three-term recurrence formula

\[ Z_n = T(n)Z_{n-1} + Z_{n-2} \quad (n \geq 2) \]

with arbitrary initial values \( Z_0, Z_1 \) and a sequence of integers,

\[ (T(n))_{n \geq 0} = (a_0, a_1, a_2, \ldots, a_\rho, T_1(k), T_2(k), \ldots, T_w(k))_{k=1}^\infty, \]

where \( \rho \geq 0 \) and \( w \geq 1 \) are fixed integers. Then, for any integers \( r \) and \( i \) with \( 0 \leq \rho \leq i < \rho + r \) and \( n \geq 2 \),

\[ K_{r-1}(M - r) \cdot z_n - \left( K_{r-1}(M)K_r(M - r) + K_{r-1}(M - r)K_{r-2}(M + 1) \right) \cdot z_{n-1} \]

\[ + (-1)^rK_{r-1}(M) \cdot z_{n-2} = 0 \]

holds for \( z_n = Z_{m+i} \). For \( T(a) > 0 \) for all \( a > \rho \) one has \( K_{r-1}(M) \neq 0 \).

In particular, with \( Z_n = q_n \), \( q_0 = 1 \), \( q_1 = a_1 \) or \( Z_n = p_n \), \( p_0 = a_0 \), \( p_1 = a_0a_1 + 1 \), this recurrence formula for \( z_n \) is satisfied by the denominators \( q_{m+i} \) and numerators \( p_{m+i} \), respectively, of the convergents of a regular continued fraction

\[ [a_0; a_1, a_2, \ldots, a_\rho, T_1(k), T_2(k), \ldots, T_w(k)]_{k=1}^\infty. \]

Three-term leaping recurrence relations which are entailed from continued fractions were studied in the case of \( e \) by Elsner ([1]). Similar relations were also studied in the case of \( e^{1/s} (s \geq 2) \) by the author ([7], [8]). Such concepts were extended to the cases of more regular continued fractions and non-regular continued fractions ([2], [3]).

However, it is not easy to find an analogous result for linear four-term recurrence relations

\[ Z_n = U_1(n)Z_{n-1} + U_2(n)Z_{n-2} + U_3(n)Z_{n-3}, \]

where \( U_1(n) \), \( U_2(n) \) and \( U_3(n) \) are general sequences of integers. In this article we shall consider the leaping recurrence relations for four-term recurrence relations where \( U_1(n) = a_1 \), \( U_2(n) = a_2 \) and \( U_3(n) = a_3 \) are integer constants. Then we can have the leaping relation

\[ Z_n = b_1Z_{n-k} + b_2Z_{n-2k} + b_3Z_{n-3k} \quad (n = 3k, 3k+1, 3k+2, \ldots) \]

for any leaping step \( k \).

## 2 Leaping convergents

Let \( \alpha \) be a real number. Continued fraction expansion of \( \alpha \) is denoted by

\[ \alpha = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}. \]

The \( n \)-th convergent is given by the irreducible rational number

\[ \frac{p_n}{q_n} = [a_0; a_1, a_2 \ldots, a_n]. \]

It is well-known that \( p_n \)'s and \( q_n \)'s satisfy the recurrence relations:

\[
\begin{align*}
p_n &= a_np_{n-1} + p_{n-2} \\
q_n &= a_nq_{n-1} + q_{n-2}
\end{align*}
\]

\( (n \geq 0) \quad p_{-1} = 1, \quad p_{-2} = 0, \quad q_{-1} = 1, \quad q_{-2} = 0. \)
Leaping convergents of continued fractions are those of every \( r \)-th convergent of continued fractions:
\[
\frac{p_{i}}{q_{i}}, \quad \frac{p_{r+i}}{q_{r+i}}, \quad \frac{p_{2r+i}}{q_{2r+i}}, \ldots, \quad \frac{p_{r+n+i}}{q_{r+n+i}}, \ldots
\]
For example, consider
\[
e^{1/s} = [1; (2k-1)s-1, 1, 1, 3s-1, 1, 1, \ldots]_{s=1}^{\infty} = [1; s-1, 1, 1, 3s-1, 1, 1, \ldots]_{s=2}^{\infty}, \quad (s \geq 2),
\]
then \( p_{3n} = 2s(2n-1)p_{3n-3} + p_{3n-6} \) and \( q_{3n} = 2s(2n-1)q_{3n-3} + q_{3n-6} \) (\( n \geq 2 \)) \([7]\).

### 3 Three-term relations

Three-term relations have been considered as in the continued fraction expansion of \( e \) \([1]\), as in that of \( e^{1/s} \ (s \geq 2) \) \([7, 8]\), as in that of the type \([1; T_{1}(k), T_{2}(k), T_{3}(k)]_{k=1}^{\infty} \) \([9, 10]\). Recently, three-term relations have been developed in the non-regular continued fractions \([2]\), and finally as in Theorem 1 here \([3]\).

On the other direction, one can simplify the general theorem, entailing some classical results. If \( T(n) = a_{1}, U(n) = a_{2} \ (a_{2} \neq 0) \) are integer constants in Theorem 1, we have the following.

**Theorem 2** If the sequence \( \{Z_{n}\}_{n} \) satisfies the three-term recurrence relation \( Z_{n} = a_{1}Z_{n-1} + a_{2}Z_{n-2} \) \( (a_{2} \neq 0) \), then for any positive integer \( k \) we have
\[
Z_{n} = k \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(k-i-1)!}{i!(k-2i)!} \cdot a_{1}^{k-2i} \cdot a_{2}^{i} \cdot Z_{n-k} + (-1)^{k+1} \cdot a_{2}^{k} \cdot Z_{n-2k} \quad (n \geq 2k).
\]

Moreover, if \( a_{1} = a_{2} = 1 \), then the sequence \( \{Z_{n}\}_{n} \) is called Fibonacci-type sequence. Moreover, if \( Z_{0} = 0 \) and \( Z_{1} = 1 \), then \( \{Z_{n}\}_{n} \) is the Fibonacci sequence \( \{F_{n}\}_{n} \).

**Corollary 2** For any positive integer \( k \) we have
\[
F_{n} = k \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(k-i-1)!}{i!(k-2i)!} \cdot F_{n-k} - (-1)^{k} \cdot F_{n-2k} \quad (n \geq 2k). \tag{1}
\]
If we put \( k = 2, 3, \ldots, 10 \), then we have
\[
F_{n} = 3F_{n-2} - F_{n-4}, \quad F_{n} = 4F_{n-3} - F_{n-6}, \quad F_{n} = 7F_{n-4} - F_{n-8}, \quad F_{n} = 11F_{n-5} + F_{n-10}, \quad F_{n} = 18F_{n-6} - F_{n-12}, \quad F_{n} = 29F_{n-7} + F_{n-14}, \quad F_{n} = 47F_{n-8} - F_{n-16}, \quad F_{n} = 76F_{n-9} + F_{n-18}, \quad F_{n} = 123F_{n-10} - F_{n-20}.
\]
There is a classical result corresponding to this corollary (Ruggles, 1963 \[11, \text{identity 105, p.92}\]):
\[
F_{n} = L_{k}F_{n-k} + (-1)^{k+1}F_{n-2k}, \tag{2}
\]
where \( F_n \) and \( L_n \) are Fibonacci number and Lucas numbers, respectively. Namely, they satisfy the three-term relations
\[
F_n = F_{n-1} + F_{n-2} \quad (n \geq 2), \quad F_0 = 0, \quad F_1 = 1;
\]
\[
L_n = L_{n-1} + L_{n-2} \quad (n \geq 2), \quad L_0 = 2, \quad L_1 = 1.
\]

Comparing (2) with (1), we have

**Corollary 3**
\[
L_k = k \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(k-i-1)!}{i!(k-2i)!} \quad (k \geq 1).
\]

**Proof of Theorem 2.** Set \( K_l = K_l(c) \). Then, \( \{K_l\}_{l \geq 0} \) satisfies the recurrence relation:
\[
K_l = a_1 K_{l-1} + a_2 K_{l-2} \quad (l \geq 2), \quad K_0 = 1, \quad K_1 = a_1.
\]

Hence, for \( l \geq 0 \) we have
\[
K_l = \sum_{i=0}^{\lfloor l/2 \rfloor} \frac{(l-i)!}{i!(l-2i)!} a_1^{l-2i} a_2^i.
\]

Applying Theorem 1 with \( \Omega(M) = a_2^3 \), we have
\[
Z_n = (K_r + a_2 K_{r-2}) \cdot Z_{n-1} - (-1)^r a_2^r \cdot Z_{n-2}.
\]

Since
\[
K_r + a_2 K_{r-2} = \sum_{i=0}^{\lfloor r/2 \rfloor} \frac{(r-i)!}{i!(r-2i)!} a_1^{r-2i} a_2^i + \sum_{i=0}^{\lfloor r/2 \rfloor - 1} \frac{(r-i-1)!}{i!(r-2i-2)!} a_1^{r-2i-2} a_2^{i+1} = r \sum_{i=0}^{\lfloor r/2 \rfloor} \frac{(r-i-1)!}{i!(r-2i)!} a_1^{r-2i} a_2^i,
\]
we obtain the desired result.

4 **Four-term relations**

Consider the four-term recurrence relation
\[
Z_n = U_1(n) Z_{n-1} + U_2(n) Z_{n-2} + U_3(n) Z_{n-3}.
\]

For the moment, the corresponding result to Theorem 1 has not been known. However, one can relax the conditions, in order to get some typical results. If \( U_1(n) = a_1, U_2(n) = a_2, U_3(n) = a_3 \) are constants for all \( n \), we have the following four-term leaping recurrence relation.

**Theorem 3** If the sequence \( \{Z_n\}_n \) satisfies the four-term recurrence relation \( Z_n = a_1 Z_{n-1} + a_2 Z_{n-2} + a_3 Z_{n-3} \) \( (a_3 \neq 0) \), then for any positive integer \( k \)
\[
Z_n = k \sum_{j=0}^{\lfloor k/2 \rfloor} \sum_{i=0}^{\lfloor (k-3j)/2 \rfloor} \frac{(k-i-2j-1)!}{i!(k-2i-3j)!} a_1^{k-2i-3j} a_2^{i} a_3 \cdot Z_{n-k} - k \sum_{j=0}^{\lfloor k/3 \rfloor} (-1)^{k+i+j} \sum_{i=0}^{\lfloor (k-3j)/2 \rfloor} \frac{(k-i-2j-1)!}{i!(k-2i-3j)!} a_1^{k-2i-3j} a_2^{i+2j} \cdot Z_{n-2k} + a_3^k \cdot Z_{n-3k}
\]
\( (n \geq 3k) \).
In 2001 F. T. Howard obtained a similar result ([5]):

\[ Z_n = J_k Z_{n-k} - a_3^k J_{n-2k} + a_3^k Z_{n-3k}, \]

where \( J_n \) satisfies

\[ J_n = a_1 J_{n-1} + a_2 J_{n-2} + a_3 J_{n-3} \quad (n \geq 3), \]
\[ J_0 = 3, \quad J_1 = a_1, \quad J_2 = a_1^2 + 2a_2. \]

\( J_n \) \((n = 1, 2, \ldots)\) are determined by

\[ J_{-n} = \frac{1}{a_3} (J_{-n+3} - a_1 J_{-n+2} - a_2 J_{-n+1}) \quad (n \geq 1). \]

Comparing (3) with Theorem 3, we obtain

**Corollary 4**

\[ J_k = k \sum_{j=0}^{\lfloor k/3 \rfloor} \sum_{i=0}^{\lfloor (k-3j)/2 \rfloor} \frac{(k-i-2j-1)!}{i!j!(k-2i-3j)!} a_1^i a_2^{k-2i-3j} a_3^{-k+i+2j}, \]
\[ J_{-k} = k \sum_{j=0}^{\lfloor k/3 \rfloor} \sum_{i=0}^{\lfloor (k-3j)/2 \rfloor} (-1)^{k+i+j} \frac{(k-i-2j-1)!}{i!j!(k-2i-3j)!} a_1^i a_2^{k-2i-3j} a_3^{-k+i+2j}. \]

In the case of \( k = 12 \),

\[ Z_n = (a_1^2 + 12a_1^0 a_2 + 54a_1^2 a_2^2 + 112a_1^3 a_2^3 + 105a_1^4 a_2^4 + 36a_1^5 a_2^5 + 2a_2^6 + 12a_1^6 a_3 + 96a_1^7 a_2 a_3 + 252a_1^8 a_2^2 a_3 + 240a_1^9 a_2^3 a_3 + 60a_1 a_2^4 a_3 + 42a_2^5 a_3 \]
\[ + 180a_1^2 a_2^2 a_3^2 + 180a_1^3 a_2^3 a_3^2 + 24a_2^2 a_3^3 + 40a_1^4 a_3^4 + 48a_1 a_2 a_3^5 + 3a_3^6) Z_{n-12}, \]
\[ + (a_1^2 - 12a_1^0 a_2 + 96a_1^2 a_2^2 - 252a_1^3 a_2^3 - 240a_1^4 a_2^4 - 60a_1^5 a_2^5 + 240a_1^2 a_3^6 + 180a_1^3 a_2 a_3^5 + 180a_1^4 a_2^2 a_3^4 + 3a_3^6) Z_{n-12} - a_2^{12} Z_{n-36} = 0. \]

Put \( a_1 = a_2 = a_3 = 1 \). Then the four-term recurrence relation \( T_n = T_{n-1} + T_{n-2} + T_{n-3} \) yields Tribonacci numbers \( \{T_n\}_{n \geq 0} \). If \( (T_0 = 0,) T_1 = T_2 = 1 \) and \( T_3 = 2 \), then the Tribonacci sequence is given by

1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768, 10609, 19513, 35890, 66012, 121415, \ldots

([11, p.527], [12, A000073]). Putting \( k = 2, 3, \ldots, 10 \) in Theorem 3, we have

\[ T_n = 3T_{n-2} + T_{n-4} + T_{n-6}, \]
\[ T_n = 7T_{n-3} - 5T_{n-6} + T_{n-9}, \]
\[ T_n = 11T_{n-4} + 5T_{n-8} + T_{n-12}, \]
\[ T_n = 21T_{n-5} + T_{n-10} + T_{n-15}, \]
\[ T_n = 39T_{n-6} - 11T_{n-12} + T_{n-18}, \]
\[ T_n = 71T_{n-7} + 15T_{n-14} + T_{n-21}, \]
\[ T_n = 131T_{n-8} - 3T_{n-16} + T_{n-24}, \]
\[ T_n = 241T_{n-9} - 23T_{n-18} + T_{n-27}, \]
\[ T_n = 443T_{n-10} + 41T_{n-20} + T_{n-30}. \]
5 Five-term relations

Consider the sequence \( \{Z_n\}_n \) satisfying the five-term recurrence relation
\[
Z_n = a_1 Z_{n-1} + a_2 Z_{n-2} + a_3 Z_{n-3} + a_4 Z_{n-4} \quad (a_4 \neq 0).
\]
Then how can we determine the integer constants \( b_1, b_2, b_3, b_4 \), satisfying
\[
Z_n = b_1 Z_{n-k} + b_2 Z_{n-2k} + b_3 Z_{n-3k} + b_4 Z_{n-4k}
\]
for any positive integer \( k \) (\( 1 < k < n/4 \))?

In the case of \( k = 5 \),
\[
Z_n = (a_5^2 + 5a_1 a_2 + 5a_1^2 a_3 + 5a_2 a_4 + 5a_1 a_4)Z_{n-5}
\]
\[
+ (a_2 - 5a_1 a_3 + 5a_1^2 a_4 - 5a_2 a_3 + 5a_1 a_4 + 5a_2 a_4)Z_{n-10}
\]
\[
- 5a_3 a_4 + 5a_1 a_2 a_4 + 5a_3 a_4 - 5a_2 a_4 + 5a_2 a_4)Z_{n-15} + a_5^2 Z_{n-20}.
\]

In the case of \( k = 6 \),
\[
Z_n = (a_6^2 + 6a_1 a_2 + 9a_1^2 a_3 + 6a_2^2 a_4 + 12a_1 a_3 + 3a_1 a_4 + 6a_2 a_4)Z_{n-6}
\]
\[
+ (-a_3^2 + 6a_1 a_2 a_3 + 9a_1^2 a_4 + 6a_2^2 a_4 + 2a_3 a_4 - 12a_1 a_2 a_3 - 3a_1 a_4 - 6a_2 a_4)
\]
\[
- 6a_3 a_4 + 12a_1 a_2 a_4 + 18a_2^2 a_4 - 3a_1 a_4 - 9a_2 a_4 + 18a_1 a_2 a_4 + 2a_4)Z_{n-12}
\]
\[
+ (a_4^2 - 6a_2 a_3 a_4 + 9a_2^2 a_4 + 6a_1 a_3 a_4 - 2a_3 a_4 - 12a_1 a_2 a_3 a_4 + 6a_2 a_3 a_4 + 3a_2 a_4 - 6a_2 a_4)Z_{n-18}
\]
\[
- a_4^2 Z_{n-24}.
\]

Tetranacci Numbers \( \{F^{(4)}_k\}_{k \geq 1} \) are the \( n = 4 \) case of the Fibonacci \( n \)-step Numbers, defined by
\[
F^{(4)}_k = F^{(4)}_{k-1} + F^{(4)}_{k-2} + F^{(4)}_{k-3} + F^{(4)}_{k-4} \quad (k \geq 5) \quad \text{with} \quad F^{(4)}_1 = F^{(4)}_2 = F^{(4)}_3 = F^{(4)}_4 = 4.
\]
The first terms are
\[
1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, 10671, 20569, 39648, 76424, 147312,
\]
\[
283953, 547337, 1055026, 2033628, 3919944, 755935, 14564533, 28074040, 54114452, 104308960, \ldots
\]
\(([12, \text{A000078}])\). They satisfy the recurrence relations:
\[
F^{(4)}_k = 3F^{(4)}_{k-2} + 3F^{(4)}_{k-4} - F^{(4)}_{k-6} - F^{(4)}_{k-8},
\]
\[
F^{(4)}_k = 7F^{(4)}_{k-3} + F^{(4)}_{k-6} + F^{(4)}_{k-9} + F^{(4)}_{k-12},
\]
\[
F^{(4)}_k = 15F^{(4)}_{k-4} - 17F^{(4)}_{k-8} + 7F^{(4)}_{k-12} - F^{(4)}_{k-16},
\]
\[
F^{(4)}_k = 26F^{(4)}_{k-5} + 16F^{(4)}_{k-10} + 6F^{(4)}_{k-15} + F^{(4)}_{k-20},
\]
\[
F^{(4)}_k = 51F^{(4)}_{k-6} + 15F^{(4)}_{k-12} - F^{(4)}_{k-18} - F^{(4)}_{k-24},
\]
\[
F^{(4)}_k = 99F^{(4)}_{k-7} - 13F^{(4)}_{k-14} + F^{(4)}_{k-21} + F^{(4)}_{k-28},
\]
\[
F^{(4)}_k = 191F^{(4)}_{k-8} - 81F^{(4)}_{k-16} + 15F^{(4)}_{k-24} - F^{(4)}_{k-32},
\]
\[
F^{(4)}_k = 367F^{(4)}_{k-9} + 127F^{(4)}_{k-18} + 19F^{(4)}_{k-27} + F^{(4)}_{k-36},
\]
\[
F^{(4)}_k = 708F^{(4)}_{k-10} + 58F^{(4)}_{k-20} + 4F^{(4)}_{k-30} - F^{(4)}_{k-40}.
\]

\(b_1, b_2\) and \(b_4\) are calculated as follows.
Theorem 4

\[ b_1 = k \sum_{\kappa=0}^{\lfloor k/4 \rfloor} \sum_{i=0}^{\lfloor (k-i-2j-3\kappa-1)/2 \rfloor} \frac{(k-i-2j-3\kappa-1)!}{i!j!\kappa!(k-2i-3j-4\kappa)!}a_1^{k-2i-3j-4\kappa}a_2^i a_3^j a_4^n, \]

\[ b_3 = k \sum_{\kappa=0}^{\lfloor (k-4\kappa)/3 \rfloor} \sum_{i=0}^{\lfloor (k-3j-4\kappa)/2 \rfloor} (-1)^i \frac{(k-i-2j-3\kappa-1)!}{i!j!\kappa!(k-2i-3j-4\kappa)!}a_1^i a_2^j a_3^{k-2i-3j-4\kappa}a_4^{i+2j+3\kappa}, \]

\[ b_4 = (-1)^{k-1}a_4^k. \]

However, it is not easy to find an explicit form of \( b_2 \). This shall be discussed in the next section.

In 2005 Latushkin and Ushakov ([6]) obtained a different form of five-term leaping relations.

\[ Z_n = H_k Z_{n-k} + \frac{H_{2k} - H_k^2}{2} Z_{n-2k} + (-a_4)^k H_{-k} Z_{n-3k} - (-a_4)^k Z_{n-4k}, \quad (4) \]

where

\[ H_n = x_1^n + x_2^n + x_3^n + x_4^n \quad (n \in \mathbb{Z}) \]

and \( x_1, x_2, x_3 \) and \( x_4 \) are the complex roots (including multiple roots) of the equation \( x^4 - a_1 x^3 - a_2 x^2 - a_3 x - a_4 = 0 \). On the other hand, the sequence \( \{H_n\}_n \) satisfies the recurrence relation:

\[ H_n = a_1 H_{n-1} + a_2 H_{n-2} + a_3 H_{n-3} + a_4 H_{n-4} \quad (n \in \mathbb{Z}). \]

The initial values are determined by

\[ H_0 = 4, \]
\[ H_1 = a_1, \]
\[ H_2 = a_1 H_1 + 2a_2 = a_1^2 + 2a_2, \]
\[ H_3 = a_1 H_2 + a_2 H_1 + 3a_3 = a_1^3 + 3a_1a_2 + 3a_3, \]
\[ H_4 = a_1 H_3 + a_2 H_2 + a_3 H_1 + 4a_4 = a_1^4 + 4a_1a_2 + 4a_1a_3 + 2a_2^2 + 4a_4. \]

Comparing their results (4) with ours in Theorem 4, we get the following.

Corollary 5

\[ H_k = k \sum_{\kappa=0}^{\lfloor (k-4\kappa)/3 \rfloor} \sum_{i=0}^{\lfloor (k-3j-4\kappa)/2 \rfloor} (-1)^i \frac{(k-i-2j-3\kappa-1)!}{i!j!\kappa!(k-2i-3j-4\kappa)!}a_1^i a_2^j a_3^{k-2i-3j-4\kappa}a_4^{i+j+2j+3\kappa}, \]

\[ H_{-k} = k \sum_{\kappa=0}^{\lfloor (k-4\kappa)/3 \rfloor} \sum_{i=0}^{\lfloor (k-3j-4\kappa)/2 \rfloor} (-1)^{i+k} \frac{(k-i-2j-3\kappa-1)!}{i!j!\kappa!(k-2i-3j-4\kappa)!}a_1^i a_2^j a_3^{k-2i-3j-4\kappa}a_4^{i+k+2j+3\kappa}. \]

Pentanacci Numbers \( \{F_k^{(5)}\}_{k \geq 1} \) are the \( n = 5 \) case of the Fibonacci \( n \)-step Numbers, defined by \( F_k^{(n)} = F_{k-1}^{(n)} + F_{k-2}^{(n)} + F_{k-3}^{(n)} + F_{k-4}^{(n)} + F_{k-5}^{(n)} \) \( (k \geq 6) \) with \( F_1^{(5)} = F_2^{(5)} = 1, F_3^{(5)} = 2, F_4^{(5)} = 4 \) and \( F_5^{(5)} = 8 \). The first terms are

\[ 1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, 6930, 13624, 26784, 52656, 103519, 203513, 400096, 786568, 1546352, 3040048, 6080096, 12174960, 24349920, 48699840, 97399680, 194799360, 389598720, 779197440, 1558394880, 3116789760, 6233579520, 12467159040, 24934318080. \]
They satisfy the recurrence relations:

\begin{align*}
F_k^{(5)} &= 3F_{k-2}^{(5)} + 3F_{k-4}^{(5)} + F_{k-6}^{(5)} + F_{k-8}^{(5)} + F_{k-10}^{(5)}, \\
F_k^{(6)} &= 7F_{k-2}^{(6)} + 4F_{k-3}^{(6)} + 4F_{k-5}^{(6)} + F_{k-7}^{(6)} + F_{k-9}^{(6)} + F_{k-11}^{(6)}, \\
F_k^{(6)} &= 15F_{k-4}^{(6)} - F_{k-6}^{(6)} + F_{k-8}^{(6)} + F_{k-10}^{(6)} + F_{k-12}^{(6)}, \\
F_k^{(5)} &= 31F_{k-5}^{(5)} - 49F_{k-10}^{(5)} + 31F_{k-15}^{(5)} - 9F_{k-20}^{(5)} + F_{k-25}^{(5)}, \\
F_k^{(6)} &= 57F_{k-6}^{(6)} + 42F_{k-12}^{(6)} + 22F_{k-18}^{(6)} + 7F_{k-24}^{(6)} + F_{k-30}^{(6)}, \\
F_k^{(6)} &= 113F_{k-7}^{(6)} + F_{k-7}^{(6)} + F_{k-12}^{(6)} + F_{k-18}^{(6)} + F_{k-24}^{(6)} + F_{k-30}^{(6)}, \\
F_k^{(5)} &= 313F_{k-8}^{(5)} - 113F_{k-16}^{(5)} + 31F_{k-24}^{(6)} + F_{k-32}^{(6)} + F_{k-40}^{(6)}, \\
F_k^{(5)} &= 11F_{k-9}^{(5)} - 9F_{k-18}^{(5)} + F_{k-27}^{(6)} + F_{k-36}^{(6)} + F_{k-45}^{(6)}, \\
F_k^{(5)} &= 43F_{k-10}^{(5)} - 49F_{k-20}^{(5)} + 141F_{k-30}^{(5)} - 19F_{k-40}^{(5)} + F_{k-50}^{(5)}.
\end{align*}

6 A form of \( b_2 \) in five-term leaping relations

An explicit form of \( b_2 \) has not been known yet. Instead, there is a way to express \( b_2 \) by matrices.

\[
b_2 = a_2 \Lambda_{k-1} - 2a_3 \Phi_{k-1} + 3a_4 \Psi_{k-1} - a_2 \Phi_{k-1} + 2a_3 \Psi_{k-1} - a_2 \Psi_{k-1} + 2a_4 \Lambda_{k-2} - 3a_4 \Lambda_{k-2}.
\]

where

\[
\Lambda_n = \begin{bmatrix} -a_2 & -a_1 & 1 & 0 \\
-a_3 & -a_2 & -a_1 & 1 \\
-a_4 & -a_3 & -a_2 & -a_1 \\
0 & -a_4 & -a_3 & -a_2 \\
\end{bmatrix},
\]

\[
\Phi_n = \begin{bmatrix} -a_1 & 1 & 0 \\
-a_3 & -a_2 & -a_1 & 1 \\
-a_4 & -a_3 & -a_2 & -a_1 \\
0 & -a_4 & -a_3 & -a_2 \\
\end{bmatrix},
\]

\[
\Psi_n = \begin{bmatrix} -a_1 & 1 & 0 \\
-a_2 & -a_1 & 1 \\
-a_4 & -a_3 & -a_2 & -a_1 \\
0 & -a_4 & -a_3 & -a_2 \\
\end{bmatrix}.
\]
Notice that

\[ \Lambda_n = -a_2 \Lambda_{n-1} + a_3 \Phi_{n-1} - a_4 \Psi_{n-1}, \]
\[ \Phi_n = -a_1 \Lambda_{n-1} + a_3 \Lambda_{n-2} - a_4 \Phi_{n-2}, \]
\[ \Psi_n = -a_1 \Phi_{n-1} + a_2 \Lambda_{n-2} - a_4 \Lambda_{n-3}. \]

$b_2$ may be expanded as follows.

\[
\begin{align*}
b_2 & = (-1)^{k-1} k \left( a_2^2 - (a_1 a_3 - a_4) a_2^{k-2} - (a_3^2 a_4) a_2^{k-3} ight) \\
& + \left( \frac{(k-3)!}{2! (k-4)!} a_2^2 a_3^2 - \frac{(k-4)!}{(k-6)} (k-6) a_1 a_3 a_4 + \frac{k-3}{2} a_4^2 \right) a_2^{k-4} \\
& + \left( \frac{(k-4)!}{(k-5)!} a_1 a_3 - (k-6) a_4 \right) a_2^{k-5} \\
& - \left( \frac{(k-4)!}{3! (k-6)!} a_1 a_3^2 - \frac{(k-5)!}{2! (k-6)!} (k-12) a_2^2 a_3 a_4 + \frac{k^2 - 15k + 60}{2} a_1 a_3 a_4^2 \right) a_2^{k-6} \\
& - \left( \frac{(k-4)!}{3! (k-6)!} a_1 a_3^3 - \frac{k-5}{2} (a_1 a_3^2 + a_2^3) \right) a_2^{k-7} \\
& + \left( \frac{(k-5)!}{4! (k-8)!} a_1 a_3^2 a_4 - \frac{(k-6)!}{3! (k-8)!} (20) a_2^2 a_3^2 a_4 \right) a_2^{k-8} \\
& + \left( \frac{(k-5)!}{4! (k-7)!} a_2^2 a_3^2 - \frac{(k-6)!}{3! (k-7)!} (k-12) a_3^2 a_4 + \frac{(k-10)(k-12)(k-14)}{3} a_1 a_3 a_4^2 \right) a_2^{k-9} \\
& + \left( \frac{(k-6)!}{3! (k-9)!} a_1 a_3^3 a_4 - \frac{(k-7)!}{2! (k-9)!} (k-15) a_2^3 a_4 \right) a_2^{k-10} \\
& - \left( \frac{(k-6)!}{3! (k-9)!} (k-23k + 140) a_1 a_3^2 a_4^2 - \frac{(k-9)!}{3! (k-9)!} (k-10)(k-12)(k-14) a_4^3 \right) a_2^{k-11} \\
& - \left( \frac{(k-7)!}{3! (k-10)!} a_2^2 a_3^3 \right) a_2^{k-12} \\
& - \cdots.
\end{align*}
\]

However, its simplified form has not been known.

## 7 \( (s+1)\)-term recurrence relations

We may extend terms to five, six, seven, and so on. In 1999 Howard got a general term leaping relation ([4]). Young also found a different form ([13]). This result holds for more-term recurrence relations, but it is not so useful practically in order to obtain an explicit form for any given $s$.

If the sequence \( \{Z_n\}_{n \geq 0} \) satisfies the relation

\[ Z_n = a_1 Z_{n-1} + a_2 Z_{n-2} + \cdots + a_s Z_{n-s} \quad (a_s \neq 0), \]

where \( Z_0, Z_1, \ldots, Z_{s-1} \) are arbitrary initial values, then we have

\[ Z_{n+i} = c_{r,r} Z_{r(n-1)+i} - c_{r,2r} Z_{r(n-2)+i} + \cdots + (-1)^{s-i} c_{r,sr} Z_{r(n-s)+i}, \]
where \( c_{r,r}, c_{r,2} \) are determined by
\[
\prod_{\nu=0}^{r-1}(1 - a_1(\zeta^\nu x) - a_2(\zeta^\nu x)^2 - \cdots - a_s(\zeta^\nu x)^s) = 1 - c_{r,r}x^r + c_{r,2r}x^{2r} - \cdots + (-1)^sc_{r,sr}x^{sr},
\]
where \( \zeta \) is a primitive \( r \)-th root of unity.

As a straight generalization of our theorems 2, 3, 4, we obtain the following.

**Theorem 5** If
\[
Z_n = b_1 Z_{n-k} + b_2 Z_{n-2k} + \cdots + b_{s-1} Z_{n-(s-1)k} + b_s Z_{n-sk},
\]
then
\[
b_1 = k \sum_{2i_1+3i_2+\cdots+si_{s-1} \leq k} \frac{(k-i_1-2i_2-\cdots-(s-1)i_{s-1}-1)!}{i_1!i_2!\cdots i_{s-1}!(k-2i_1-3i_2-\cdots-si_{s-1})!} a_1^{i_1}a_2^{i_2}\cdots a_{s-1}^{i_{s-1}},
\]
\[
b_{s-1} = k \sum_{2i_1+3i_2+\cdots+si_{s-1} \leq k} (-1)^{i_1-1} \frac{(k-i_1-2i_2-\cdots-(s-1)i_{s-1}-1)!}{i_1!i_2!\cdots i_{s-1}!(k-2i_1-3i_2-\cdots-si_{s-1})!} a_1^{i_1}a_2^{i_2}\cdots a_{s-1}^{i_{s-1}},
\]
where
\[
I = \begin{cases} i_1 + i_2 + \cdots + i_{s-2} + i_{s-1} & \text{if } s \text{ is odd;} \\ i_1 + i_2 + \cdots + i_{s-3} & \text{if } s \text{ is even,} \end{cases}
\]
and
\[
b_s = \begin{cases} a_s^k & \text{if } s \text{ is odd;} \\ (-1)^{k-1}a_s^k & \text{if } s \text{ is even.} \end{cases}
\]

8 Periodicity

In [2, Theorem 3] a result about periodicity of three-term leaping relations is obtained.

**Theorem 6** Given a three-term recurrence formula
\[
Z_n = T(n)Z_{n-1} + U(n)Z_{n-2} \quad (n \geq 2)
\]
with arbitrary initial values \( Z_0, Z_1 \) and two sequences of integers \((T(n))_{n \geq 0}\) and \((U(n))_{n \geq 0}\), which both are (ultimately) periodic modulo \( m \) with periods of length \( r \), say
\[
(T(n) \mod m)_{n \geq 0} = (a_0, a_1, a_2, \ldots, a_p, T_1, T_2, \ldots, T_r),
\]
\[
(U(n) \mod m)_{n \geq 0} = (b_0, b_1, b_2, \ldots, b_p, U_1, U_2, \ldots, U_r).
\]
Then, the sequence \((Z(n))_{n \geq 0}\) is (ultimately) periodic modulo \( m \). If \( \rho \in \{0,1\} \) and \( U(n) = 1 \) for all \( n \geq \rho \), then the sequence \((Z(n))_{n \geq 0}\) is periodic modulo \( m \).

This result can be extended to the case of any term leaping relations.

**Theorem 7** Given a \( (s+1) \)-term recurrence formula
\[
Z_n = T_1(n)Z_{n-1} + T_2(n)Z_{n-2} + \cdots + T_s(n)Z_{n-s} \quad (n \geq 2)
\]
with arbitrary initial values \( Z_0, Z_1, \ldots, Z_s \) and \( s \) sequences of integers \((T_j(n))_{n \geq 0}\) \((j = 1, 2, \ldots, s)\), which all are (ultimately) periodic modulo \( m \) with periods of length \( r \), then the sequence \((Z(n))_{n \geq 0}\) is (ultimately) periodic modulo \( m \).
References


Graduate School of Science and Technology
Hirosaki University, Hirosaki, 036-8561, Japan
komatsu@cc.hirosaki-u.ac.jp