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Differential equations and rational approximations of polylogarithms

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Abstract

• The monodromy's study of Fuchsian hypergeometric differential equation provides a natural framework for the explicit determination of rational approximations of polylogarithmic functions .Thus , we can obtain almost without calculation explicit determination of many polynomials and hypergeometric power series related to their Padé approximations .
From now on , using a classical way , one can study the arithmetic nature of numbers related to the values taken by these functions.

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1 Introduction

♠ In this paper I want to explain the origin of many formulas which are related to the simultaneous rational approximations of polylogarithmic functions.
Let us recall that :

Definition 1

$$Li_q(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^q}$$

For $q = 1$, one recognizes the power series expansion to $-\log(1-z)$.
For $q = 2$ this function is called the dilogarithmic function .

1.1 Arithmetic motivations

♠ The arithmetic motivation for searching such effective rational approximations comes from proving irrationality or transcendence of numbers arising as values of polylogarithmic functions , such as $Li_q(1/p))$, $p \in \mathbb{Z}$,

$$Li_q(1) = \zeta(q),$$

(for q integer $q \geq 2$)) .

$$\zeta(2), \zeta(3), \cdots \ etc,$$

• We sall now describe the preliminaries for the main result of this paper . (Marc Huttner :Israel Math Journal 2006),[Hu].

1.2 Riemann-Hilbert problem

♠ Find a very natural way to the explicit construction of functional linear forms in polylogarithmic functions using the construction of a fuchsian "hypergeometric " differential equations with prescribed singular points $0, 1, \infty$ and prescribed monodromy.
We solve in this particular case a "Riemann-Hilbert problem ".

Remark 1 *Let us recall that the Riemann-Hilbert problem is : Prove that there always exists a Fuchsian linear differential equation of order $q+1$ such that its singular points and monodromy operator are given.*
In general this fuchsian equation involves on accessory parameters and apparent singularities (i.e singular points for the differential equation but not for the solutions!)
For our special case there exists a solution, we shall prove that this equation does not involve accessory parameters and apparent singularities. This operator is thus unique!

We use a new explicit construction which replaces and generalizes many constructions often given without proofs by many authors. (See Apéry [Ap],Nesterenko,[Ne] ,Gutnik [Gu], Ball-Rivoal [Ba,Ri],Zudilin ,[Zu] .

1.3 Pochhammer symbol , hypergeometric power series

Definition 2 *In the following if $\alpha \in \mathbb{C}$ we put $(\alpha)_0 = 1$ and if $n \geq 1$,*

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)$$

Definition 3

$$
{}_{q+1}F_q\left(\begin{matrix} a_0, a_1, \cdots, a_q \\ b_1, b_2, \cdots, b_q \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{q}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{z^n}{n!} \tag{1}
$$

denotes the hypergeometric power series .

1.4 Hypergeometric differential equation,Levelt's construction

♠ The hypergeometric power series is the holomorphic solution at 0 of the following differential equation of order $q+1$. ,

$$
\begin{gathered} \mathcal{H}yp((a)_i,(b)_i) \\ ((\theta + b_1 - 1)(\theta + b_2 - 1)\cdots(\theta + b_q - 1)- \\ z(\theta + a_0)(\theta + a_2)\cdots(\theta + a_q))y(z) = 0. \end{gathered} \tag{2}
$$

The natural domain of definition of the solutions of the ordinary differential equation (ODE) is the Riemann -sphere $\mathbb{CP}_1$.
By examination the ODE $\mathcal{H}yp((a)_i,(b)_i)$ has $0, 1, \infty$ at its only regular singular points .
${}_{q+1}F_q$ can be continued to a meromorphic function on $Z = \mathbb{CP}_1 - \{0, 1, \infty\}$ which is generally multivalued .
• The solution space of any order ODE on $\mathbb{CP}_1$ is determined by the characteristic exponents associated to a symbol called Riemann-P-scheme (see for example [AAR]) , which indicates the location of the singular points , and the exponents relative to each singularity .
(These exponents do not depend of the basis of solution choosen!)
• The equation $\mathcal{H}yp((a)_i,(b)_i)$ is free of accessory parameters and the Riemann-P-symbol related to this equation is

Theorem 1

$$
{}_{q+1}F_q\left(\begin{matrix} a_0, a_1, \cdots, a_q \\ b_1, b_2, \cdots, b_q \end{matrix} \middle| z\right) \propto P \begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ 0 & a_0 & 0 & \\ 1-b_1 & a_1 & 1 & \\ 1-b_2 & a_2 & 2 & \vert z \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & q-1 & \\ 1-b_q & a_q & d & \end{pmatrix} \tag{3}
$$

$$d = \sum_{j=1}^{q} b_j - \sum_{j=0}^{q} a_j$$

The notation $\propto$ indicates that the unique analytic solution $f(z)$ (hypergeometric power series) $f(z) =_{q+1} F_q(z)$) belongs to the zero exponent at $z = 0$ and satisfies $f(0) = 1$.
The main point is that at $z = 1$ there exist q holomorphic linearly independent solutions of $\mathcal{H}yp((a)_i, (b)_i)$. This result is very important and is characteristic of the hypergeometric ODE , (Levelt) [Le].

Remark 2 *When $d \in \mathbb{Z}$, one solution at $z = 1$ is in general logarithmic i.e can be written $\psi(z) = u(z) + (1-z)^d[v(z)\log(1-z) + w(z)]$ where u is a polynomial of degree $q-1$ and v resp w are analytic functions at $z = 1$.*

1.5 Padé problem

♠ Find the $\sigma = q(n+1)$ coefficients of the polynomials $A_k(z)$,of degree n ($1 \leq k \leq q$) and the remainder $R_\infty(z)$ such that for given $\sigma_\infty \geq n+1$, the linear form :

$$R_\infty(z) = A_0(z) + \sum_{k=1}^{q} A_k(z) Li_k(1/z).$$

satisfies $Ord_\infty R_\infty(z) = \sigma_\infty$.

1.6 Rivoal's problem

• Recall that : $Ord_\infty R_\infty(z) = \sigma_\infty$. i.e

$$R_\infty(z) = \frac{1}{z^{\sigma_\infty}}(c_0 + c_1\frac{1}{z} + \cdots)$$

with $c_0 \neq 0$. (The polynomial $A_0(z)$ is completely determined and of degree $\leq n-1$.)

Construct (if possible) these polynomials such that $A_1(1) = 0$ i.e $R_\infty(1)$ exists. and also $A_{q-1}(1) = A_{q-3}(1) = \cdots = A_2(1) = 0$.
The following assumption :

$$\sigma = \sigma_\infty + \sigma_1 + \sigma_0 \tag{4}$$

where σ_1 and σ_0 are positive integer (related to analytic continuation of $R_\infty(z)$ at $z = 0$ rep $z = 1$) will be needed to prove the following theorem :[Huttner.Israel Math Journal,[Hu]]

Theorem 2 (Main theorem) ♠ *Under the assumption (3), the polynomial $A_q(z)$ and the remainder $R_\infty(z)$ are solutions of the Fuchsian differential equation :*

$$\theta^q(\theta+1-\sigma_0) - z(\theta+\sigma_\infty)(\theta-n)^q = 0 \tag{5}$$

$R_\infty(z)$ is analytic in the vicinity of $z=\infty$ and belongs to the exponent σ_∞ at $z=\infty$. As usual , we put $\theta = z\frac{d}{dz}$.
$R_\infty(z)$ is an hypergeometric power series!

$$R_\infty(z) = C_\infty(n)(1/z)^{\sigma_\infty}\times$$

$${}_{q+1}F_q\left(\begin{matrix}\sigma_\infty,\cdots,\sigma_\infty,\sigma_\infty+\sigma_0\\ \sigma_\infty+n,\cdots,\sigma_\infty+n\end{matrix}\middle|1/z\right) \tag{6}$$

where $C_\infty(n)$ denotes a constant which depends on σ_∞ and σ_0.

$$A_q(z) = {}_{q+1}F_q\left(\begin{matrix}-n,-n,\cdots,-n,\sigma_\infty\\ 1,\cdots,1,1-\sigma_0\end{matrix}\middle|z\right) \tag{7}$$

• To obtain a polynomial (hypergeometric) solution at $z=0$, we must suppose that $\sigma_0 = 0$. (In this case the polynomial $A_q(z) \in \mathbb{Z}[z]$ or $1-\sigma_0 \leq -n$. *i.e* ,$\sigma_0 > 1+n$.) (See the well-poised-case where we have the relation :

$$\sigma_\infty + 1 - \sigma_0 = 1 - n$$

In particular the study of this differential equation gives the rational approximation related to Rivoal's theorem .

Theorem 3 (Rivoal's Theorem) *For any even $q \geq 4$,*

$$dim_{\mathbb{Q}}(\mathbb{Q}+\mathbb{Q}\zeta(3)+\mathbb{Q}\zeta(5)+\cdots+\mathbb{Q}\zeta(q-1)) \geq \frac{1+o(1)}{1+\log 2}\log(q)$$

2 Polylogarithmic functions and local systems

♠ Now we review the necessary mathematical background which allows to understand this lecture :
In the following we put :

$$Z = \mathbb{P}_1(\mathbb{C}) - \{0,1,\infty\}$$

Let us recall that for q integer , $q \geq 1$

$$Li_q(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^q},\ z \in \mathbb{C},\quad |z| \leq 1 \tag{8}$$

(If $q \geq 2$ and < 1 if $q = 1$).
has an analytic continuation to the cut plane $X = \mathbb{C} - [1, +\infty[$.
For $q \geq 2$, we have

$$\theta(Li_q(z)) = Li_{q-1}(z)$$

In this case ,$y(z) = Li_q(z)$ is the holomorphic solution of the non-homogeneous differential equation :

$$(1-z)\frac{d}{dz}(\theta^{q-1})(y) = 1$$

Remark 3 *A basis of solutions at $z = 0$ of this equation is*

$$1, \log z, \frac{(\log z)^2}{2}, \cdots, \frac{(\log z)^{q-1}}{(q-1)!}, Li_q(z)$$

The polylogarithmic functions , $Li_q(z)$ has an analytic continuation to X and may be conceived of as a 'multivalued ' function on Z (i.e. function on W the universal covering of Z) .
Let us recall also the following integral formulae

$$Li_1(z) := -\log(1-z) = \int_0^z \frac{dt}{1-t}$$

and for the *higher logarithm* :

$$Li_{q+1}(z) := \int_0^z \frac{Li_q(t)}{t} dt.$$

We use the analytic continuation of $Li_1(z), Li_2(z) \cdots, Li_q(z)$ along loops γ_1 circling $z = 1$, and γ_0 circling $z = 0$.

- Analytic continuation along γ_1 gives :

$$Li_k(z) \to Li_k(z) + \frac{(2i\pi)^{k-1}}{(k-1)!}(\log z)^{k-1}$$

Using monodromy , it is easy to see that the $q+1$ fonctions

$$1, \log(1-z), Li_2(z) \cdots, Li_q(z)$$

are $\mathbb{Q}(z)$ linearly independent .Thus , we obtain a local system

$$\mathcal{P}L_i(q) =: \mathbb{C}(z)\{\log(1-z), \cdots, Li_q(z)\}$$

which is of rank $q+1$ over $\mathbb{C}(z)$.

Remark 4 • *The connections formulae for the $Li_q(z)$ between $z = 0$ and $z = \infty$ involve Bernoulli polynomials in* $\log z$ *). That give the analytic continuations of $R_\infty(z)$ at $z = 0$ and $z = 1$.*
The monodromy group of this local system is well known ; it is in particular unipotent .

3 Periods

♠ We use the analytic continuation of $Li_1(z), Li_2(z) \cdots, Li_q(z)$ along loops γ_1 and γ_0.
The second row is a result of the monodromy transform of the first row along loop γ_1, the third row along loop γ_0, i.e analytic continuation along

$$\gamma_1, \gamma_0 \circ \gamma_1 \cdots, \gamma_0^{q-2} \circ \gamma_1, \gamma_0^{q-1} \circ \gamma_1.$$

We obtain the following matrix of "periods" :

Theorem 4

$$\Lambda(z) = \begin{pmatrix} 1 & Li_1(z) & & \cdots Li_q(z) \\ 0 & 2i\pi \cdots & & 2i\pi \log^{(q-1)} z/(q-1)! \\ & \ddots & & \\ 0 & 0 & (2i\pi)^{q-1} & (2i\pi)^{q-1} \log z \\ 0 & \cdots & 0 \cdots & (2i\pi)^q \end{pmatrix} \tag{9}$$

3.1 Proofs :Analytic construction of linear forms of polylogarithmic functions

♠ Let us recall the main steps of this proof which is almost the same as in [Hu]). (In this paper we study the approximation at infinity , i.e. z is replaced by the local parameter $1/z$).
• Consider the linear form :

Definition 4

$$R_\infty(z) = A_0(z) + \sum_{k=1}^{q} A_k(z) Li_k(1/z).$$

Now this form gives rise to linear forms obtained by use of analytic continuation of $R_\infty(z)$ along loops based in a vicinity of $z = 1$ resp $z = 0$ (i.e. is monodromy around the points $z = 1$ and $z = 0$)

$$\begin{pmatrix} R_\infty(z) \\ R_1(z) \\ \vdots \\ \vdots \\ R_q(z) \end{pmatrix} = \Lambda(z) \begin{pmatrix} A_0(z) \\ \vdots \\ A_k(z) \\ \vdots \\ A_q(z) \end{pmatrix} \tag{10}$$

Now ,from a local system of rank $q+1$ we can construct a differential equation whose solutions are given by a basis of this local system.

Theorem 5 (Classical theorem) • *Let $f_1(z), \cdots, f_{q+1}(z)$ be a system of multivalued and regular holomorphic functions on Z such that its Wronskian $det(f_i^{(j)}) \neq 0$ and such that the analytic continuations of the f_j's along the loops γ_j define automorphisms of the space of functions spanned by f_k's .*
Then there exists a $(q+1)^{th}$ order differential equation with coefficients in $\mathbb{C}(z)$ such that the system $f_1(z), \cdots, f_{q+1}(z)$ of functions is its fundamental system. (The matrix of analytic continuations of the f_j's along loops γ_j are called monodromy matrices).

Using this theorem ,we obtain :

Theorem 6 • *$R_1(z), \cdots, R_q(z) = (2i\pi)^q A_q(z)$ satisfy the same Fuchsian differential equation of order $q+1$ as $R_\infty(z)$.*

3.2 Applications of Levelt's construction to Padé problem

As the analytic continuation of $R_\infty(z)$ along γ_1 is

$$R_\infty(z) \to R_\infty(z) + 2i\pi \sum_{k=1}^{q} A_k(z)(\frac{(\log(1/z))^{k-1})}{(k-1)!})$$

• We put now

$$R_1(z) = 2i\pi(\sum_{k=1}^{q} A_k(z)(\frac{(\log(1/z)^{k-1}}{(k-1)!})$$

Using analytic continuation of $R_1(z)$ along γ_0 gives

$$R_2(z) = (2i\pi)^2 \sum_{k=2}^{q} A_k(z)(\frac{(\log(1/z))^{k-2}}{(k-2)!}; \; R_3(z) = \cdots$$

That gives $\Lambda \times (A(z))$ where

$$A(z) = (A_0(z), A_1(z) \cdots, A_q(z))^t$$

• The exponents at $z = \infty$ are :

$$\sigma_\infty, -n, \cdots, -n.$$

• At $z = 0$ one finds

$$\sigma_0, 0, \cdots, 0.$$

(σ_0 is the exponent given by analytic continuation at 0 of $R_\infty(z)$.)
• At $z = 1$:

$$0, 1, \cdots, q-1, \sigma_1$$

Fuchs relation for Fuchsian differential equations of order $(q+1)$ gives :(Fuchs relation) $\sigma_0 + \sigma_\infty + \sigma_1 - qn + \frac{q(q-1)}{2} = \frac{q(q+1)}{2}$

$$\sigma_0 + \sigma_\infty + \sigma_1 = q(n+1)$$

(There does not exist apparent singularities and we find exactly the number of coefficient of the polynomials $A_k(z)$.)
• Let σ_0 , and at $z = 1$ (σ_1) , be the exponents related to the analytic continuations of $R_\infty(z)$ (which depend on additional assumptions on the polynomials $A_k(z)$.)
• The Riemann scheme related to this equation gives the main theorem! :

Theorem 7 **(Main Riemann scheme)**

$$P\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \sigma_0 & \sigma_\infty & \sigma_1 & \\ 0 & -n & 0 & \\ 0 & -n & 1 & |z \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \\ 0 & -n & q-1 & \end{pmatrix}$$

which can be written :

$$(1/z)^{\sigma_\infty} P\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \sigma_0 & \sigma_\infty + \sigma_0 & \sigma_1 & \\ -\sigma_\infty - n & \sigma_\infty & 1 & \\ -\sigma_\infty - n & \sigma_\infty & 2 & |1/z \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \\ -\sigma_\infty - n & \sigma_\infty & 0 & \end{pmatrix}$$

• The elements of this local system are solutions of the following differential equation:

$$(\theta^q(\theta + 1 - \sigma_0) - z(\theta + \sigma_\infty)(\theta - n)^q)(y) = 0\,.$$

Within a multiplicative constant this give the formulae of the main theorem for the remainder as well as the Fuchsian differential equation .
• If one puts $A_q(z) = \sum_{j=0}^{n} c(j) z^j$, the other polynomials are obtained by

the use of Frobenius method for solving Fuchsian linear differential equations , i.e ,for $1 \leq k \leq q-1$,

$$A_{q-k}(z) = \frac{d^k}{dt^k}[\sum_{j=0}^{n} c(j+t)z^j]|_{t=0}$$

Let us recall that the' logarithmic' solutions of the Fuchsian differential equation are given by

$$R_k(z) = \frac{d^k}{dt^k}[\sum_{j=0}^{n} c(j+t)z^{j+t}]|_{t=0}.$$

• The Padé case is related to $\sigma_0 = \sigma_1 = 0$.

3.3 D-modules

♠ But if $\sigma_0 \geq 1+n$, i.e. if there exist relations between the analytic continuation of the power series $R_\infty(z)$ at $z = \infty$ and at $z = 0$, we find that the rank of the $D-$ module $\frac{\mathbb{Q}(z)(\theta)}{L(\theta)} \cong$

$$\mathbb{Q}(z)[Li_1(1/z), \cdots, Li_q(1/z)].$$

is q , (not of rank $q+1$. as expected!)
• The previous fact has been verified by Rivoal himself and has been generalized by Nesterenko .
There exists also an elementary proof using a decomposition in partial fraction of $R_\infty(z)$ [Ba,Ri] .
For a proof , we can use the following relations :

$$(\theta + a_0)_p/(a_0)_p \, {}_{q+1}F_q \left(\begin{array}{c} a_0, a_1, \cdots, a_q \\ b_1, b_2, \cdots, b_q \end{array} \middle| z \right) =$$

$${}_{q+1}F_q \left(\begin{array}{c} a_0 + p, a_1, \cdots, a_q \\ b_1, b_2, \cdots, b_q \end{array} \middle| z \right)$$

If , for instance $a_0 + p = b_1$, we obtain

$${}_qF_{q-1} \left(\begin{array}{c} a_1, \cdots, a_q \\ b_2, \cdots, b_q \end{array} \middle| z \right).$$

3.4 The well-poised case

♠ We consider the differential operator

$$H(\theta) = (\theta + a_0)_p/(a_0)$$

for $a_0 = \sigma_\infty; p = \sigma_\infty - n - 1$.
In this case we can write ,

$$R_\infty(z) = H(\theta)(\tilde{R}(z))$$

$\tilde{R}(z)$ being a solution of a Fuchsian differential equation of order q and $H(\theta)$ commutes with the monodromy .
We obtain a shift for the linear combination of polylogarithmic functions $Li_k(1/z)$ with, for $1 \leq k \leq q-1$: the same polynomials .
The new polynomial A_{q-1} replaces the previous polynomial $A_q(z)$ i.e $A_q(z) = 0$,etc .
• The new linear form becomes :

$$R_\infty(z) = \sum_{j=1}^{q-1} A_j(s)Li_j(1/z) + \bar{A}_0(z)$$

$$\bar{A}_0(z) = -[A_q(z)Li_{q-1}(1/z) + \cdots A_2(z)Li_1(1/z)]_{n-1}$$

(polynomial part at the order $n-1$). This gives the "well-poised-case".
• In the literature concerning special functions , [AAR] : if the parameters of the hypergeometric power series satisfy

Definition 5 $a_0 + 1 = a_1 + b_1 = \cdots = a_q + b_q$ *the power-series is said well-poised.*
• *It is said very-well-poised if it is well-poised and* $a_1 = \frac{1}{2}a_0 + 1$

Remark 5 *In the present problem , in the very-well poised case , one finds that the first polynomial* $A_{q-1}(z)$ *satisfies* $A_{q-1}(1) = 0$.
The differential equation satisfied by this polynomial is of order $q+1$ *but the local system is of rank* $r_L = q-2$ *over* $\mathbb{C}[z]$.

• Let us consider the relation

$$\sigma_\infty - \sigma_0 + 1 = 1 - n \tag{11}$$

This relation means that in the above differential equation $y(z)$ is a solution if and only if $z^n y(1/z)$ is also a solution. In this case, the remainder $R_\infty(z)$ can be written [Well-poised remainder]

$$R_\infty(z) = (1/z)^{\sigma_\infty} \cdot {}_{q+1}F_q\left(\begin{matrix} 2\sigma_\infty, \cdots, \sigma_\infty, \sigma_\infty \\ \sigma_\infty + n + 1, \cdots, \sigma_\infty + n + 1 \end{matrix} \middle| 1/z \right)$$

• The polynomial $A_q(z)$ satisfies the relation

Theorem 8 (A reciprocal polynomial) $A_q(z) = (-1)^{(q+1)n} z^n A_q(1/z)$.

Let us write $A_q(z) = \sum_{j=0}^{n} c_j z^j$.
For $0 \leq j \leq n$, we find , $c_j = c_{n-j}$.
Since these polynomials are solutions of a Fuchsian differential equation , the other polynomials are computed by Frobenius method.
• For $1 \leq k \leq q-2$ the polynomial coefficients of $A_k(z)$ satisfy the relations :

$$\frac{d^k}{dt^k}(c_{j+t})|_{t=0} =$$

$$\frac{d^k}{dt^k}(c_{n-(j+t)})|_{t=0} = (-1)^k \frac{d^k}{dt^k}(c_{j+t}))|_{t=0}.$$

We find :

$$A_{q-k}(z) = (-1)^{(q+1)n+k} z^n A_{q-k}(1/z).$$

3.5 Arithmetic applications

♠ For $k = 2 \cdots q-1$, the polynomials $A_k(z)$ are such that

$$A_{q-2}(1) = A_{q-4}(1) \cdots A_2(1) = A_1(1) = 0.$$

In particular if $q - 1 = 2a + 1$ is odd , we obtain the famous Rivoal's relation on linear form of $\zeta(2k+1)$ [Ba,Ri]. The remainder can thus be written :

Theorem 9

$$R_\infty(1) = A_{2a+1}(1)\zeta(2a+1) + \cdots + A_3(1)\zeta(3) + \tilde{A}_0(1).$$

We have multiplied the remainder by a normalized constant related to various integral values which represent $R_\infty(z)$ and also by an common denominator D_n such that :
$A_q(z) \in \mathbb{Z}[z]$ and $d_n^k A_{q-k}(z) \in \mathbb{Z}[z]$
• We put

$$\sigma_\infty = rn + 1, \sigma_0 = \sigma_\infty + n$$

with the parameter r satisfying : $1 \leq r \leq \frac{q-1}{2}$. ($\sigma_1 \geq 1$.
These assumptions permits us to compute the remainder $R_\infty(z)$ at $z = 1$).
In this case ,

$$A_q(z) =_{q+1} F_q \left(\begin{matrix} rn+1, -n, \cdots, -n \\ -(r+1)n, 1, \cdots, 1 \end{matrix} \middle| z \right).$$

The remainder is given by

$$R_\infty(1) = C(n, r, q)._{q+1}F_q \left(\begin{matrix} (2r+1)n+2, rn+1, \cdots, rn+1) \\ (r+1)n+2, \cdots, (r+1)n+2 \end{matrix} \middle| 1 \right)$$

$$C(n,q,r) = n!^{q-1-2r}\frac{(rn!)^q\,((2r+1)n+1)!}{((r+1)n+1)!^q}$$

• The remainder can also be written using Euler's integral

$$R_\infty(z) = \frac{(2r+1)n)!}{n!^{2r+1}}\int_{[0,1]^q}[\frac{\prod_{l=1}^q(t_l^r(1-t_l)}{(1-t_1\cdots t_q)^{2r+1}}]^n dt_1\cdots dt_q$$

4 Apéry, Gutnik, Nesterenko , $\zeta(2)$ and $\zeta(3)$

♠ Let us recall Beukers's and Gutnik 's method ,[Be],[Gu] concerning simultaneous approximations of $\zeta(2)$ and $\zeta(3)$.
• Linear Algebra shows that there exists four polynomials

$$A_3(z), A_2(z), A_1(z), A_0(z)$$

of degree n such that :

$$R_1(z) = A_3(z)Li_2(1/z) + A_2(z)Li_1(1/z) + A_1(z)$$
$$R_2(z) = 2A_3(1/z)Li_3(z) + A_2(Li_2(1/z) + A_0(z)$$

satisfying $Ord_\infty R_1(z) \geq n+1$, $Ord_0 R_2(\infty) \geq n+1$ and $A_2(1) = 0$.

Remark 6 *The main idea to motivate the introduction of $R_2(z)$ comes from "Frobenius method of perturbing the power series".*
In this aim we introduce the function :

$$Li_k(z,s) = \sum_{n=1}^{\infty}\frac{z^{n+s}}{(n+s)^k}$$

where s denotes a 'formal' variable .

Since ,

$$\frac{\partial Li_k(z,s)}{\partial s}|_{s=0} = Li_k(z)\log z - kLi_{k+1}(z)$$

Using the following function :

$$R_1(1/z,s) = A_3(z)Li_2(1/z,s) + A_2(z)Li_1(1/z,s) + A_1(z,s)(1/z)^s$$

• An easy computation shows that :

$$\frac{\partial R_1(1/z,s)}{\partial s}|_{s=0} =$$

$$R_1(1/z)\log(1/z) - R_2(z)$$

with $A_1(z) = A_1(z,s)|_{s=0}$ and

$$A_0(z) = \frac{\partial A_1(z,s)}{\partial s}|_{s=0}.$$

• We put now

$$\tilde{R}_2(z) = \log(1/z).R_1(z) - R_2(z) \tag{12}$$

We can construct a linear differential operator L of order at least 4 such that at $z = \infty$.

Since

$$\tilde{R}_2(z) = \log(1/z).R_1(z) - R_2(z)$$

is a (logarithmic) solution of $L = 0$.
• Monodromy around 0 shows that $L(R_1(z)) = 0$.
Now if we put :

$$R_3(z) = A_3(z)\log(1/z) + A_2(z),$$

monodromy around 1 shows that $L(R_3(z)) = 0$.
Monodromy around 0 for $R_4(z) = A_3(z)$ yields $L(R_4(z)) = 0$.

Theorem 10 *The 'Levelt basis' of solutions of L at 0 is*

$$\tilde{R}_2(z), R_1(z), R_3(z), R_4(z)$$

They are linearly independent solutions at $z = \infty$ of a Fuchsian differential equation of order 4 .
• *The Riemann scheme of L is :*

$$P\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ 0 & -n & 0 & \\ 0 & -n & 1 & |z \\ 0 & n+1 & 2 & \\ 0 & n+1 & 1 & \end{pmatrix}.$$

The unique differential hypergeometric equation related to this Riemann scheme is

$$\theta^4 - z(\theta - n)^2(\theta + n + 1)^2 = 0.$$

• This Riemann scheme gives the famous Apéry's polynomial ,[Ap]

$$A_3(z) = {}_4F_3\left(\begin{matrix} -n, -n, n+1, n+1 \\ 1,1,1 \end{matrix} |z\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 z^n$$

• We also see that $R_3(1) = 0$ *i.e* $A_2(1) = 0$ (see the Riemann scheme)!) We find the form of the remainder, only by studying this Riemann scheme ! One finds that $R_1(z)$ is equal (with the choice of a multiplicative normalisation 's constant) to

$$\frac{n!^4}{(2n)!^2}(1/z)^{n+1}\, {}_4F_3 \left(\begin{matrix} n+1, n+1, n+1, n+1 \\ 2n+1, 2n+1, 1 \end{matrix} \, \Big| 1/z \right).$$

If one puts ,

$$R_1(z) = \frac{n!^4}{(2n+1)!^2} \frac{1}{z^{n+1}} r_1(z)$$

and $r_1(z) = \sum_{n=0}^{\infty} c_n (1/z)^n$,

• The 'logarithmic' solution belonging to the exponent $n+1$ is given by

$$r_2(z) = \frac{\partial}{\partial t}\left(\sum_{k=0}^{\infty} c_{n+t}(1/z)^{n+t}\right)\Big|_{t=0}.$$

4.1 $\zeta(3)$ is irrational !

• Since $\log 1 = 0$, we find:

$$r_2(1) = -\sum_{k=1}^{\infty} \frac{\partial}{\partial k}\left[\frac{(k-n)_n^2}{(k)_{n+1}^2}\right]$$

(which gives Beukers or Nesterenko's integral for the remainder.) Since , $d_n^3.A_0(1) \in \mathbb{Z}$,we obtain

$$(2A_3(1)\zeta(3) + A_0(1))d_n^3 = r_2(1).d_n^3$$

Since ,

$$\lim_{n \to \infty} d_n^3 r_2(1) = 0.$$

the irrationality of $\zeta(3)$ is proved !

We can conclude that in many cases, the study of the Riemann scheme gives a complete answer for the determination of simultaneous rational approximation of polylogarithmic functions.

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